A Spin on Wave Dark Matter

Mustafa A. Amin
RICE

A Spin on Wave Dark Matter

15 min

A lower bound on dark matter mass

with Mehrdad Mirbabayi 2211.09775

10 min

Spin of wave DM from astrophysics?

with Jain	2109.04892
Jain, Zhang	2111.08700
Jain, Karur, Mocz	2203.11935
Jain	2211.08433
Long, Schiappacasse	2301.11470
Jain, Thomas, Wanischarungarung	2304.01985

motivation

dark matter mass ?

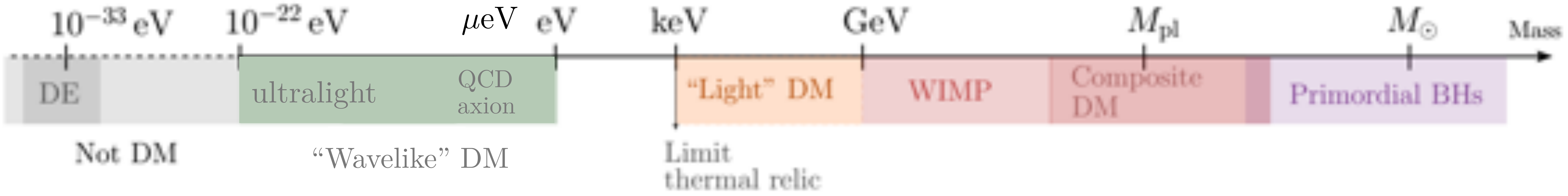
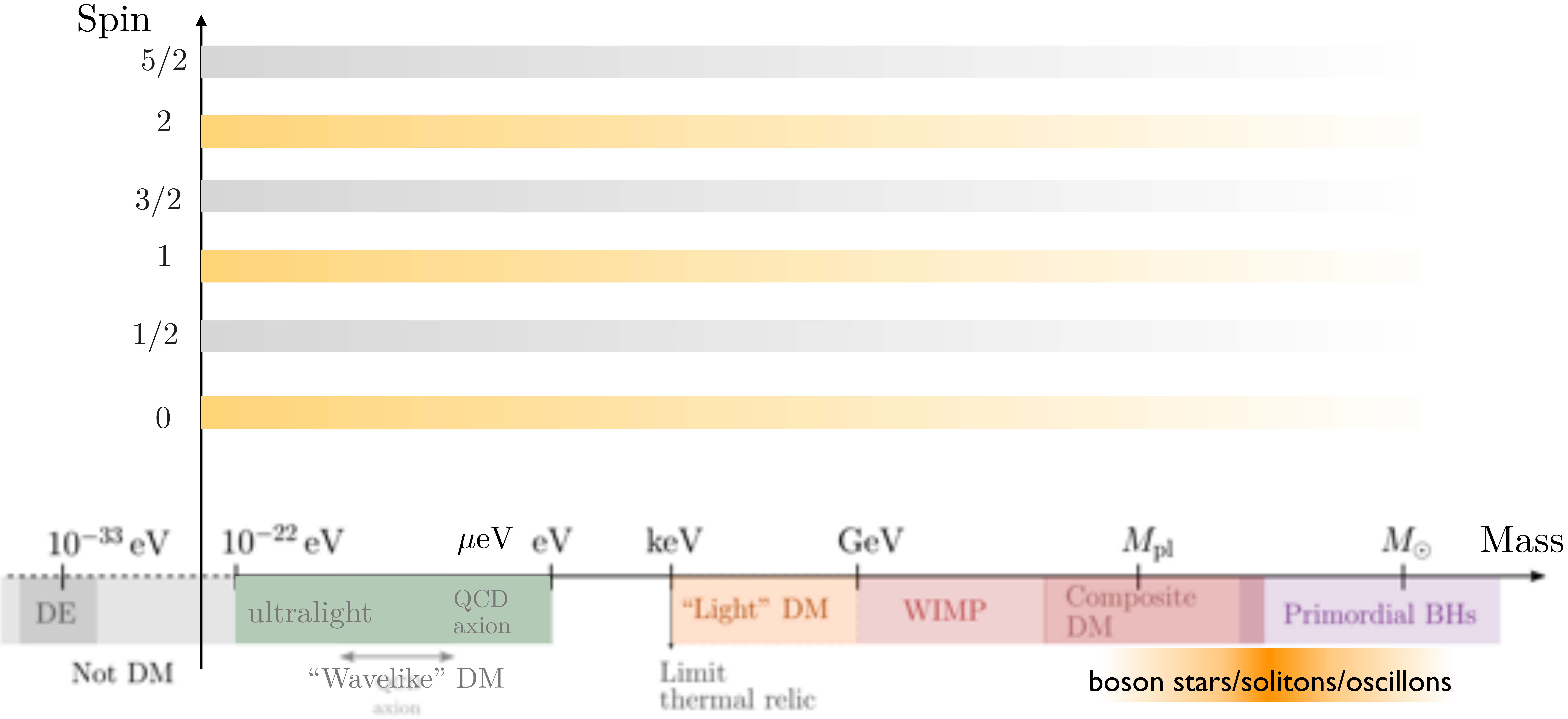
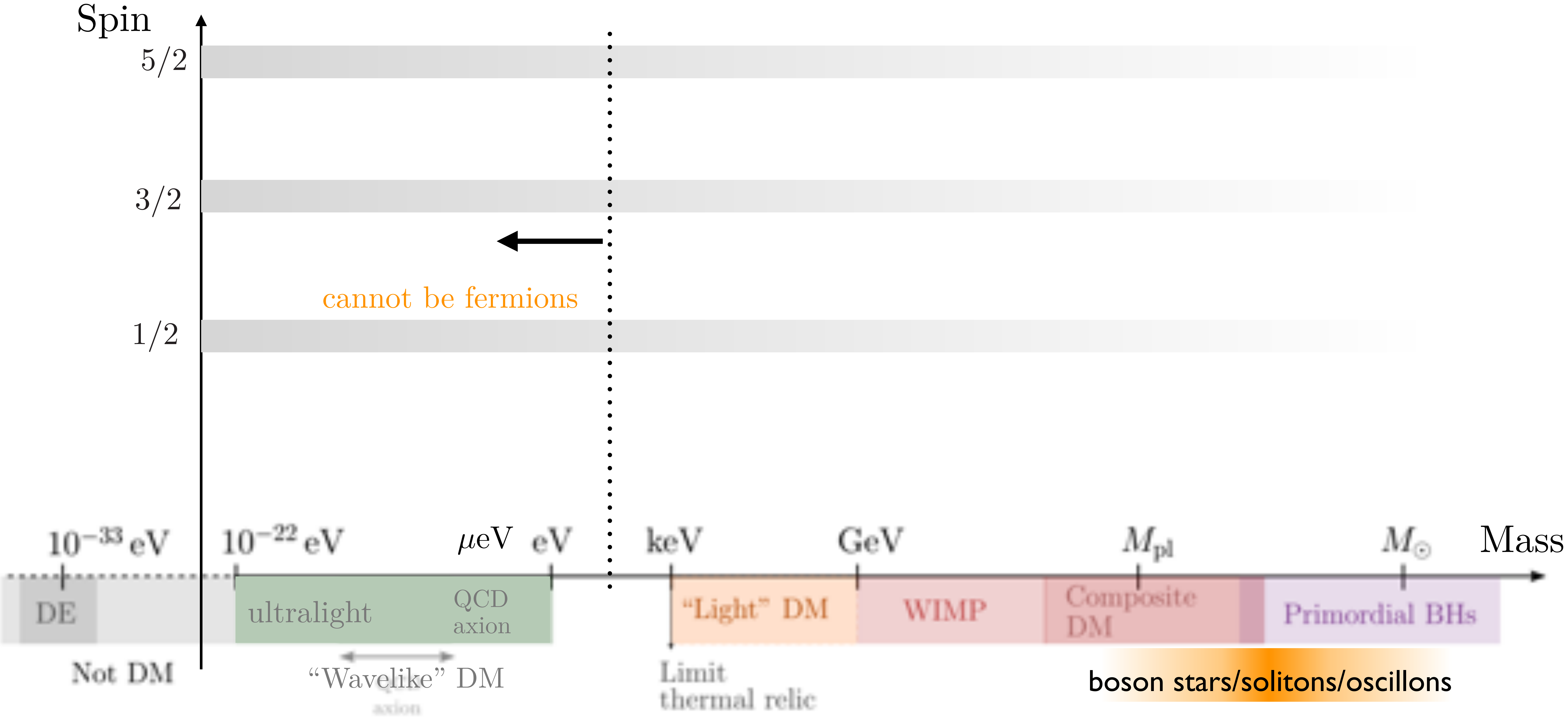


Image credit E. Ferreira (I have modified it for my purpose)

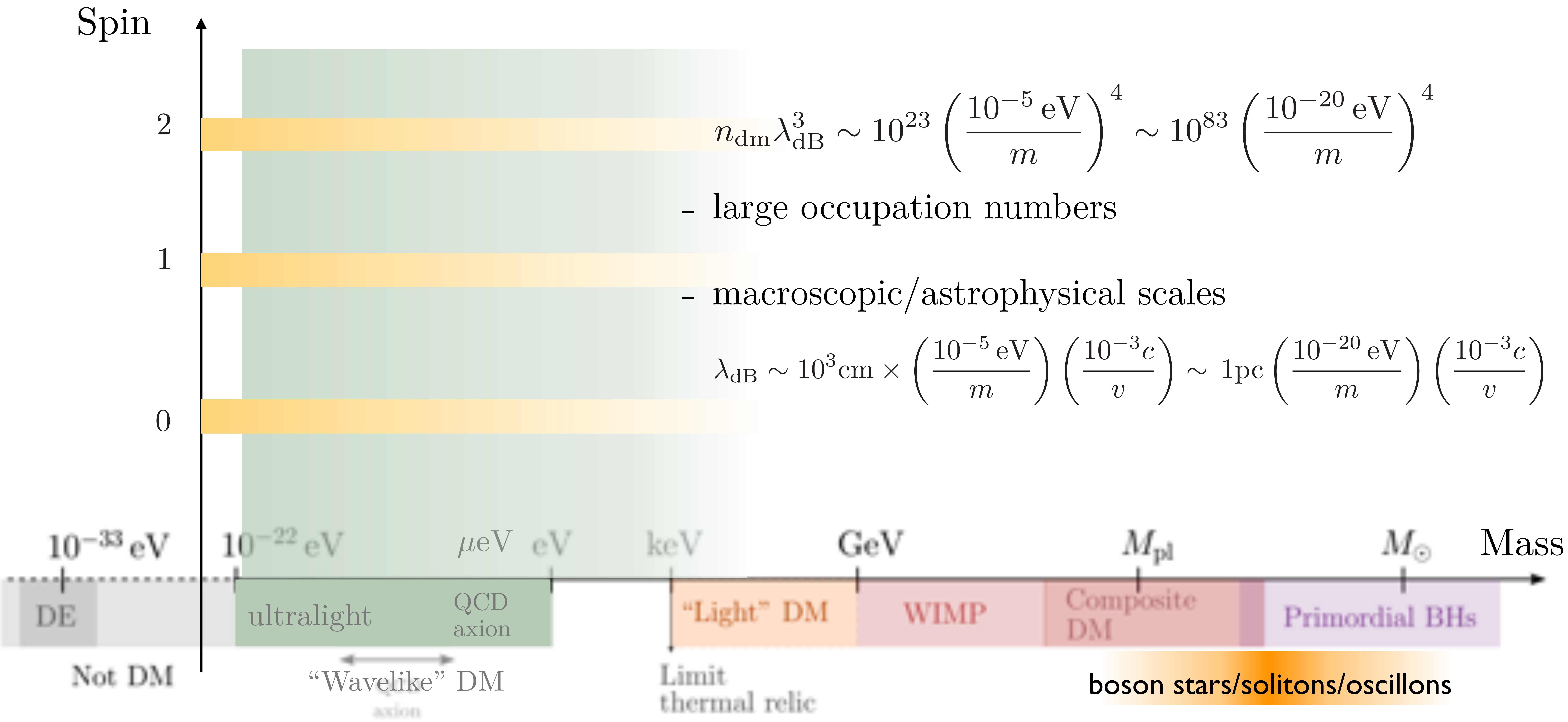
dark matter *spin* ?



dark matter spin ?



light, bosonic wave dark matter



$$n_{\rm dm}\lambda_{\rm dB}^3 \sim 10^{23}\left(\frac{10^{-5}\,{\rm eV}}{m}\right)^4 \sim 10^{83}\left(\frac{10^{-20}\,{\rm eV}}{m}\right)^4$$

- large occupation numbers
- macroscopic/astrophysical scales

$$\lambda_{\rm dB} \sim 10^3{\rm cm} \times \left(\frac{10^{-5}\,{\rm eV}}{m}\right)\left(\frac{10^{-3}c}{v}\right) \sim 1{\rm pc}\left(\frac{10^{-20}\,{\rm eV}}{m}\right)\left(\frac{10^{-3}c}{v}\right)$$

15 min

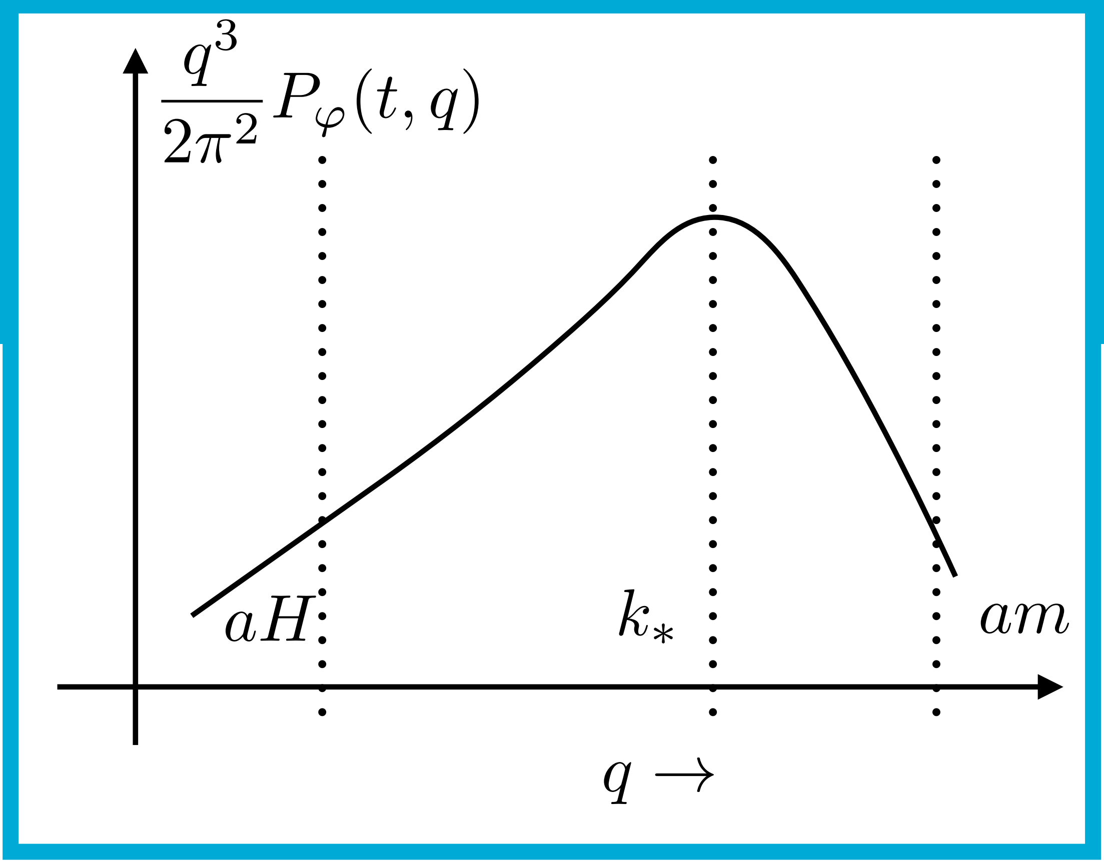
A lower bound on dark matter mass

Mustafa A. Amin RICE

with Mehrdad Mirbabayi (ICTP Trieste)

arXiv:2211.09775

main point



Dark matter density dominated by $\boxed{\text{sub-Hubble}}$ field modes

$$\implies m \gtrsim 10^{-18}\,\text{eV}$$

our argument

Dark matter density dominated by sub-Hubble field modes

1. white-noise excess in isocurvature density pert.
2. free-streaming suppression in adiabatic density pert.

1. and 2. not seen for $k < k_{\mathrm{obs}} \sim 10\,\mathrm{Mpc}^{-1}$

$$m \gtrsim 10^{-18}\,\mathrm{eV}$$

comparison with literature

$m \gtrsim 2 \times 10^{-21}\,\mathrm{eV}$ Irsic et. al (2017) — Lyα

$m \gtrsim 3 \times 10^{-21}\,\mathrm{eV}$ Nadler et. al (2021) — MW satellites

$m \gtrsim 3 \times 10^{-19}\,\mathrm{eV}$ Dalal & Kravtsov (2022) — dynamical heating of stars

$m \gtrsim 4 \times 10^{-21}\,\mathrm{eV}$ Powell et. al (2023) — lensing

$m \gtrsim 10^{-18}\,\mathrm{eV}$ MA & Mirbabayi (2022)

*Above are model independent constraints, stronger constraints exist for particular models (Irsic, Xiao & McQuinn, 2020)

some details

*to us, results were “intuitively convincing” but quantitative calculation is non-trivial

*analytic calculation of density spectra, see appendix of MA & Mirbabayi (2022)

average density from field

$\varphi(t, \boldsymbol{x})$

light, but non-relativistic scalar field during rad. dom.

$$\bar{\rho}(t) \approx m^2 \int d\ln q \, \frac{q^3}{2\pi^2} P_\varphi(t, q)$$

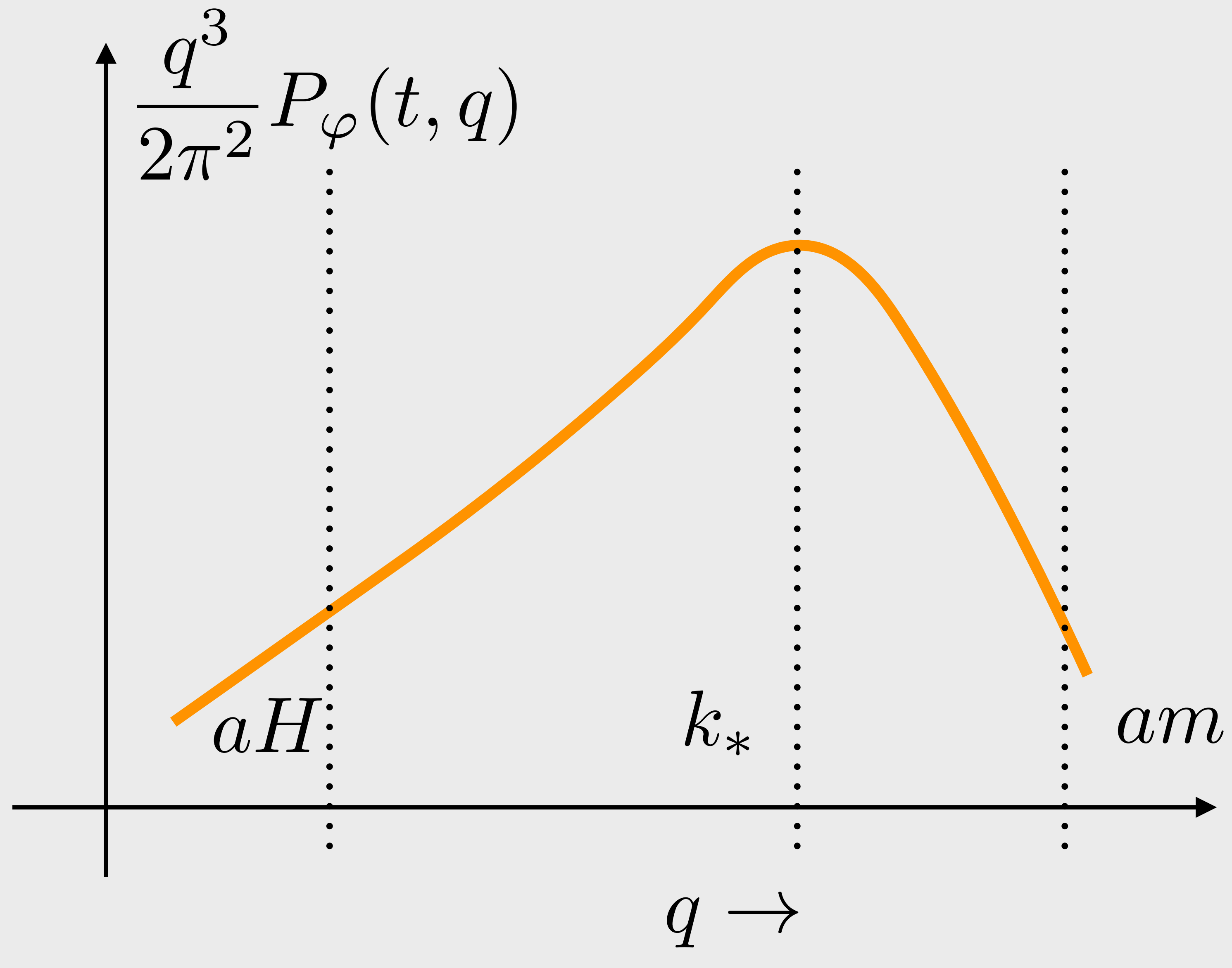
dark matter density close to matter radiation eq.

$$\frac{q^3}{2\pi^2} P_\varphi(t, q)$$

power spectrum of field, peaked at k_*

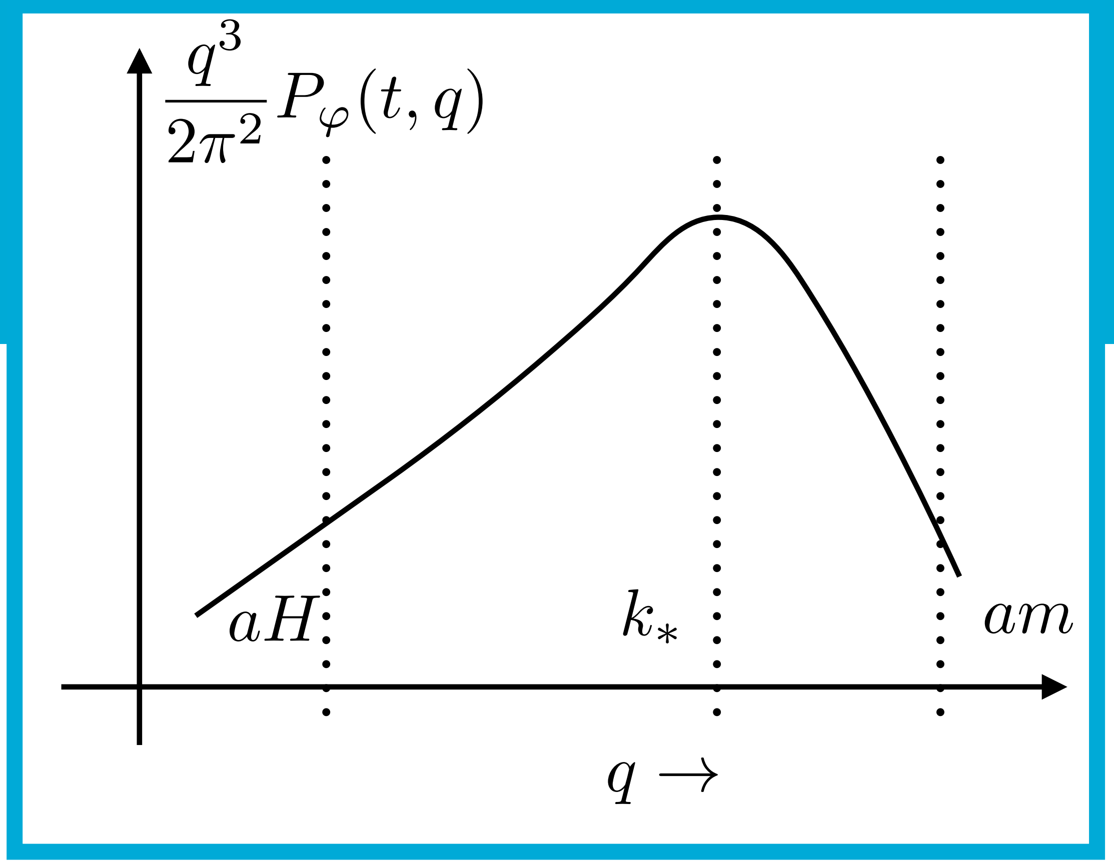
$a(t)H(t) \ll k_*$ holds for field produced after inflation

$k_* \ll a(t)m$ eventually non-relativistic to be DM



Note: no significant zero mode of the field

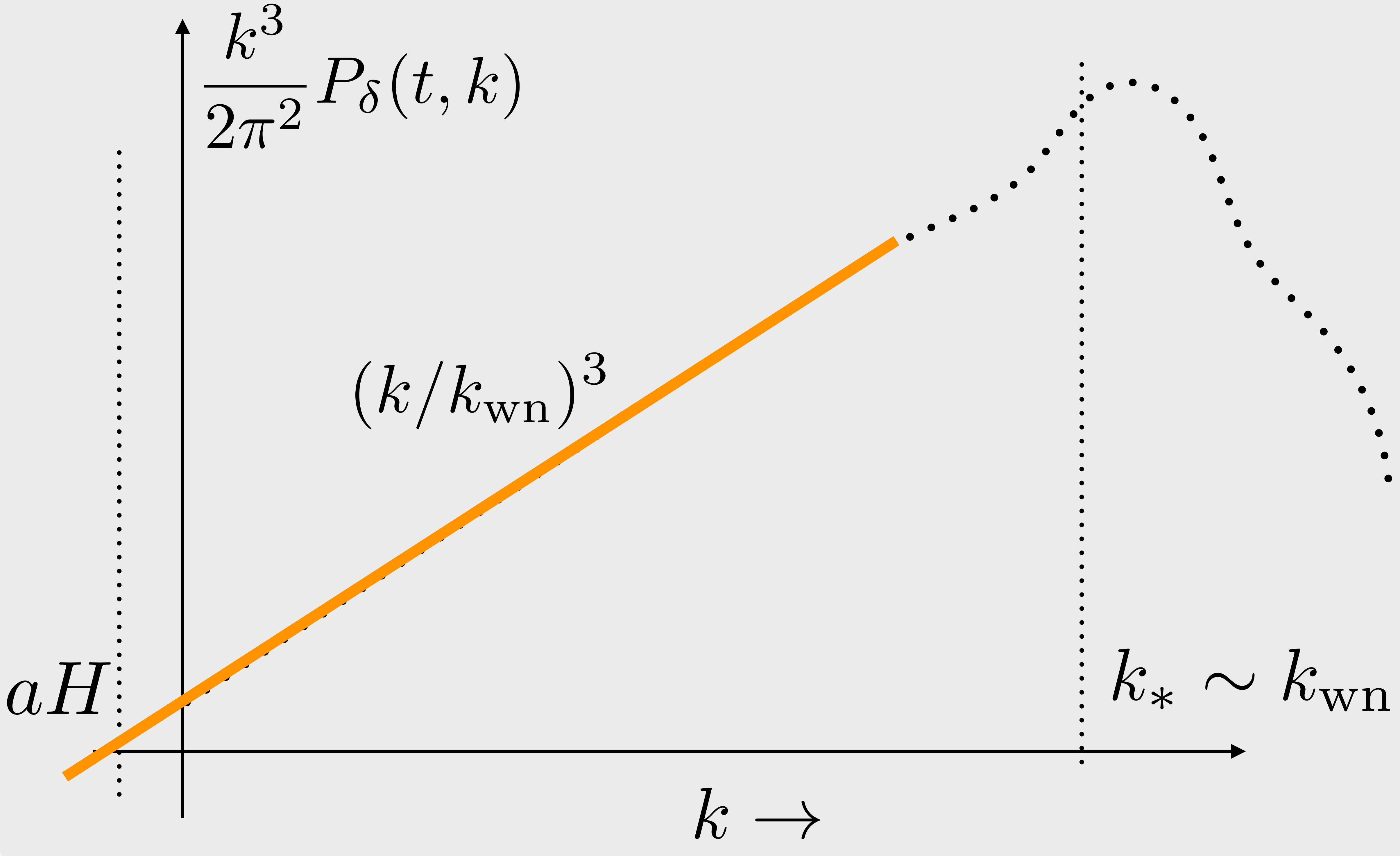
density power spectrum (isocurvature)



$$P_\delta^{(\mathrm{iso})}(t, k) \approx \frac{m^4}{\bar{\rho}^2(t)} \int d\ln q \, \frac{q^3}{2\pi^2} \left[P_\varphi(q, t)\right]^2 \equiv \frac{2\pi^2}{k_{\mathrm{wn}}^3}$$

independendent of k for $k \ll k_*$

k_{wn} is defined by the above relation

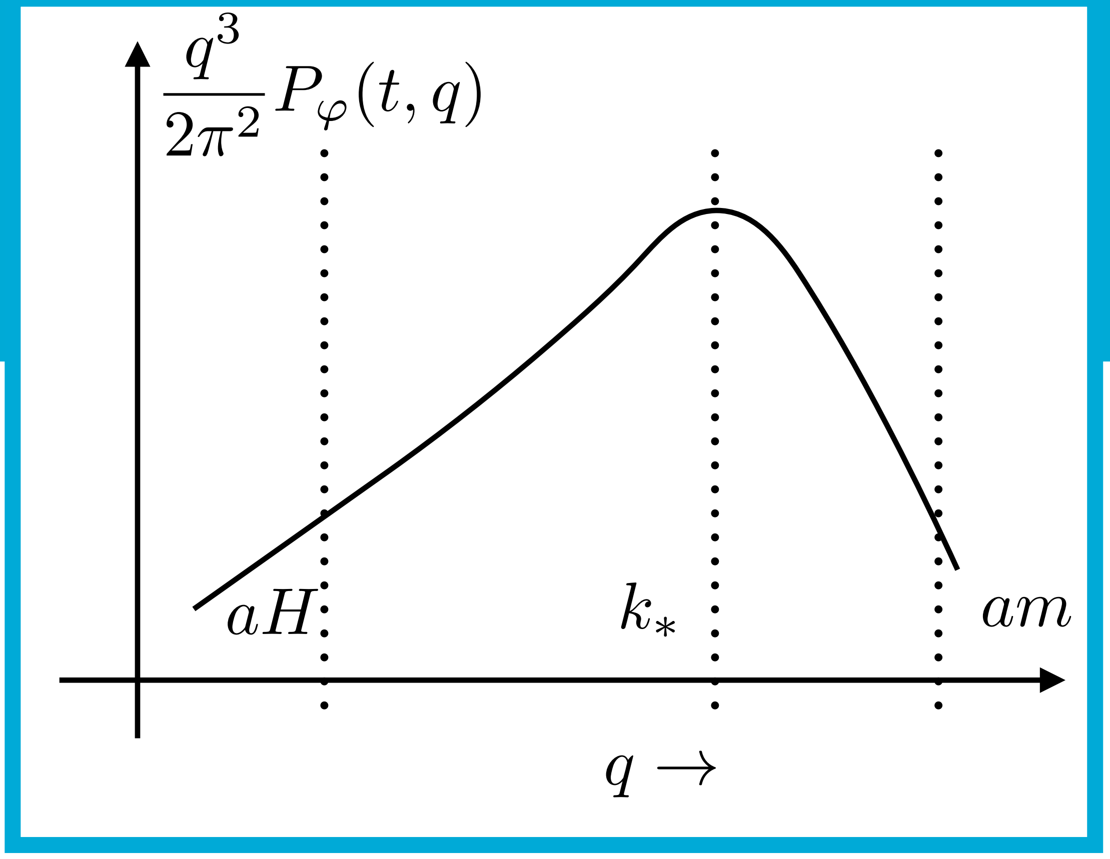


*ignore gravitational potentials on these scales during radiation domination

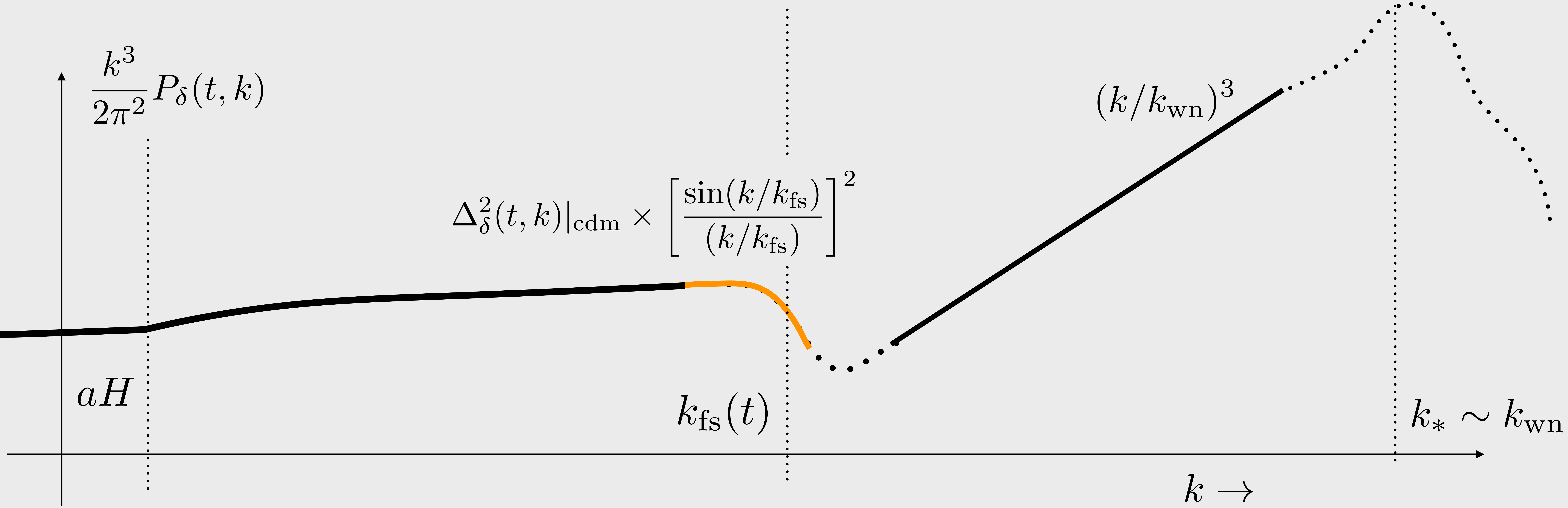
density power spectrum (adiabatic)

density perturbations in DM sourced by gravitational potentials in rad.

free streaming !



field power at $k_* \implies k_{\rm fs}(t) \approx \dfrac{a^2 H m}{k_* \ln(2am/k_*)}$



our argument — quantitative

Dark matter density dominated by sub-Hubble field modes

1. white-noise isocurvature excess in isocurvature density pert. $k_{\rm dev} \approx 10^{-3} k_*$
2. free-streaming suppression in adiabatic density pert. $k_{\rm fs}(t) \approx \dfrac{a^2 H m}{k_* \ln(2am/k_*)}$

1\. and 2. not seen for $k < k_{\rm obs} \sim 10\,{\rm Mpc}^{-1}$ e.g. [Lyα]

$\boxed{k_{\rm dev}, k_{\rm fs} \gtrsim k_{\rm obs}}$

$$m \gtrsim 10^{-18}\,{\rm eV}$$

Note that we did not need to know k_*!

strengths

"model independent" -- applies to all gravitationally interacting,

non-relativistic fields (scalar, vector, tensor ...)

"**loophole**" — inflationary production with infrared spectra (not sub-Hubble)

for vectors (and tensors?), even inflationary production leads to sub-Hubble spectra

$$k_{\rm fs} \ll k_J \sim a\sqrt{mH} \implies \text{stronger bound}$$

$$m_{\rm bound} \propto k_{\rm obs}^2 \implies \text{look at MW satellites}$$

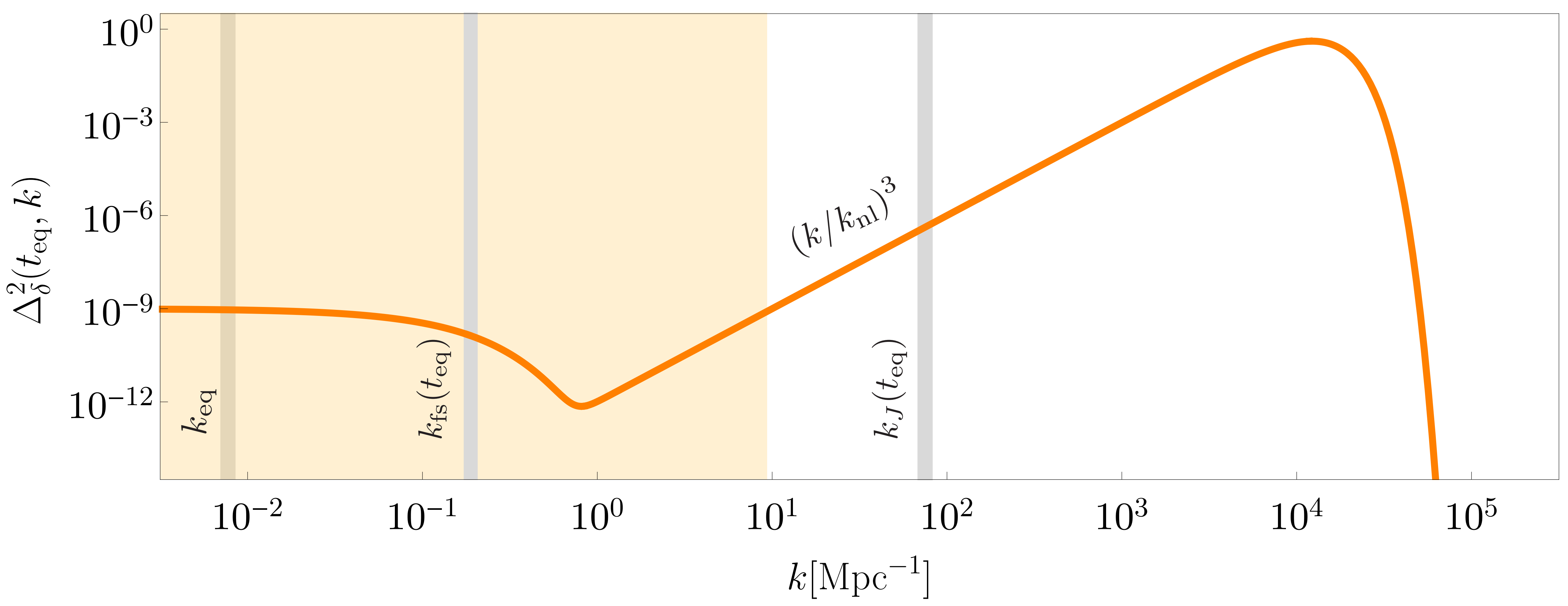
with Nadler and Wechsler

summary

Dark matter density dominated by sub-Hubble field modes

$\implies m \gtrsim 10^{-18}\,\mathrm{eV}$

bound good, detection better



extra small-scale structure

formation of mini-clusters/halos/solitons

some exciting phenomenology related to spin!

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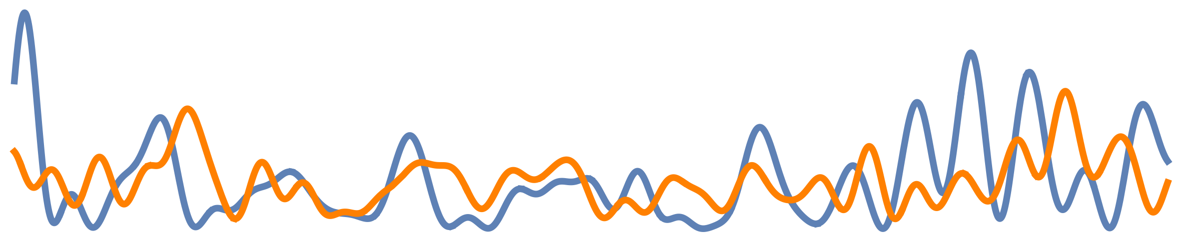
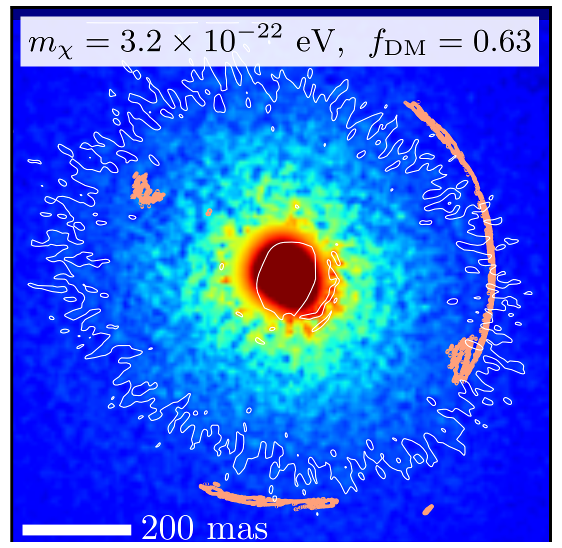
If I forget to mention it, for non-relativistic limit of vector DM, see Adshead and Lozanov (2021). For a nice new production mechanism, see Adshead, Lozanov and Weiner (2023)

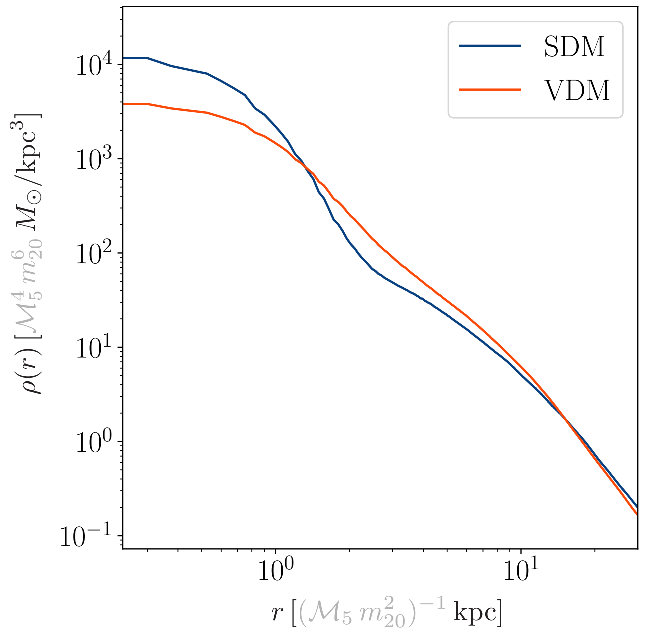
spin and dark matter sub-structure

Phenomenology

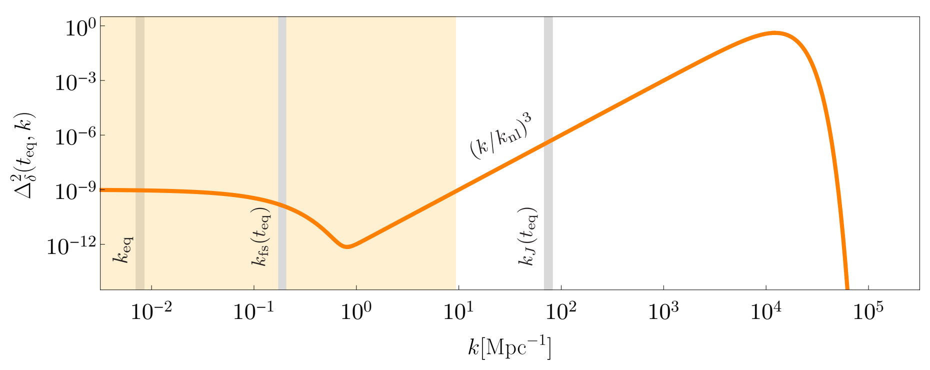
- reduced interference



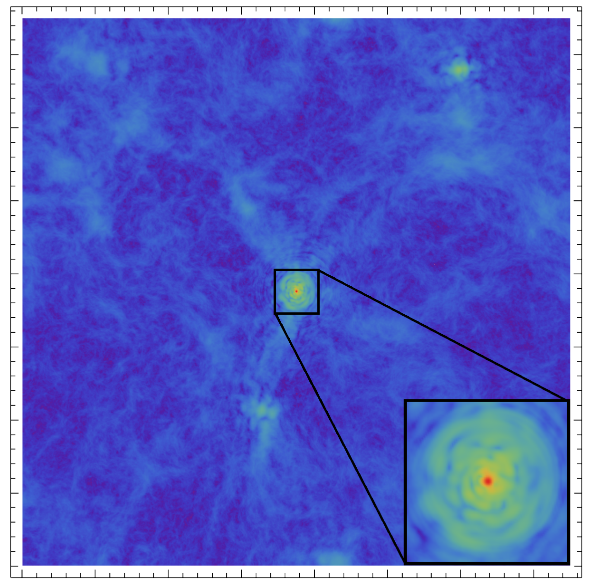


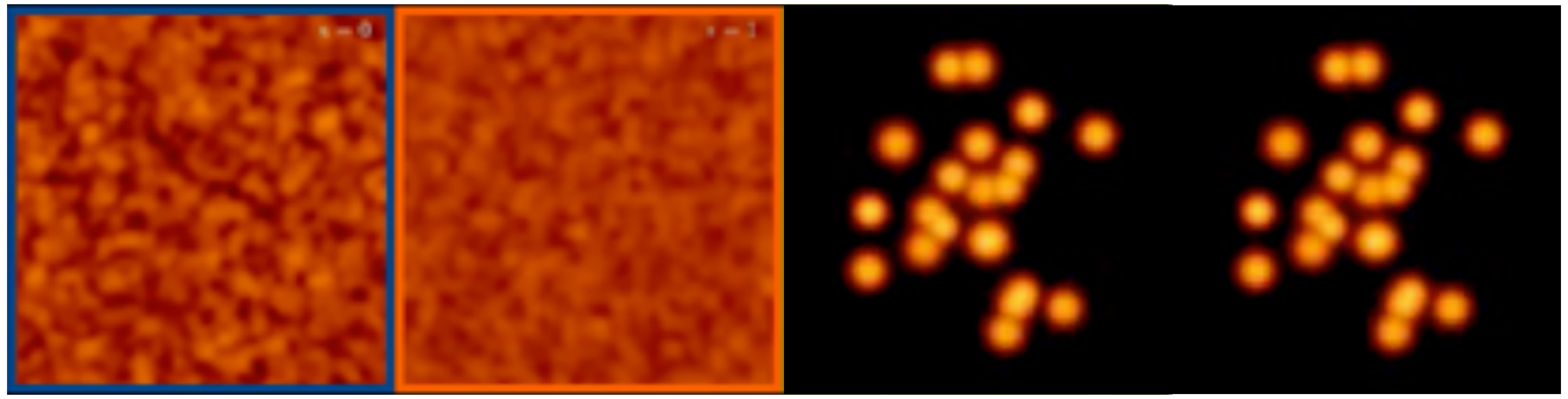


- growth of structure, nucleation time-scales

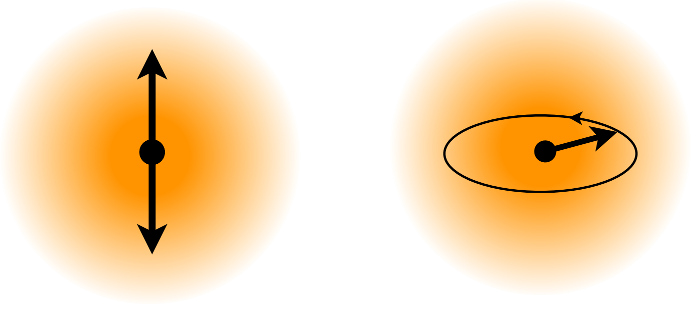


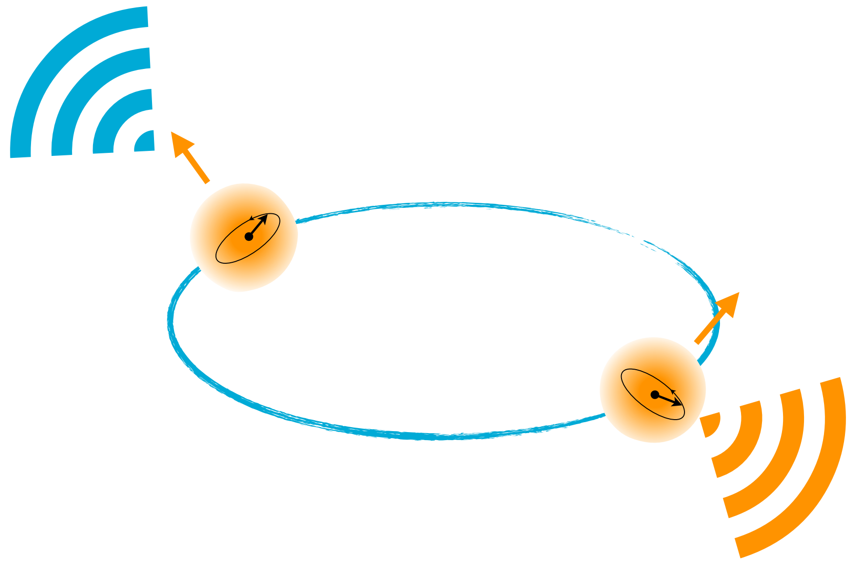
$\Longrightarrow$





- polarized solitons, with macroscopic spin

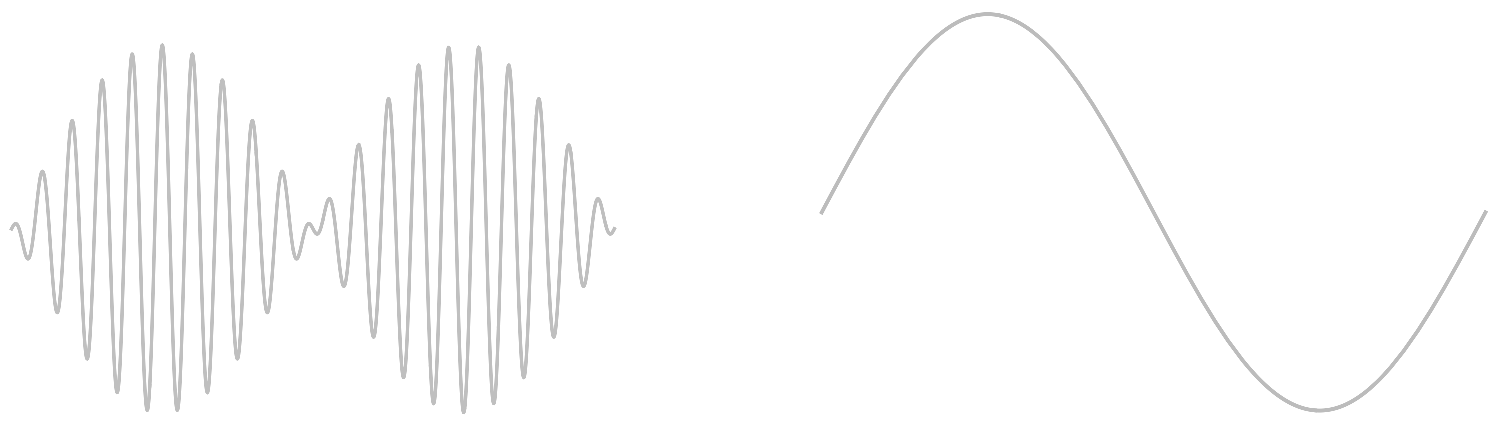




non-relativistic limit = multicomponent Schrödinger-Poisson

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left[-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\,\mathcal{G}_{\mu\nu}\mathcal{G}_{\alpha\beta} + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\,g^{\mu\nu}W_\mu W_\nu + \frac{c^3}{16\pi G}R + \ldots\right]$$ + non-grav, interactions

$$\mathcal{G}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$



split in "fast" and "slow" parts

non-relativistic limit

$$\boldsymbol{W}(t,\boldsymbol{x}) \equiv \frac{\hbar}{\sqrt{2mc}}\Re\left[\boldsymbol{\Psi}(t,\boldsymbol{x})e^{-imc^2t/\hbar}\right]$$

$$\mathcal{S}_{nr} = \int \mathrm{d}t\mathrm{d}^3x\left[\frac{i\hbar}{2}\boldsymbol{\Psi}^\dagger\dot{\boldsymbol{\Psi}} + \text{c.c.} - \frac{\hbar^2}{2m}\nabla\boldsymbol{\Psi}^\dagger\cdot\nabla\boldsymbol{\Psi} + \frac{1}{8\pi G}\Phi\nabla^2\Phi - m\,\Phi\,\boldsymbol{\Psi}^\dagger\boldsymbol{\Psi}\right]$$

Spin - 0	eg. Schive et. al (2014)
Spin - 1	Adshead & Lozanov (2021)
Spin - 2s+1	Jain & MA (2021)

non-relativistic limit = multicomponent Schrödinger-Poisson

$[\boldsymbol{\Psi}]_i = \psi_i \text{ with } i = 1, 2, 3$ vector case

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\Psi} = -\frac{\hbar^2}{2m}\nabla^2\boldsymbol{\Psi} + m\,\Phi\,\boldsymbol{\Psi}\,, \qquad \nabla^2\Phi = 4\pi G m\,\boldsymbol{\Psi}^\dagger\boldsymbol{\Psi}$$

$[\boldsymbol{\Psi}]_i = \psi_i \text{ with } i = 1$ scalar case

at this level this is just $2s{+}1$ equal mass scalar fields
but not when non-gravitational interactions are included!

conserved quantities

$$[\boldsymbol{\Psi}]_i = \psi_i \text{ with } i = 1, 2, 3$$

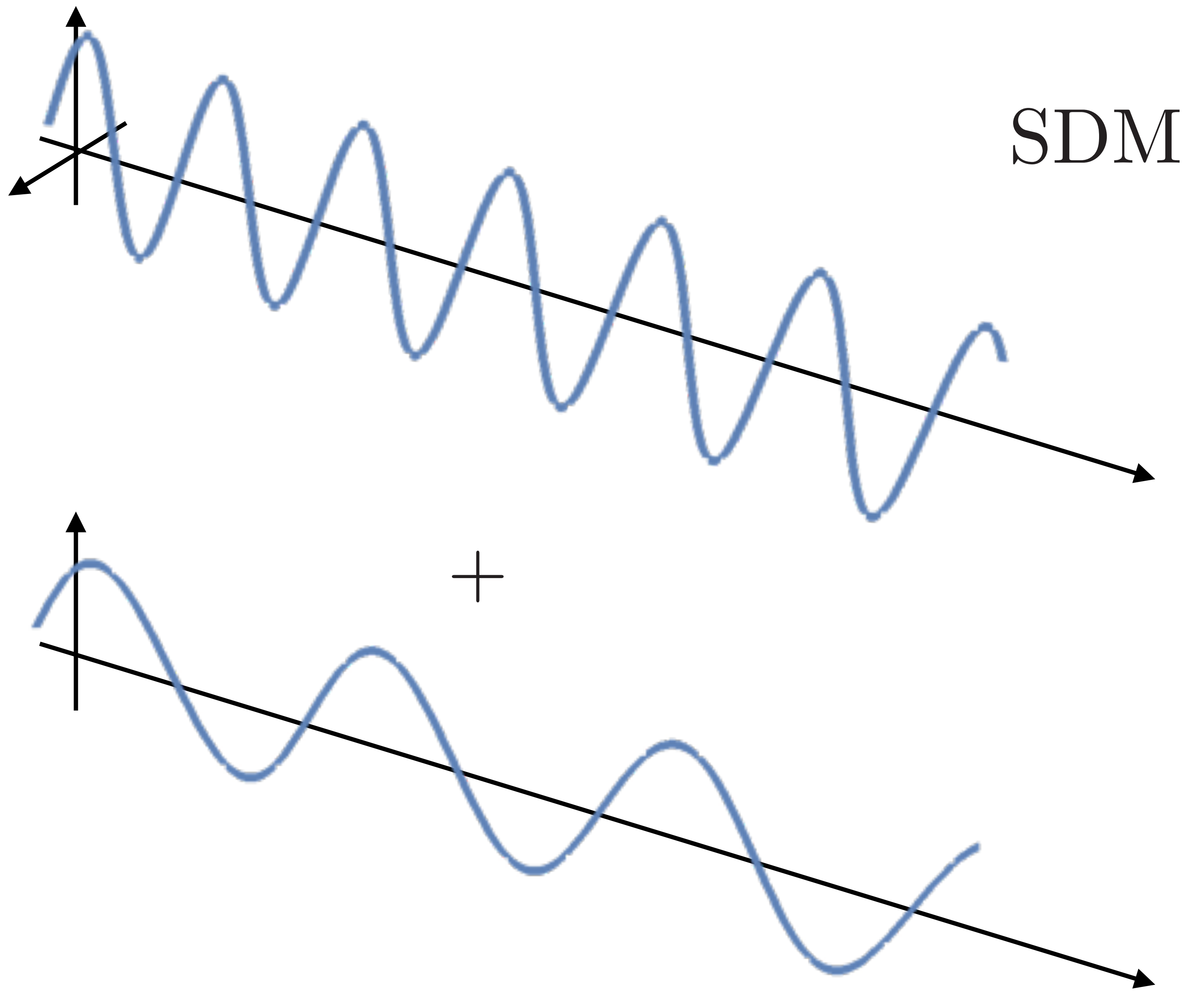
$$N = \int \mathrm{d}^3x \boldsymbol{\Psi}^\dagger \boldsymbol{\Psi}, \quad \text{and} \quad M = mN, \qquad \text{(particle number and rest mass)}$$

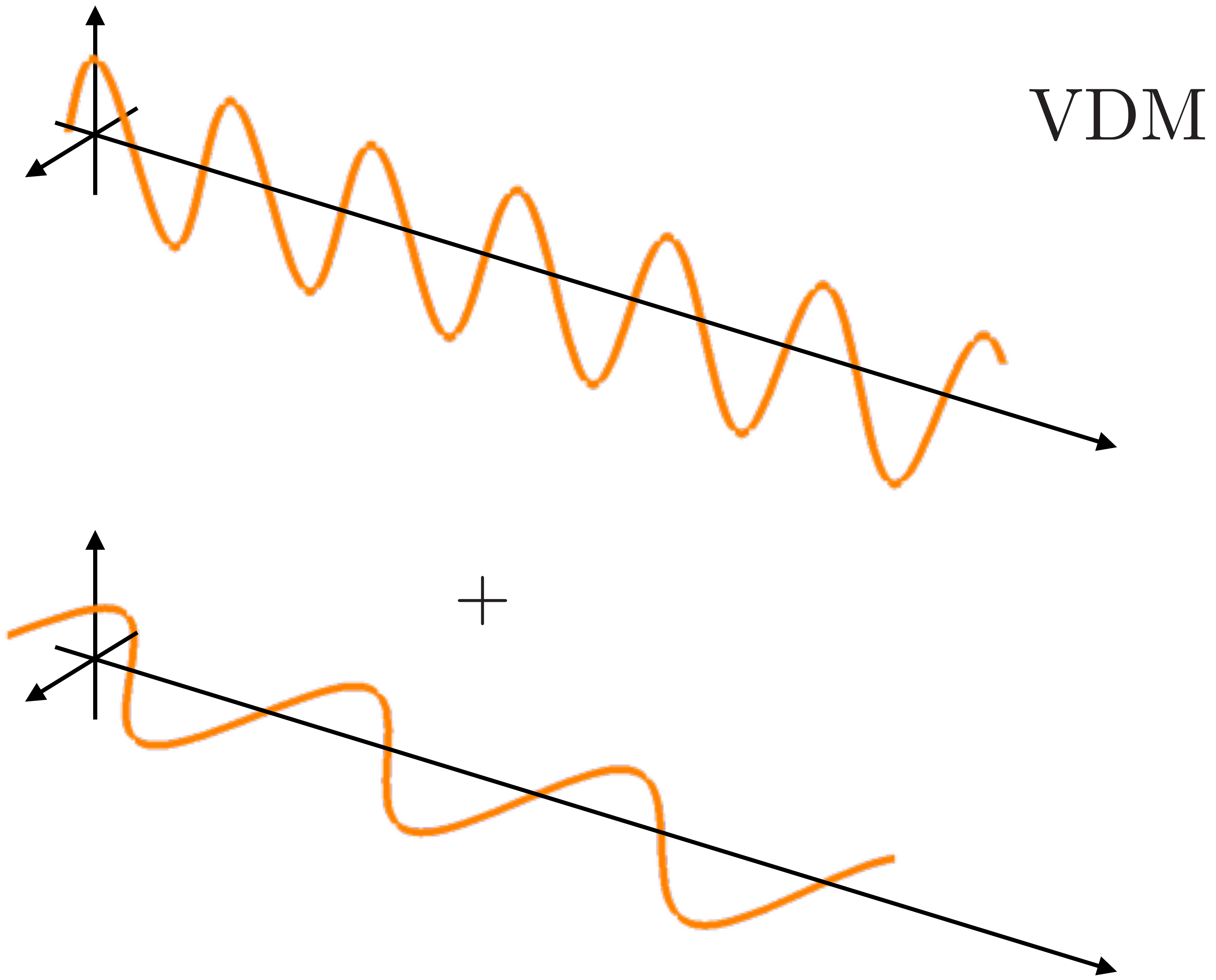
$$E = \int \mathrm{d}^3x \left[\frac{\hbar^2}{2m} \nabla \boldsymbol{\Psi}^\dagger \cdot \nabla \boldsymbol{\Psi} - \frac{Gm^2}{2} \boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \int \frac{\mathrm{d}^3y}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \boldsymbol{\Psi}^\dagger(\boldsymbol{y}) \boldsymbol{\Psi}(\boldsymbol{y}) \right], \qquad \text{(energy)}$$

$$\boldsymbol{S} = \hbar \int \mathrm{d}^3x \, i \boldsymbol{\Psi} \times \boldsymbol{\Psi}^\dagger, \qquad \text{(spin angular momentum)}$$

$$\boldsymbol{L} = \hbar \int \mathrm{d}^3x \, \Re \left(i \, \boldsymbol{\Psi}^\dagger \nabla \boldsymbol{\Psi} \times \boldsymbol{x} \right). \qquad \text{(orbital angular momentum)}$$

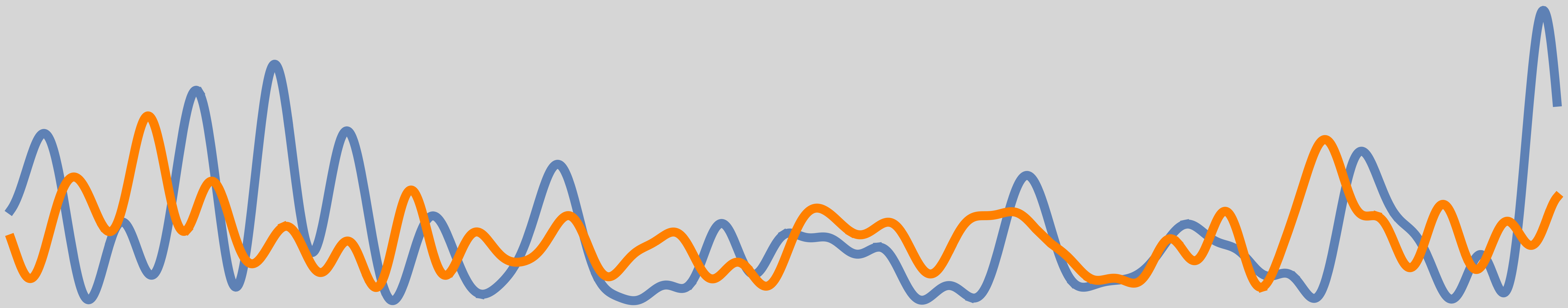
wave interference





$$|\Psi_a(\boldsymbol{x}) + \Psi_b(\boldsymbol{x})|^2 \neq |\Psi_a(\boldsymbol{x})|^2 + |\Psi_b(\boldsymbol{x})|^2$$

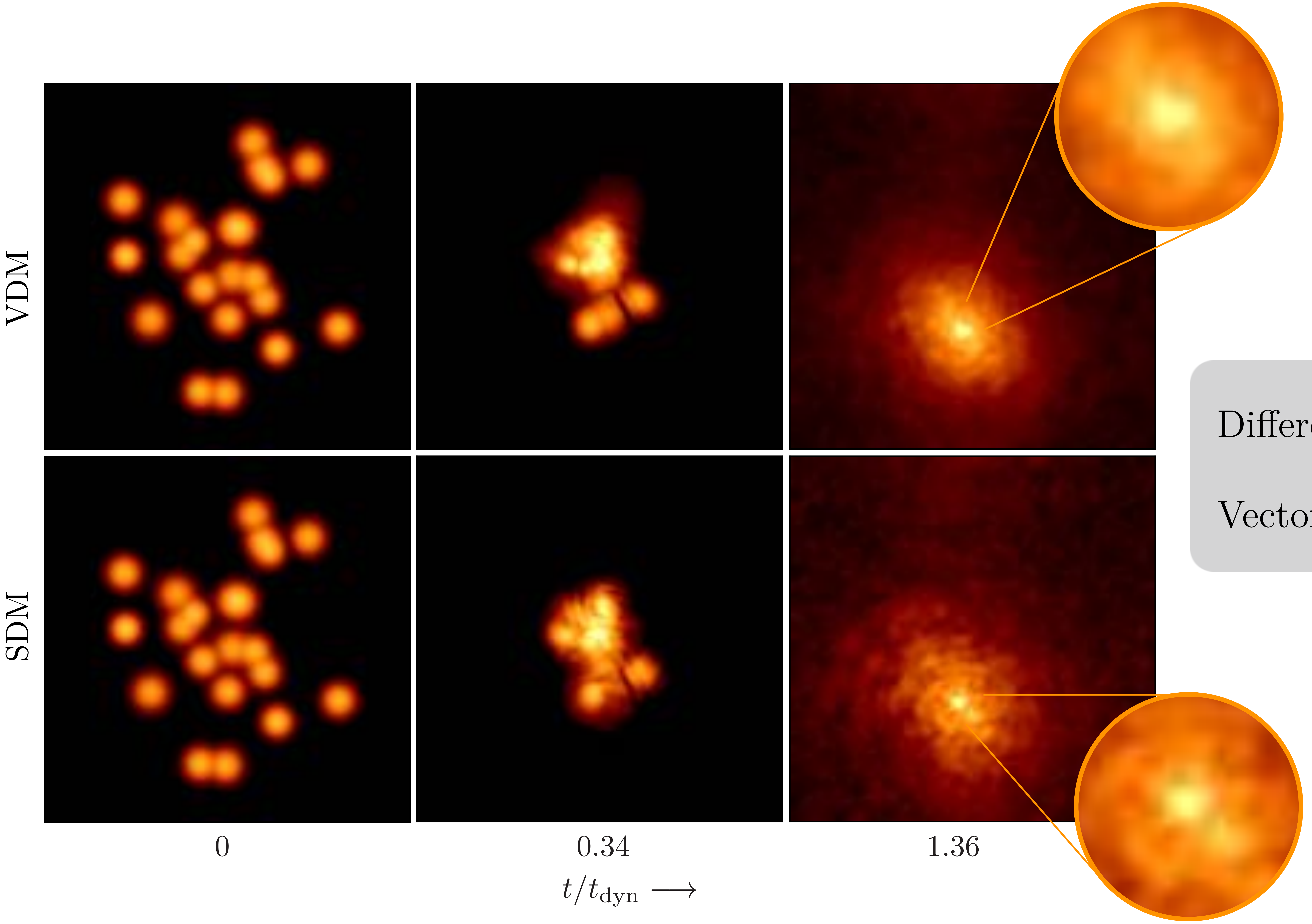
$$|\Psi_a(\boldsymbol{x}) + \Psi_b(\boldsymbol{x})|^2 = |\Psi_a(\boldsymbol{x})|^2 + |\Psi_b(\boldsymbol{x})|^2$$



reduced interference

$$\frac{\delta\rho}{\rho} \propto \frac{1}{\sqrt{2s+1}}$$

MA, Jain, Karur & Mocz (2022)



Difference between

Vector & Scalar Dark Matter

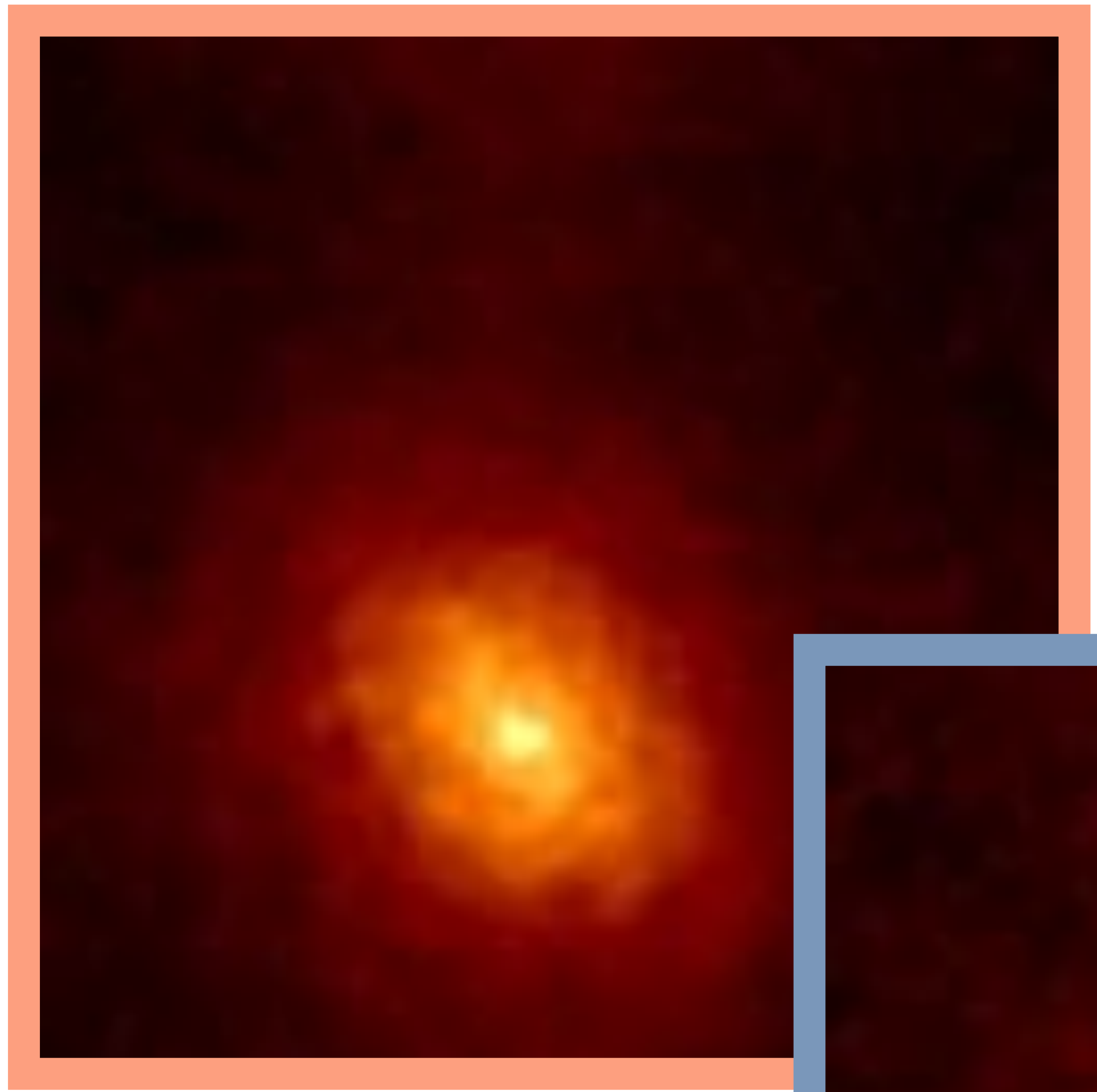
MA, Jain, Karur & Mocz(2022)

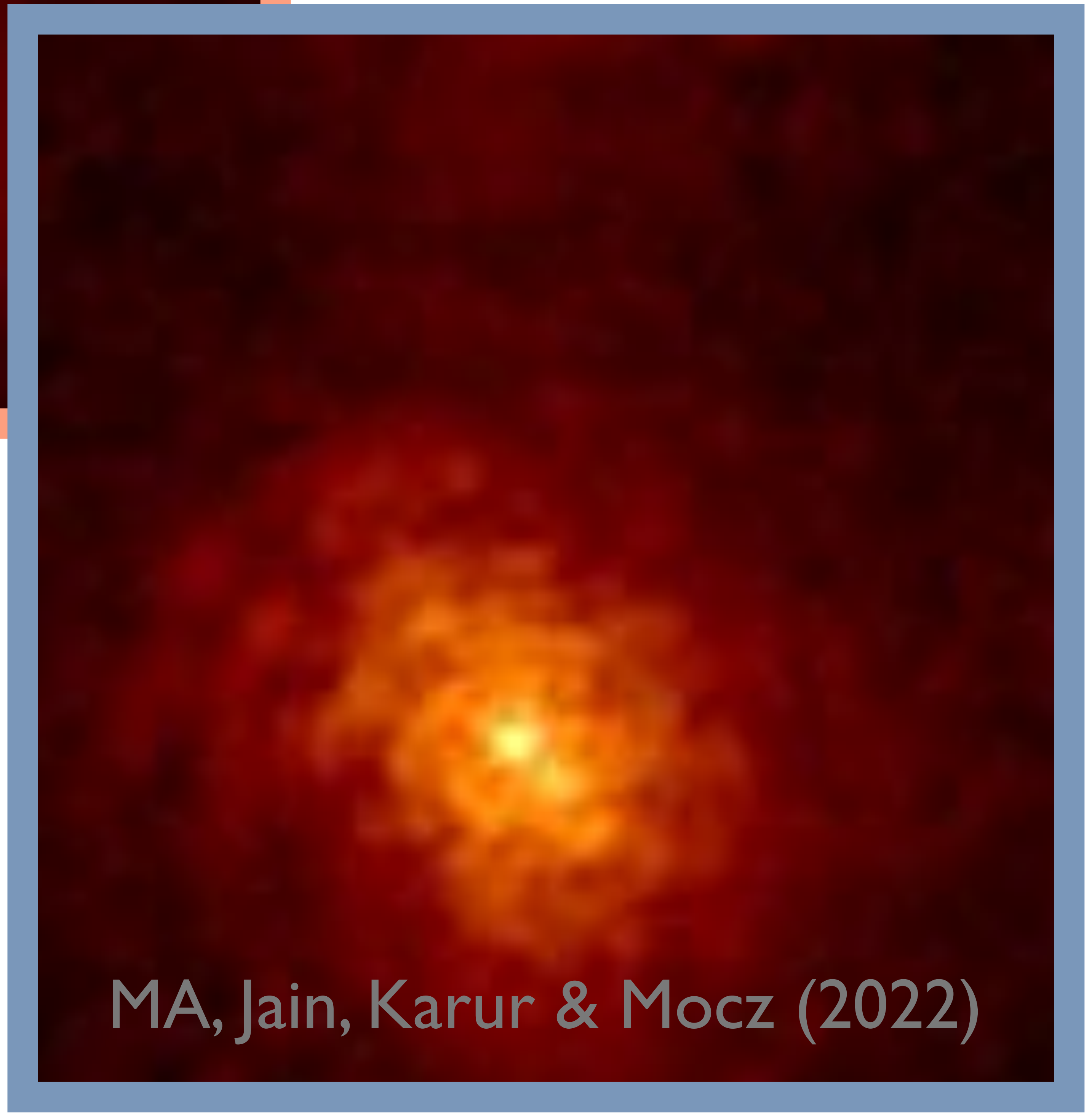
gravitational implications (examples)

- dynamical heating of stars

$$m \gtrsim \frac{1}{(2s+1)^{1/3}} \left[3 \times 10^{-19} \mathrm{eV}\right]$$

Dalal & Kratsov (2022)





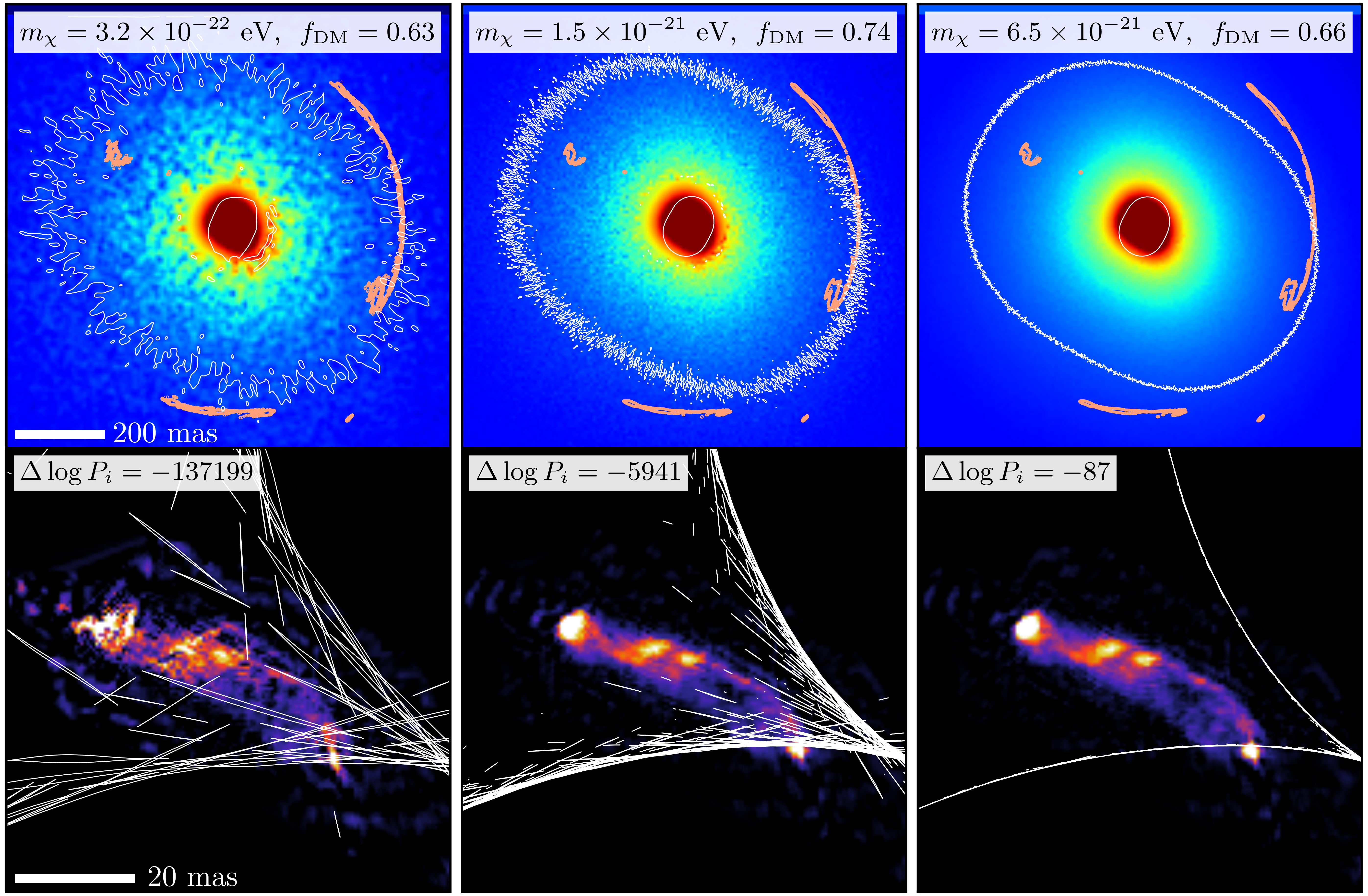
MA, Jain, Karur & Mocz (2022)

gravitational implications (examples)

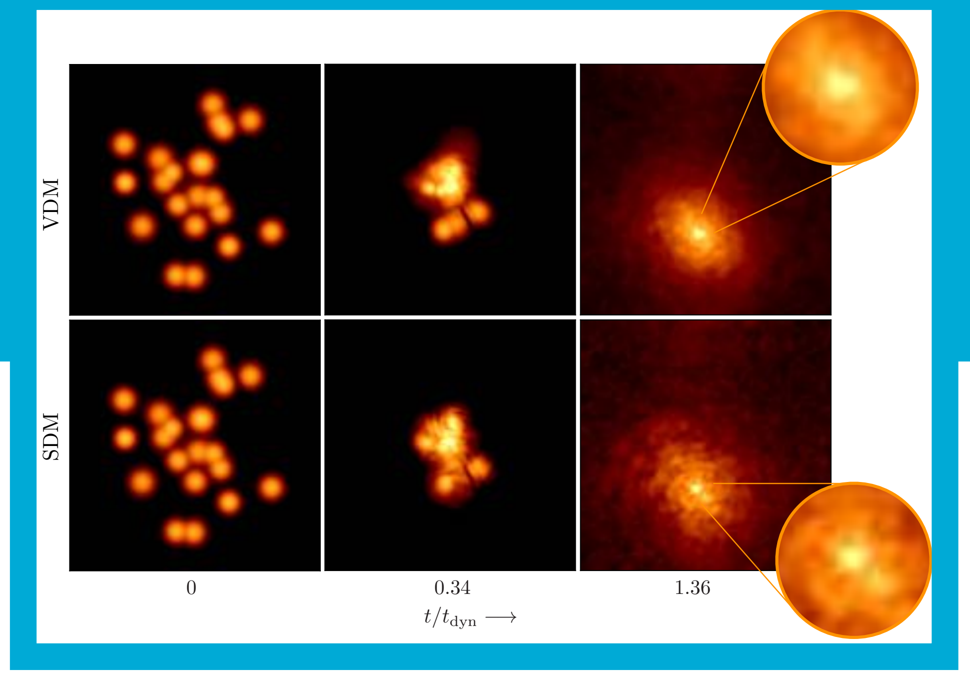
- lensing

$$m \gtrsim \frac{1}{(2s+1)} \left[4.4 \times 10^{-21}\ \mathrm{eV}\right]$$

Powell et. al (2023)

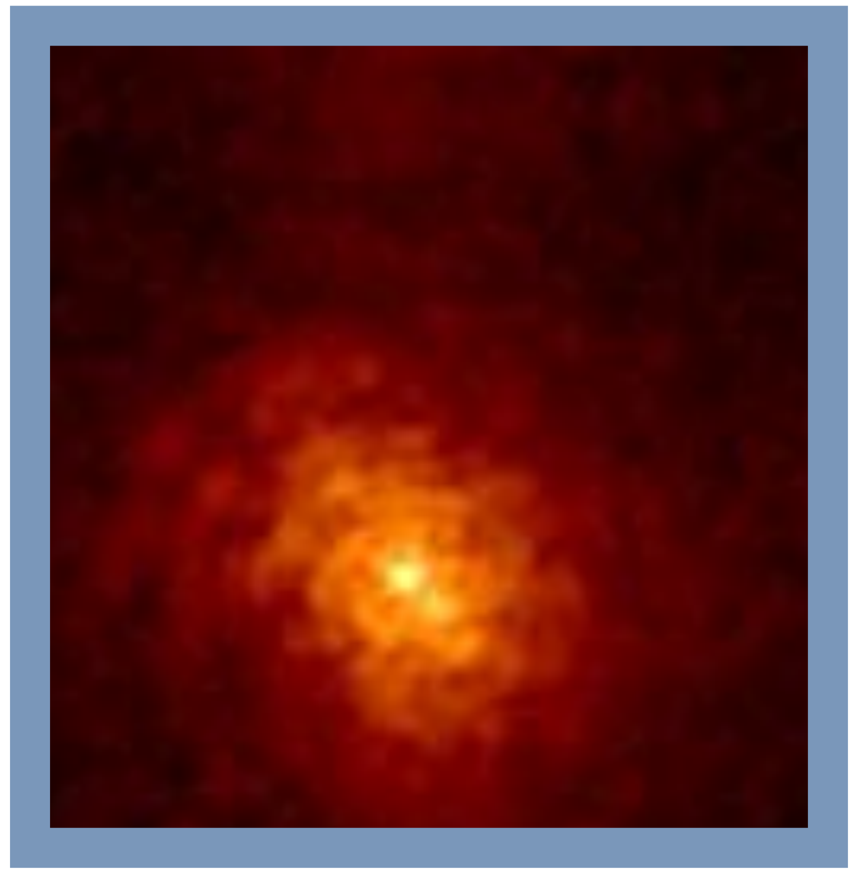


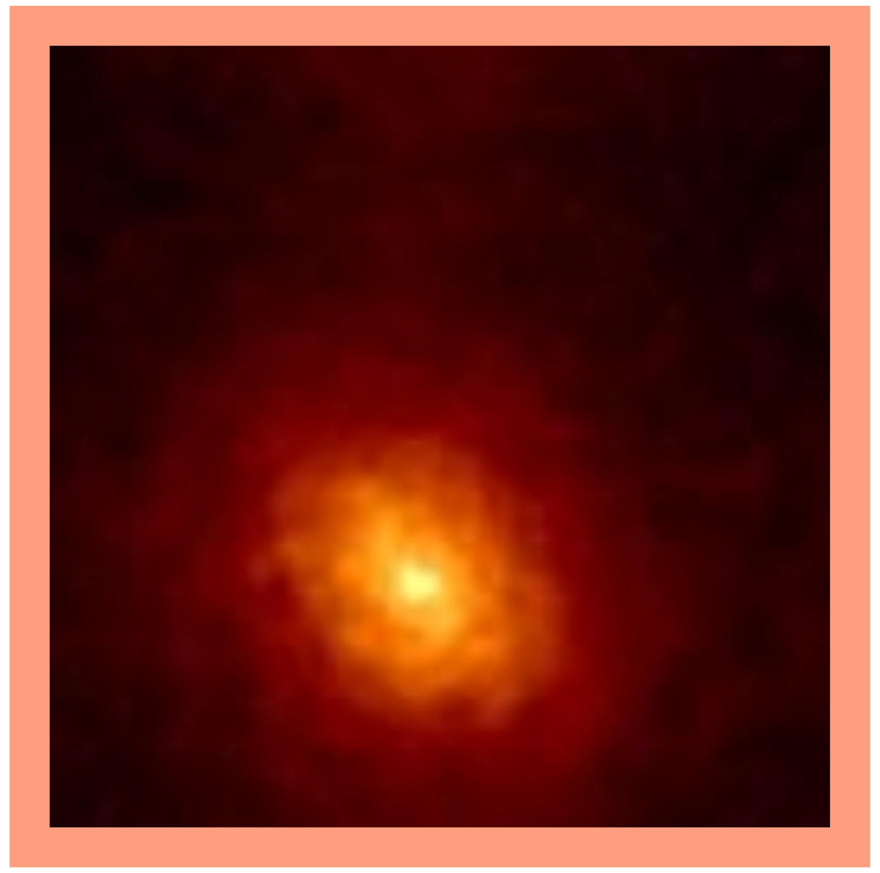
radial density profiles

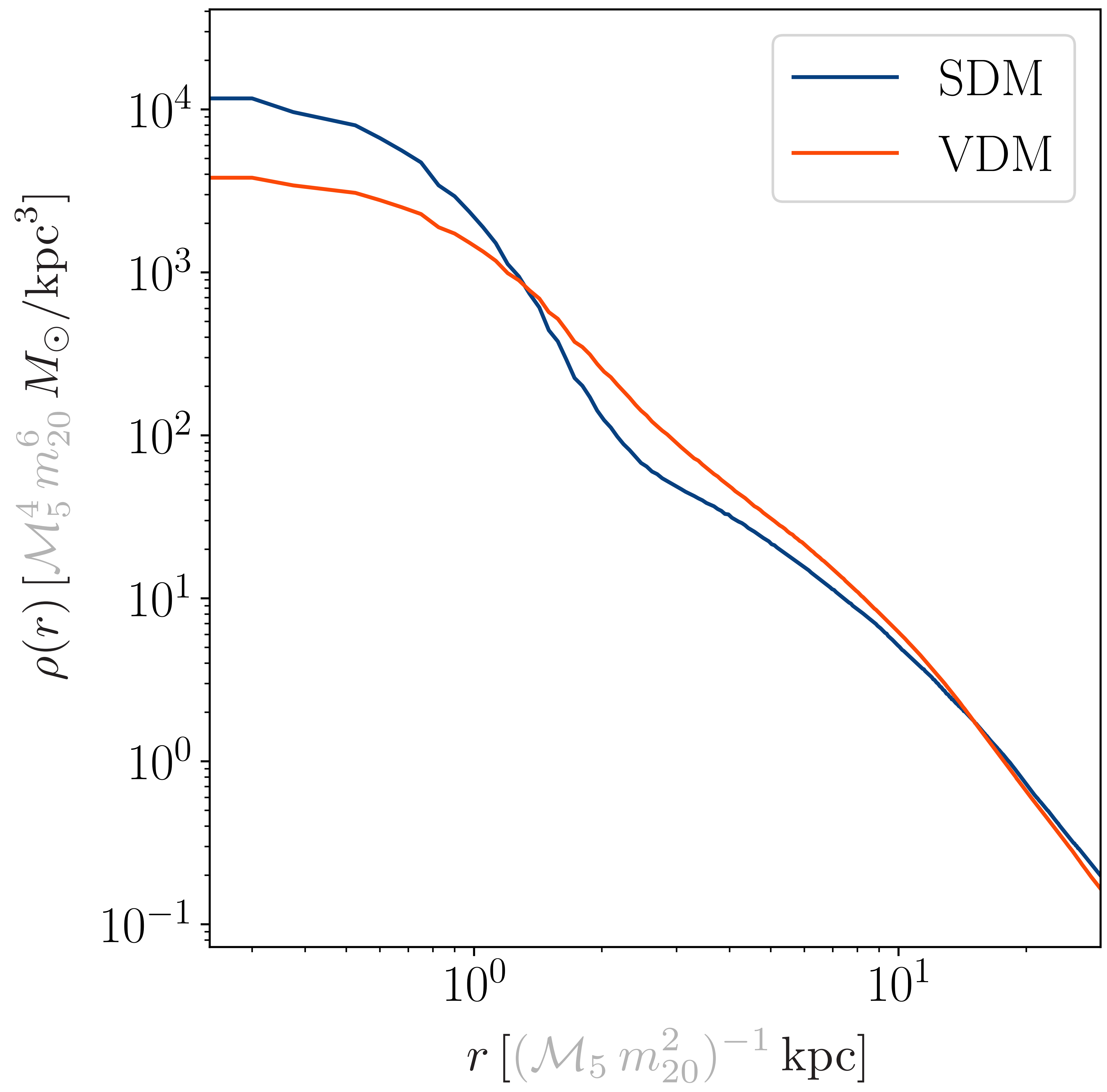


scalar vs. vector dark matter

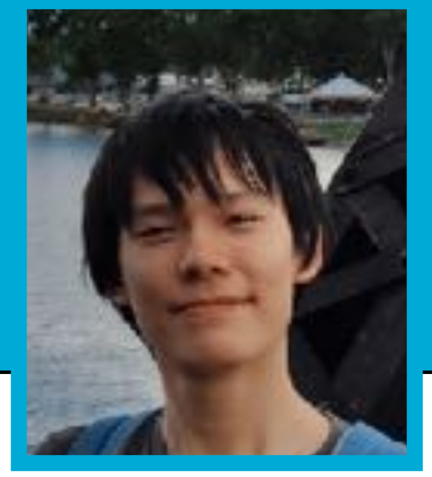
- less dense & broader core
- smoother transition to $r^{-(2\text{–}3)}$ tail





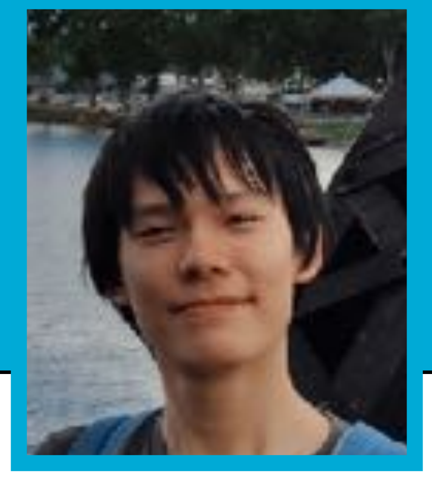


condensation in the kinetic regime



with M. Jain, J. Thomas, Wanichwecharungruang (2023)

condensation in the kinetic regime



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condensation in the kinetic regime



with M. Jain, J. Thomas, Wanichwecharungruang (2023)

condensation in the kinetic regime

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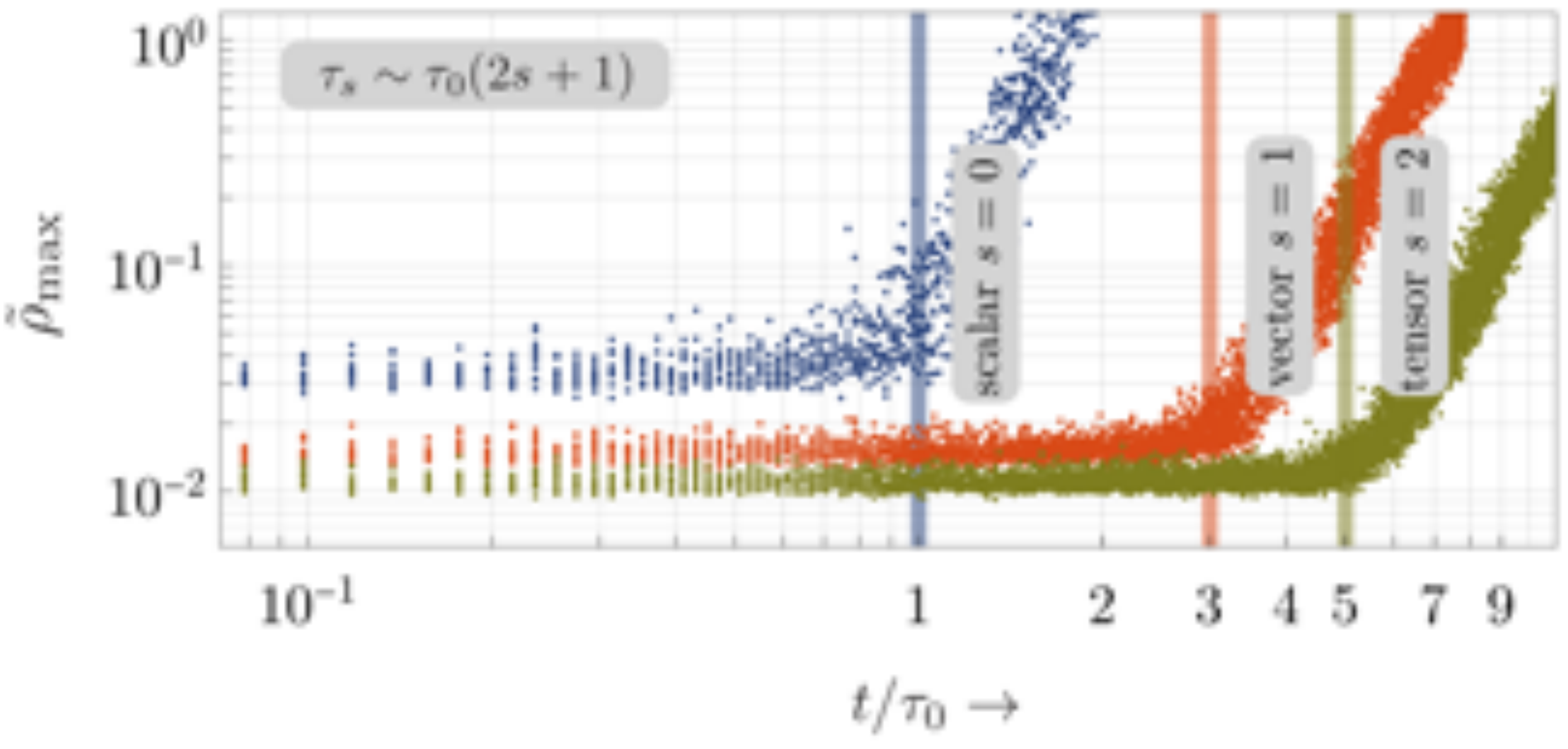
condensation in the kinetic regime

- nucleation time scale

$$\tau_{\rm s} \sim (2s+1)\tau_{\rm s=0}$$

$$\tau_{s=0} = [n\sigma_{\rm gr} v \mathcal{N}]^{-1}$$

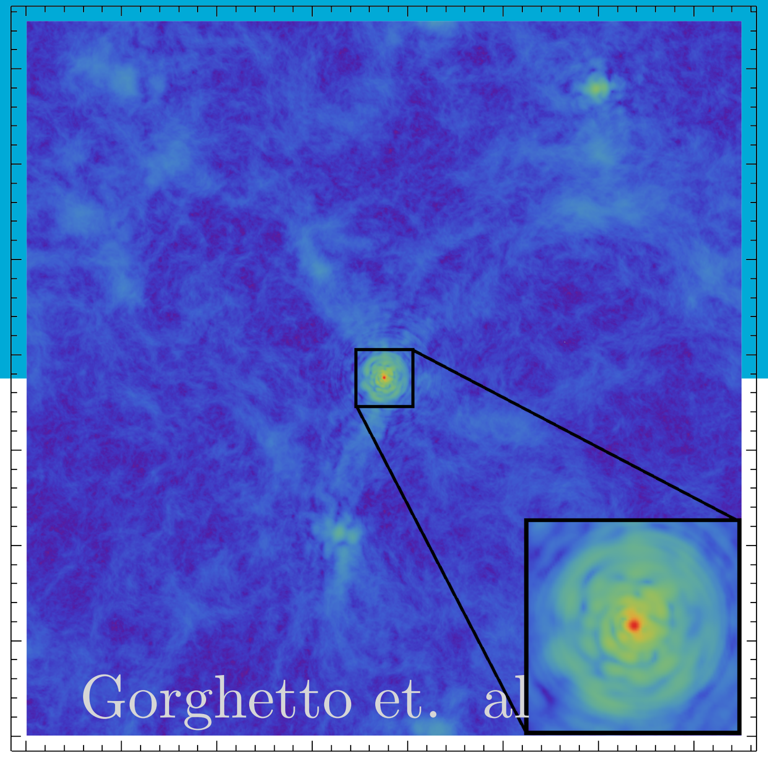
$$\sigma_{\rm gr} \sim (Gm/v^2)^2, \qquad \mathcal{N} \sim n\lambda_{\rm dB}^3$$



$$\tau_0 \sim \left(\frac{m}{10^{-22}\,{\rm eV}}\right)^3 \left(\frac{\sigma}{100\,{\rm km}\,s^{-1}}\right)^6 \left(\frac{10^8 M_\odot {\rm kpc}^{-3}}{\bar{\rho}^3}\right)^2 \times 10{\rm Gyrs}$$

see Levkov et. al (2018) for scalar case

condensation in the kinetic regime



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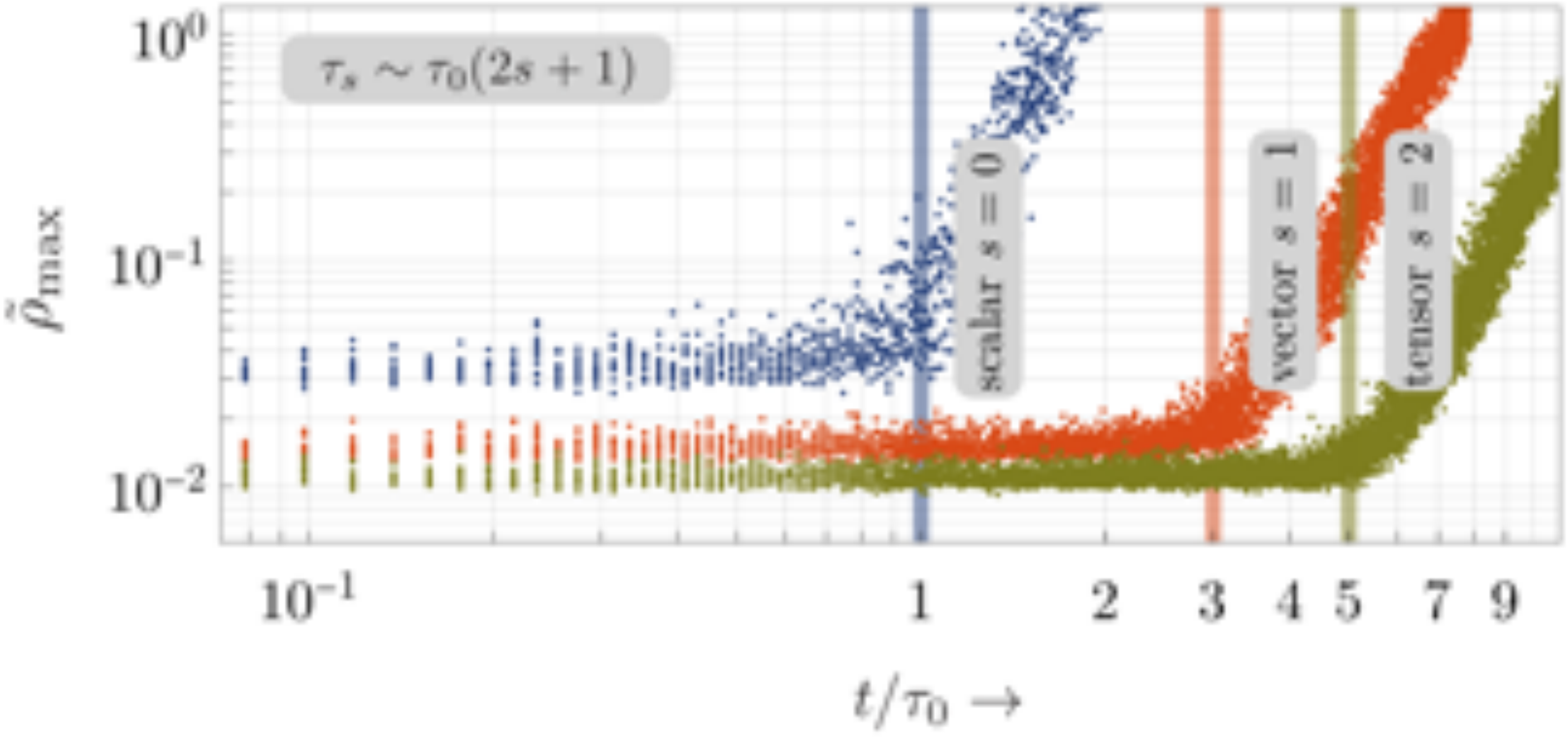
$$\tau_{\rm s} \sim (2s+1)\tau_{\rm s=0}$$

$$\tau_{\rm s=0} \sim n\sigma_{\rm gr} v \mathcal{N}$$

$$\sigma_{\rm gr} \sim (Gm/v^2)^2, \qquad \mathcal{N} \sim n\lambda_{\rm dB}^3$$



$$\tau_0 \sim \left(\frac{m}{10^{-22}\,{\rm eV}}\right)^3 \left(\frac{\sigma}{100\,{\rm km}\,s^{-1}}\right)^6 \left(\frac{10^8 M_\odot {\rm kpc}^{-3}}{\bar{\rho}^3}\right)^2 \times 10{\rm Gyrs}$$

kinetic nucleation of solitons — multi-component case

Jain, MA, Thomas, Wanichwecharungruang (2023)

$$i\frac{\partial}{\partial t}\psi_a = -\frac{1}{2m_a}\nabla^2\psi_a + m_a\Phi\,\psi_a$$

$$\nabla^2\Phi = 4\pi G\sum_a m_a\,\psi_a^*\psi_a.$$

$$\Downarrow$$

$$\frac{\partial f^a_{\boldsymbol{v}_a}}{\partial t} = \sum_b m_b^3\frac{\Lambda}{2\pi}\frac{(4\pi m_a m_b G)^2}{m_a}\nabla_{v_a^i}\left[\frac{\mathcal{D}^{ab}_{ij}}{2m_a}\nabla_{v_a^j} f^a_{\boldsymbol{v}_a} + \frac{\mathcal{F}^{ab}_i}{m_b} f^a_{\boldsymbol{v}_a}\right]$$

where $\mathcal{D}^{ab}_{ij} = \int \frac{\mathrm{d}\tilde{\boldsymbol{v}}_b}{(2\pi)^3}\, f^b_{\tilde{\boldsymbol{v}}_b}\,\frac{\delta_{ij}-\hat{u}_i\hat{u}_j}{u}\, f^b_{\tilde{\boldsymbol{v}}_b}$ and $\mathcal{F}^{ab}_i = f^a_{\boldsymbol{v}_a}\int \frac{\mathrm{d}\tilde{\boldsymbol{v}}_b}{(2\pi)^3}\,\frac{\hat{u}_i}{u^2}\, f^b_{\tilde{\boldsymbol{v}}_b}$

$$\Downarrow$$

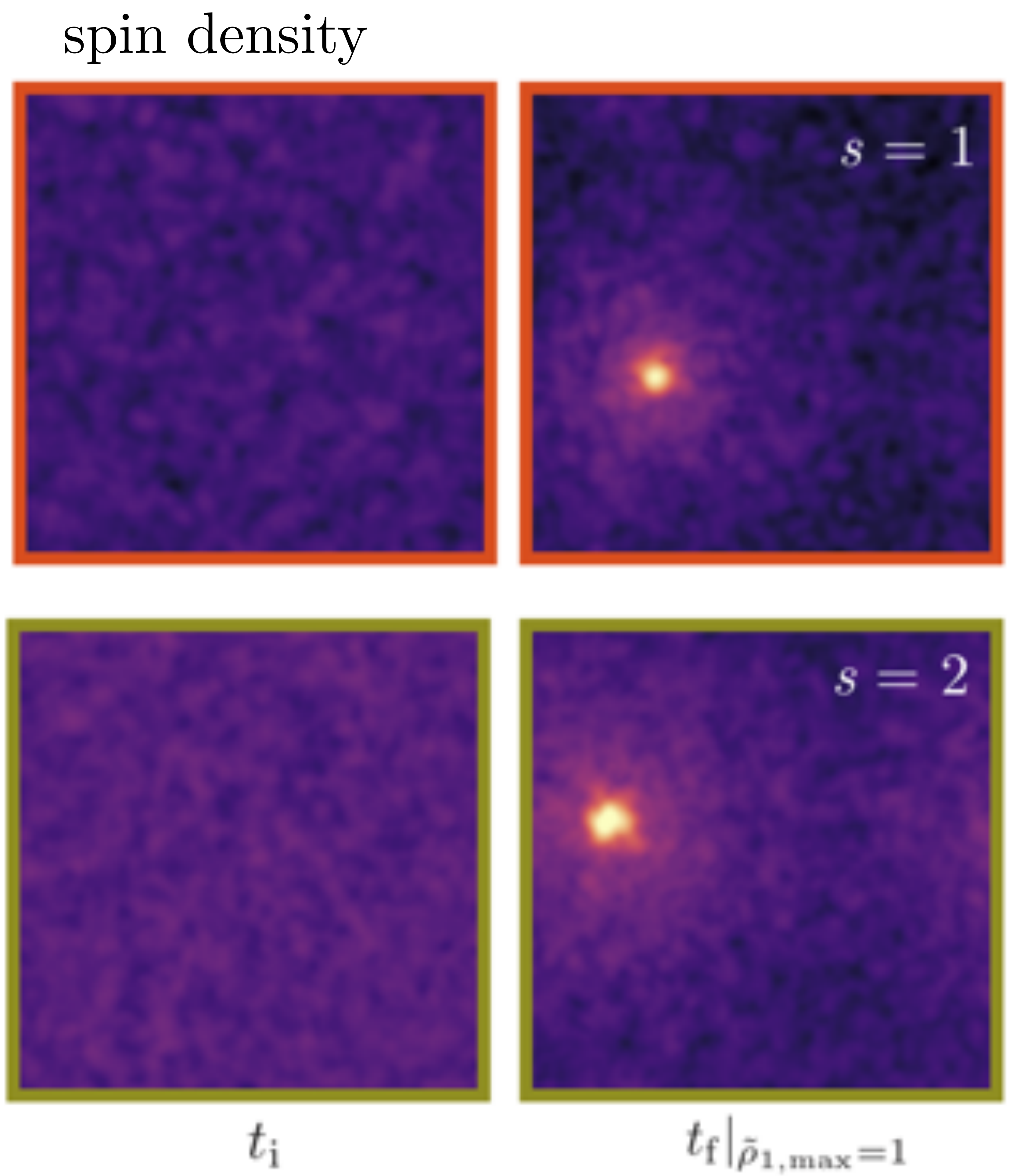
$$\Gamma_a = \sum_b \frac{\Lambda\,(4\pi G)^2\bar{\rho}_b}{\sigma_a^3\sigma_b^3}\left[2\frac{\bar{\rho}_a}{m_a^3} - \beta_{ab}\frac{\bar{\rho}_b\sigma_a}{m_b^3\sigma_b}\right]$$

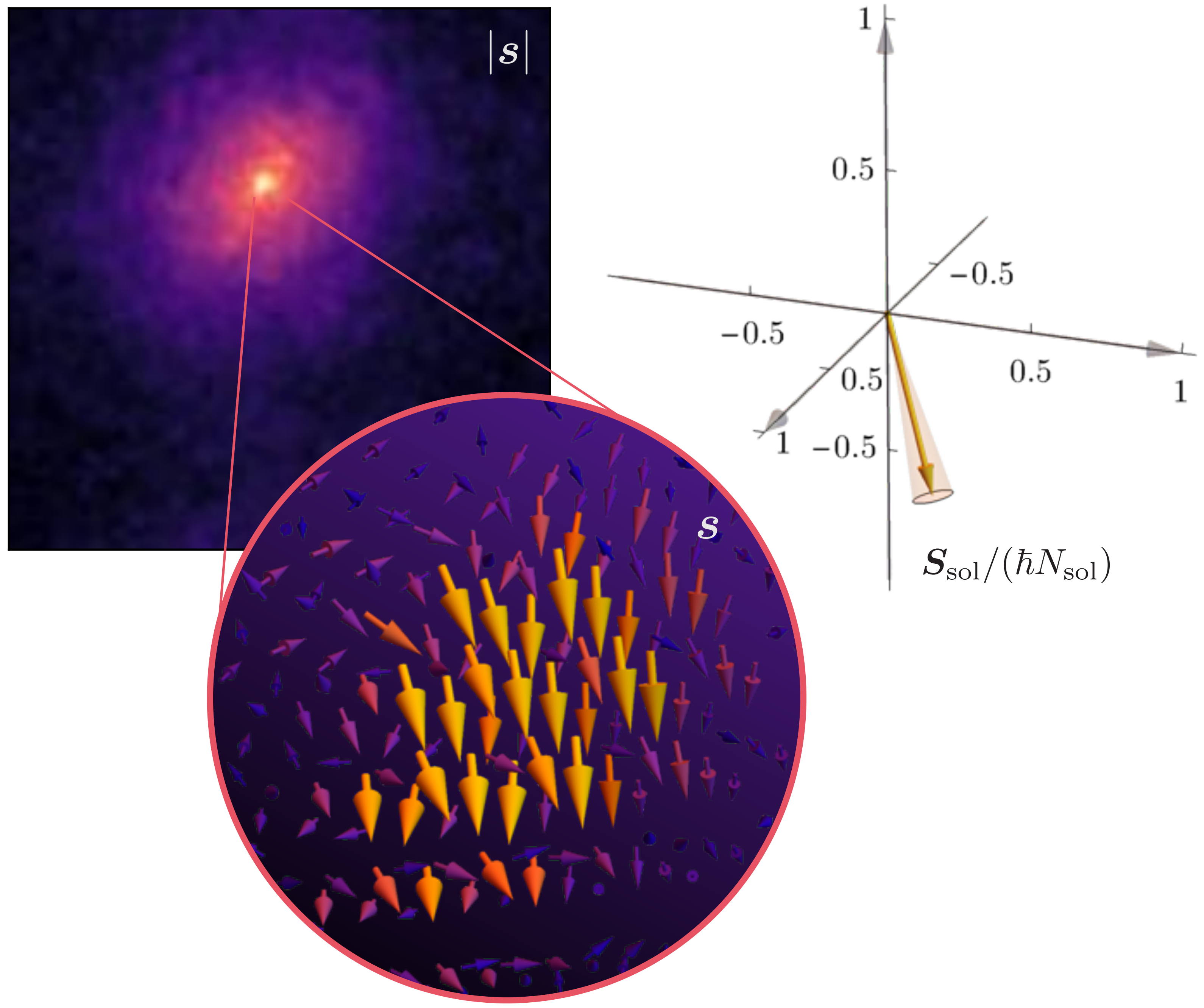
$$\Leftarrow$$

$$\Gamma_{(s)} = \Gamma_0/(2s+1)$$

what are these blobs?

born to spin !





$$S_{core} \sim \hbar \frac{M_{core}}{m}$$

Even when initial total spin is negligible

MA, Jain, Karur & Mocz (2022)

Jain, MA, Thomas, Wanichwecharungruang (2023)

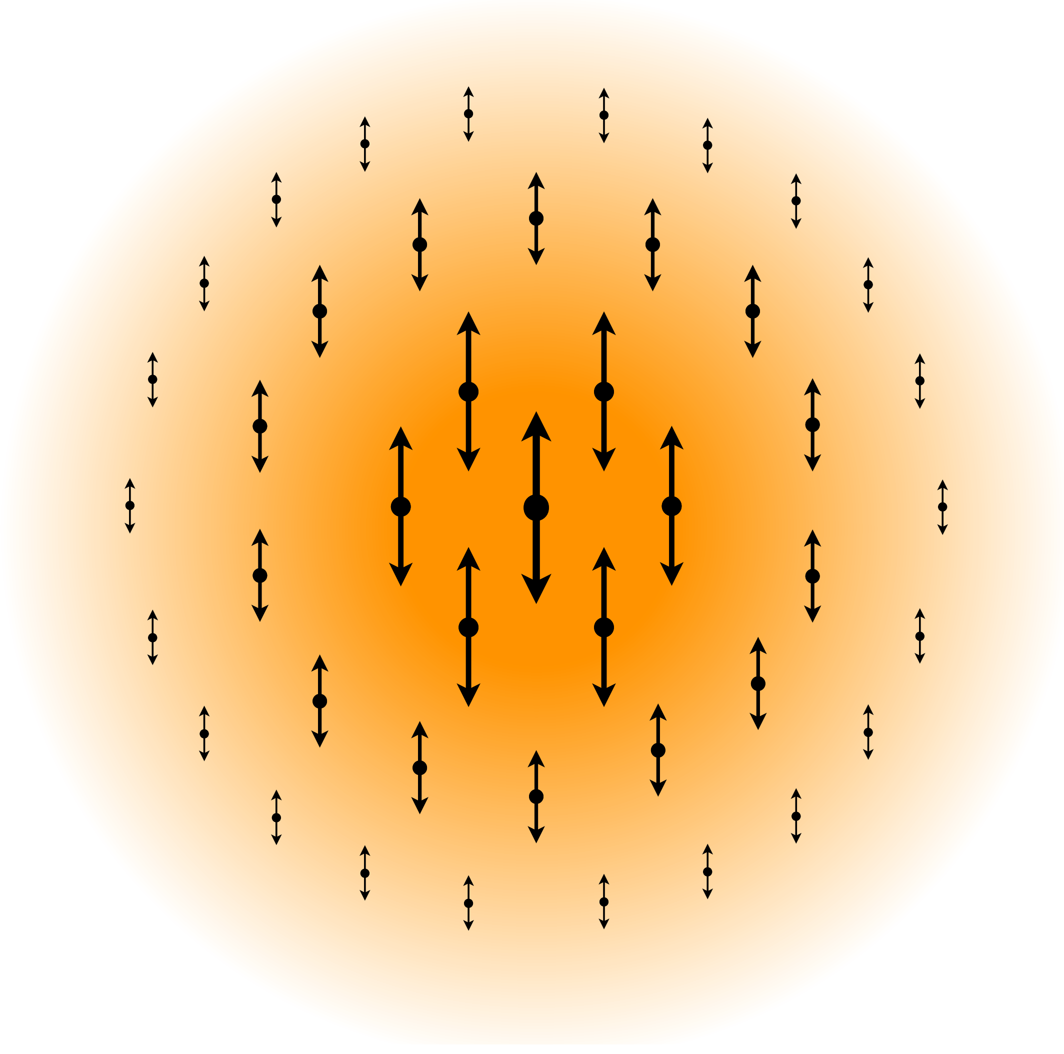
"polarized" vector solitons (with macroscopic spin)

$$\boldsymbol{S}_{\rm sol} = i(\boldsymbol{\epsilon} \times \boldsymbol{\epsilon}^{\dagger})\frac{M_{\rm sol}}{m}\hbar$$

macroscopic spin $\quad \boldsymbol{S}_{\rm tot}/\hbar = \lambda N \hat{z}$

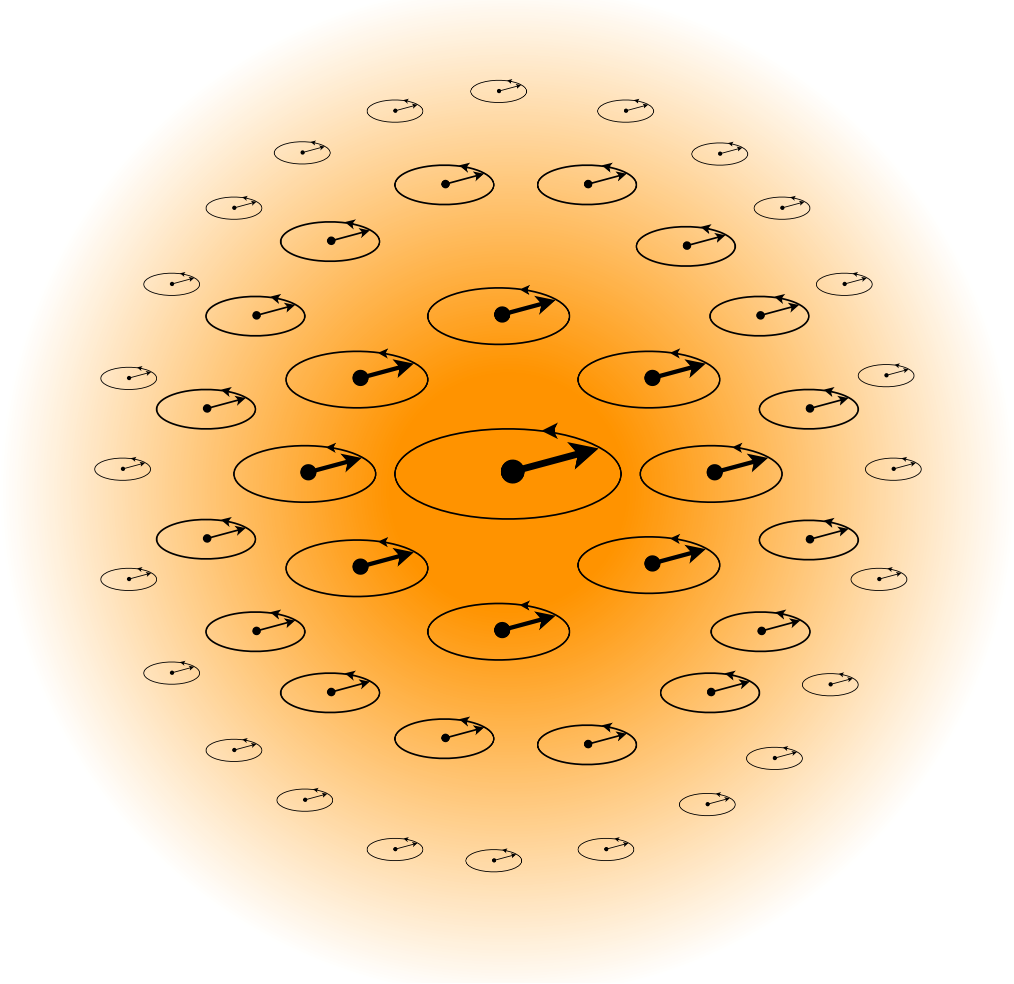
$N =$ # of particles in soliton

Jain & MA (2021)



$$\boldsymbol{\epsilon}^{(0)}_{1,\hat{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{S}_{\rm tot} = 0\hat{z}$$



$$\boldsymbol{\epsilon}^{(\pm 1)}_{1,\hat{z}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

$$\boldsymbol{S}_{\rm tot} = \hbar\frac{M_{\rm sol}}{m}\hat{z}$$

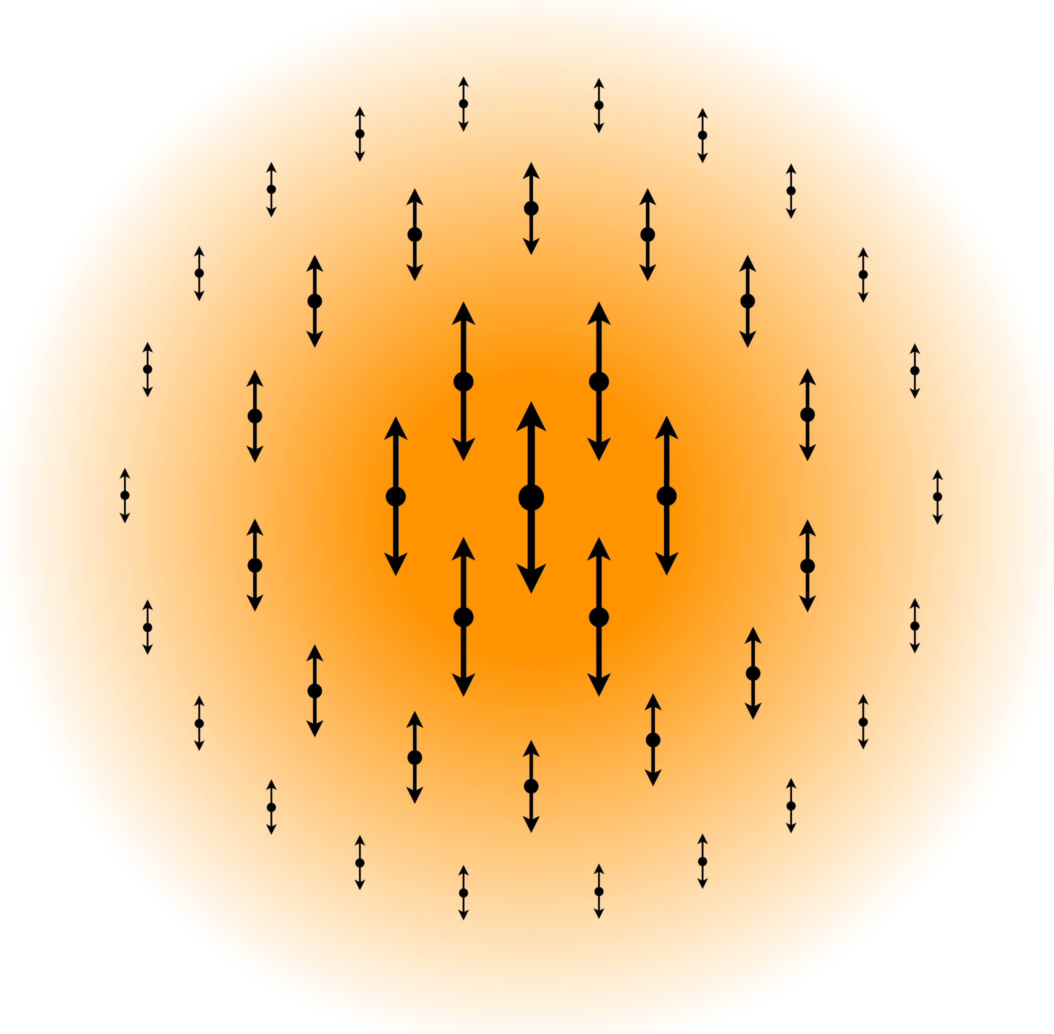
“polarized” vector solitons

$$\boldsymbol{W}(t,\boldsymbol{x}) \equiv \frac{\hbar}{\sqrt{2mc}}\Re\left[\boldsymbol{\Psi}(t,\boldsymbol{x})e^{-imc^2t/\hbar}\right]$$

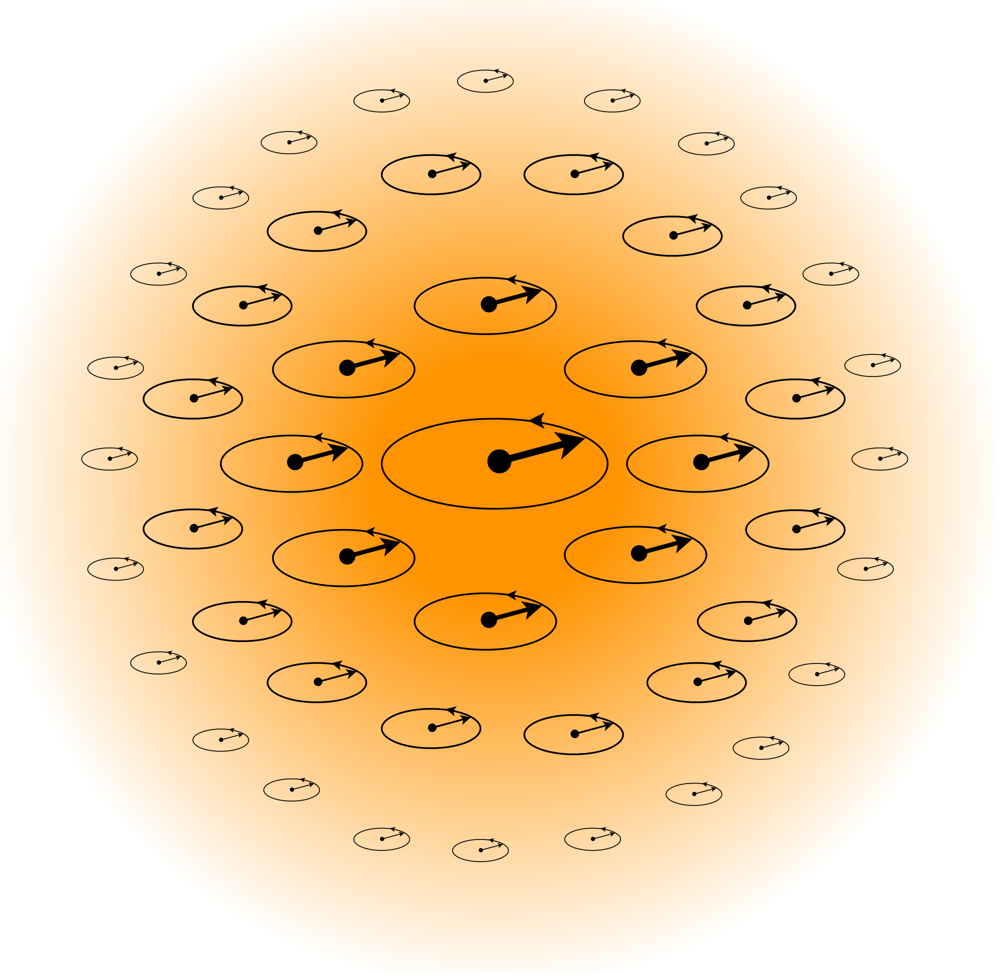
macroscopic spin

$$\boldsymbol{S}_{\rm tot} = \hbar\frac{M_{\rm sol}}{m}\hat{z}$$

- all lowest energy for fixed M
- bases for partially-polarized solitons

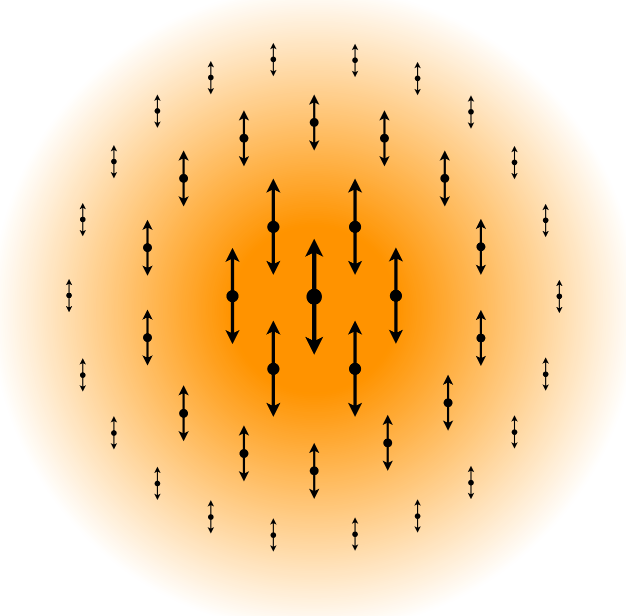
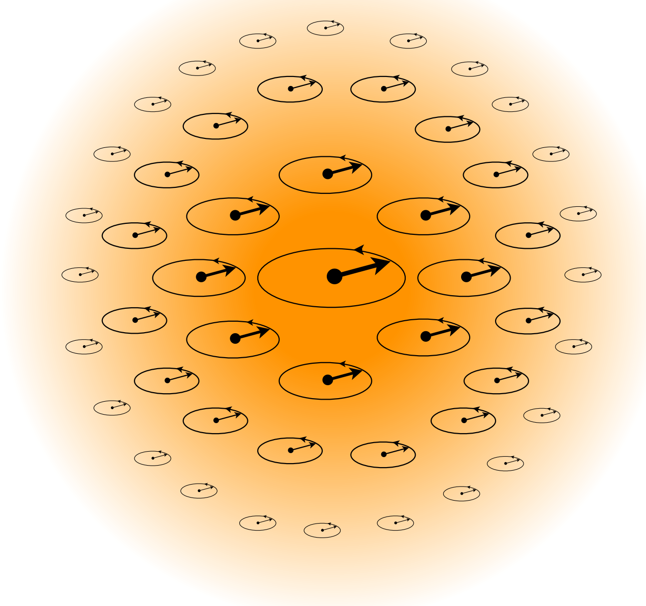
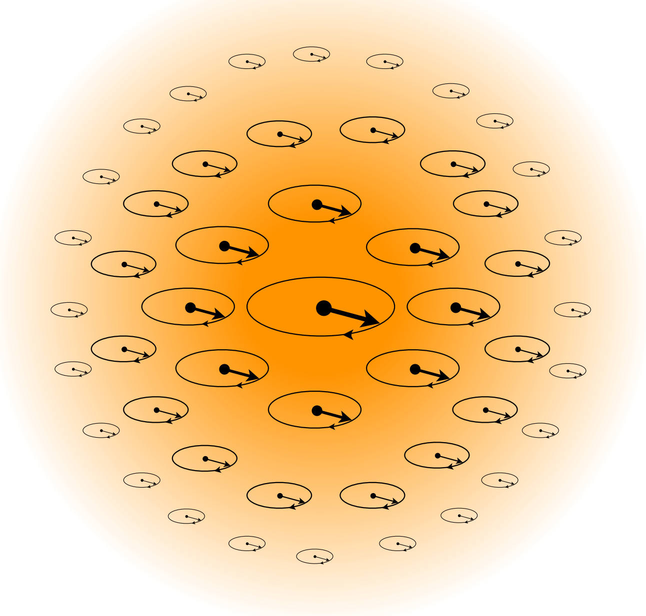


$$\boldsymbol{S}_{\rm tot} = 0\hat{z}$$



$$\boldsymbol{S}_{\rm tot} = \hbar\frac{M_{\rm sol}}{m}\hat{z}$$

$$0 \leq |\boldsymbol{S}_{\rm tot}| \leq \frac{M_{\rm sol}}{m}\hbar$$

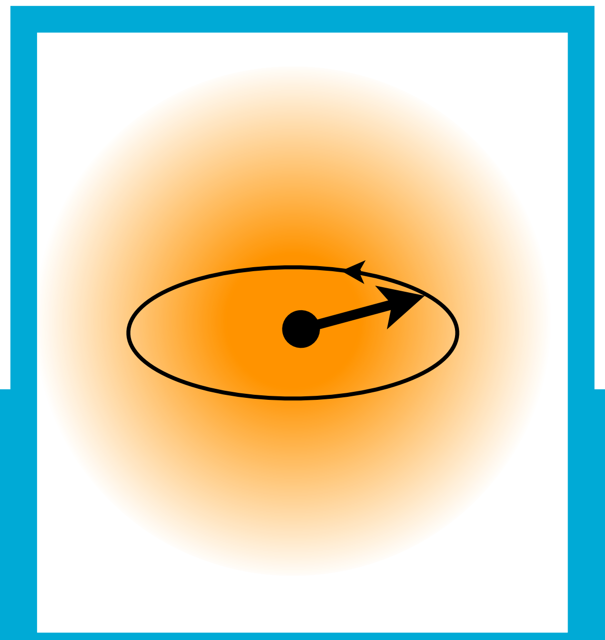
c_0  $+c_1$  $+c_{-1}$ 

degeneracy lifted by self-interactions

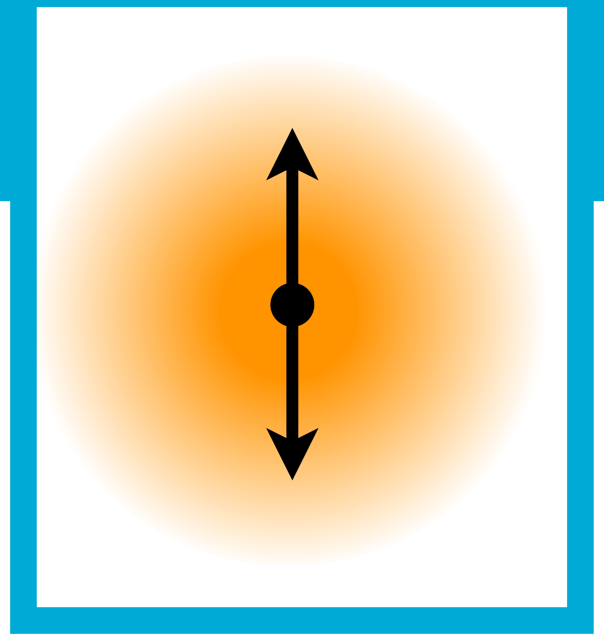
 

Jain (2022)

Zhang, Jain & MA (2022)



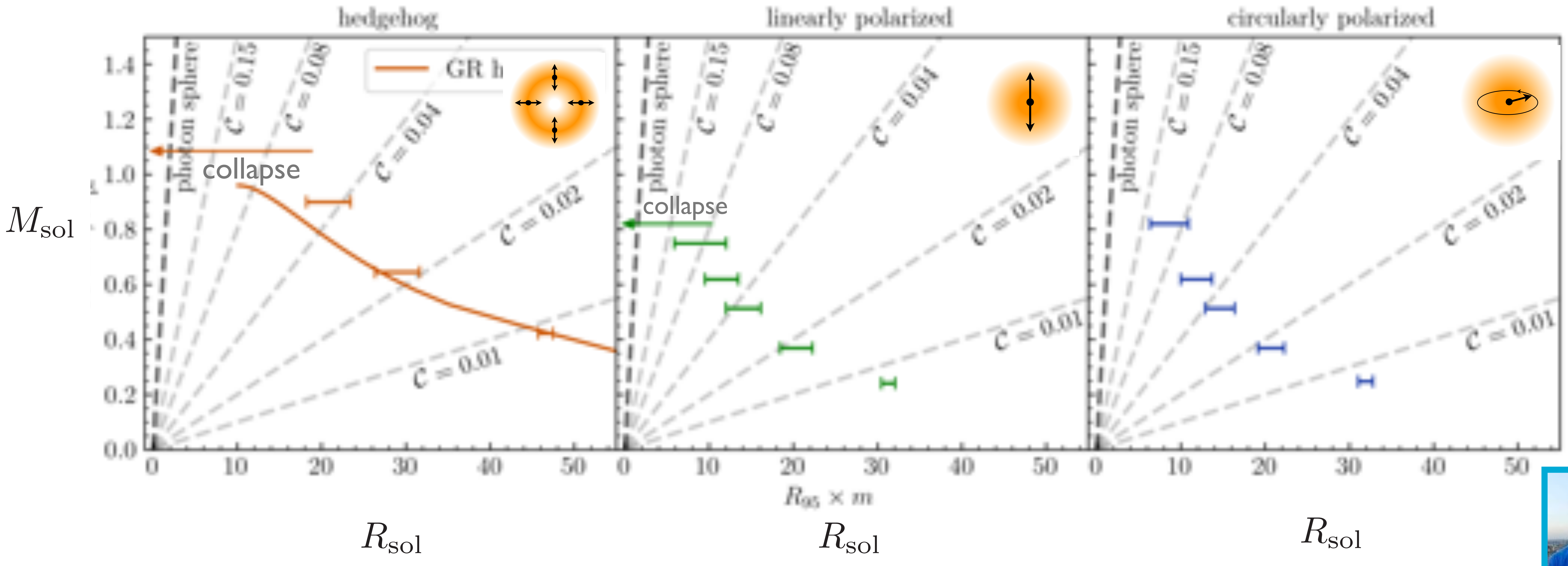
probing intrinsic spin of solitons



compactness of polarized solitons

more resistance to collapse to BH for circularly polarized stars

$$\mathcal{C} < \mathcal{C} < \mathcal{C}$$

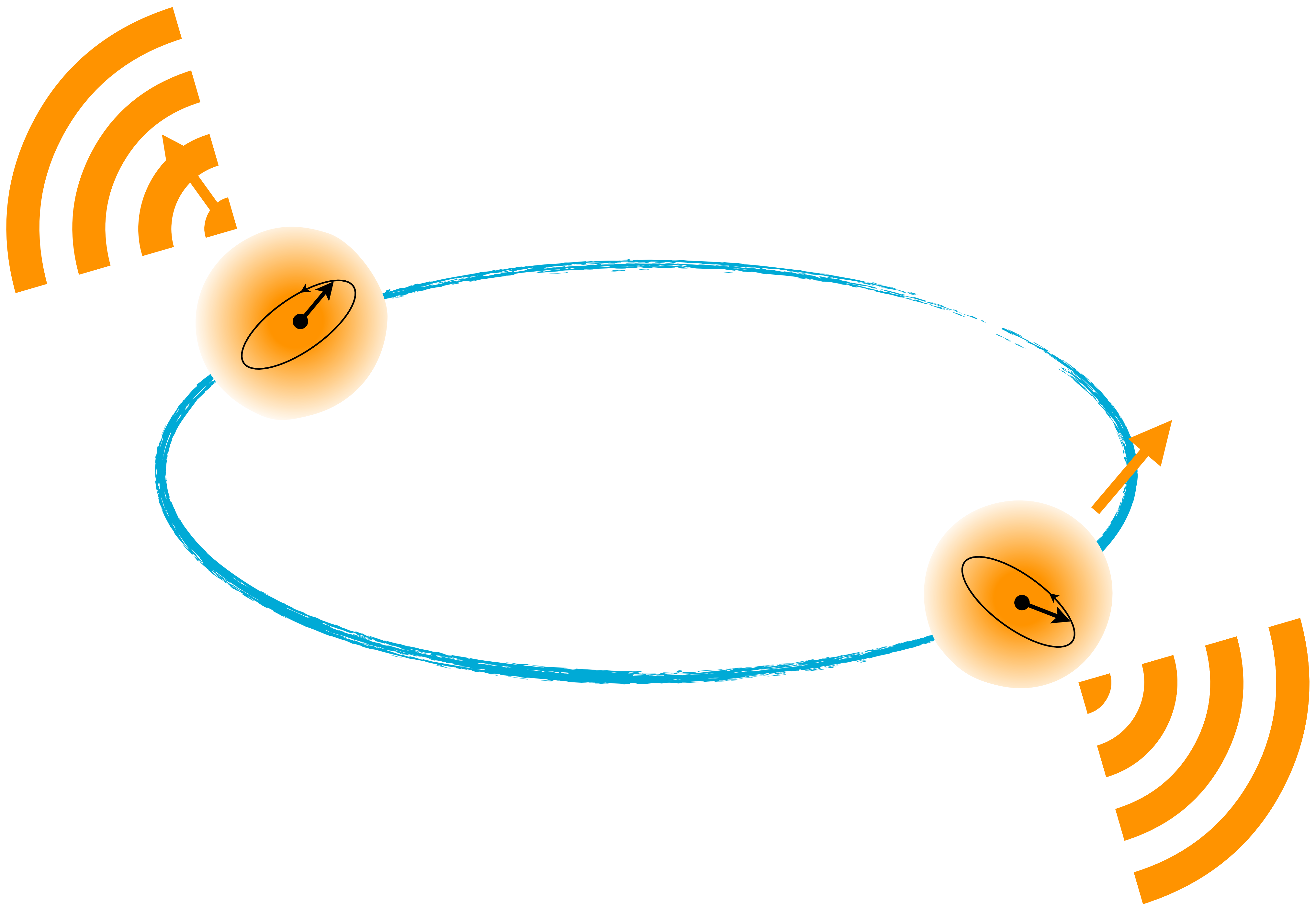


$\mathcal{C} = GM/Rc^2$

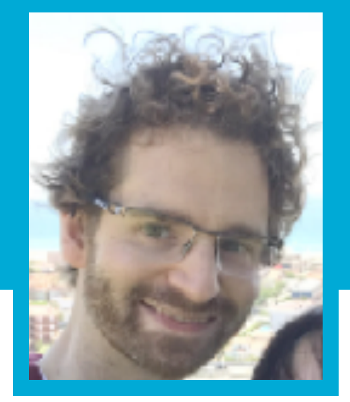
with Thomas Helfer & Zipeng Wang (soon!)

gravitational waves and spin

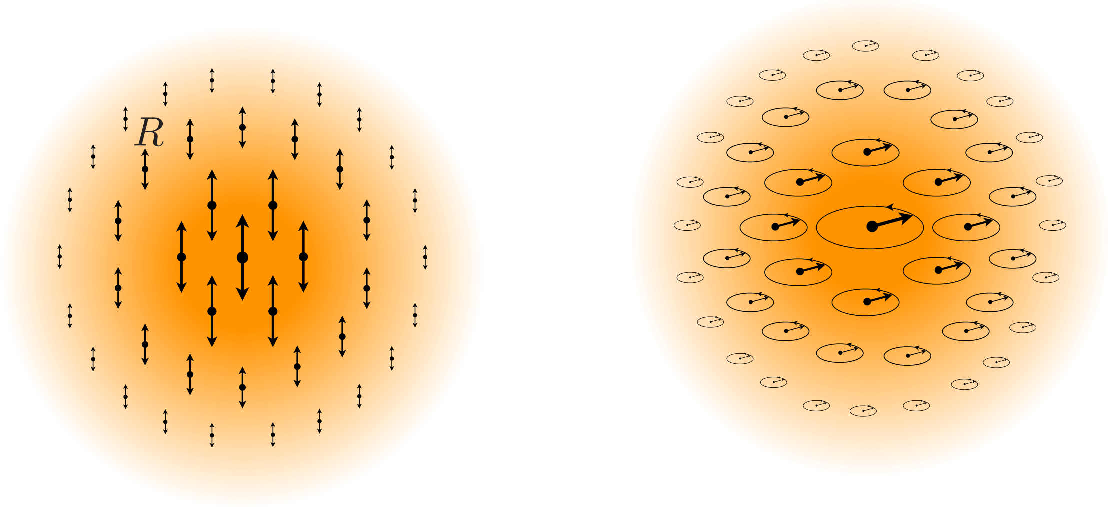
$$V = -\frac{GM_1M_2}{r}\left[1 + \mathcal{O}(v^2/c^2) - \frac{2}{rc}[\hat{\boldsymbol{r}} \times (\boldsymbol{v}_1 - \boldsymbol{v}_2)] \cdot \sum_{a=1}^{2} \frac{\boldsymbol{S}_a}{M_a}\right.$$
$$\left. + \frac{1}{r^2c^2}\left\{\frac{\boldsymbol{S}_1}{M_1}\cdot\frac{\boldsymbol{S}_2}{M_2} - 3\left(\frac{\boldsymbol{S}_1}{M_1}\cdot\hat{\boldsymbol{r}}\right)\left(\frac{\boldsymbol{S}_2}{M_2}\cdot\hat{\boldsymbol{r}}\right) + \sum_{a=1}^{2}\frac{C_{ES^2}^{(a)}}{2M_1M_2}\left[S_a^2 - 3(\boldsymbol{S}_a\cdot\hat{\boldsymbol{r}})^2\right]\right\} + \dots\right]$$



spin of soliton & polarization of photons



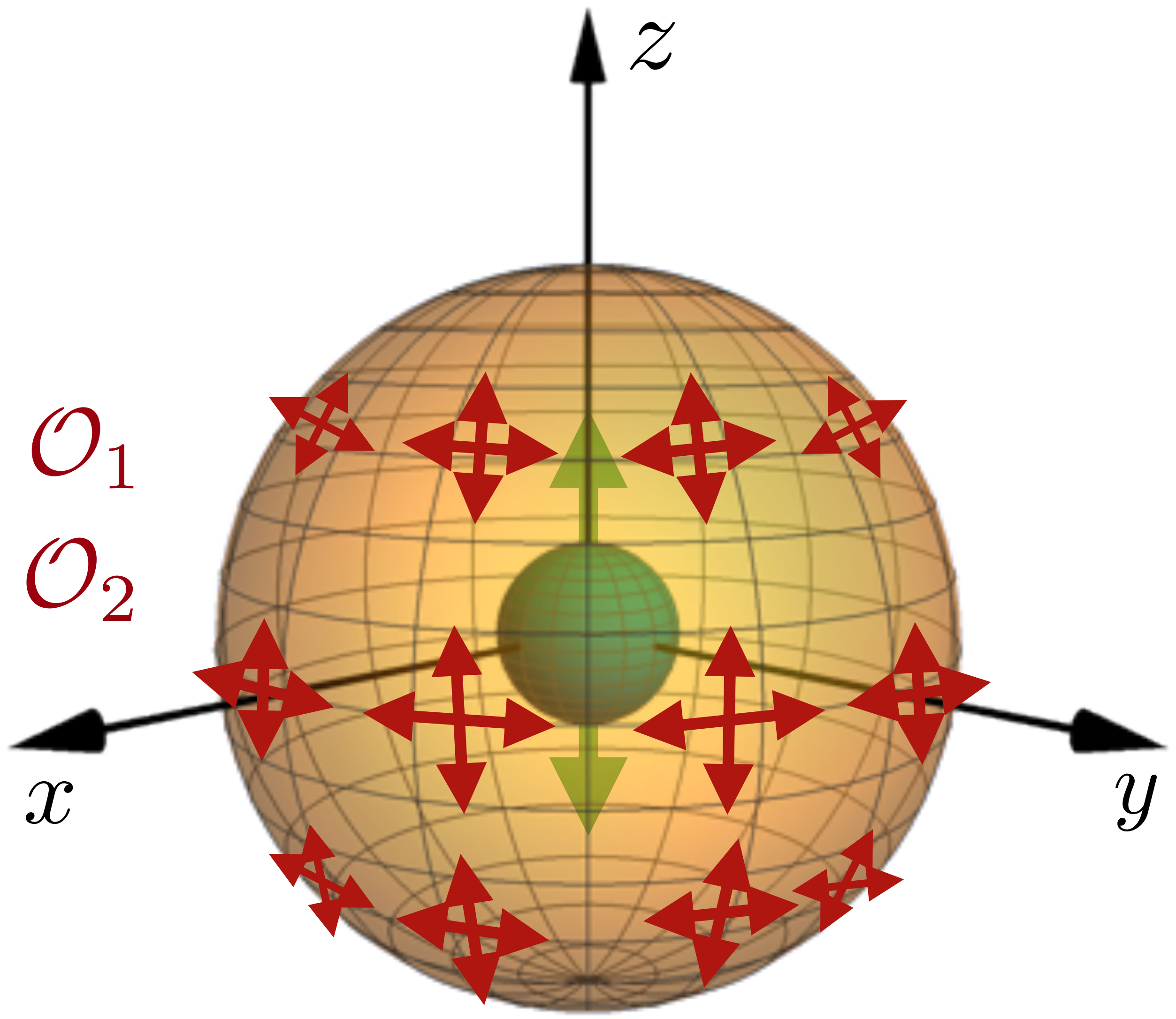
with Schiappacasse & Long (2022)

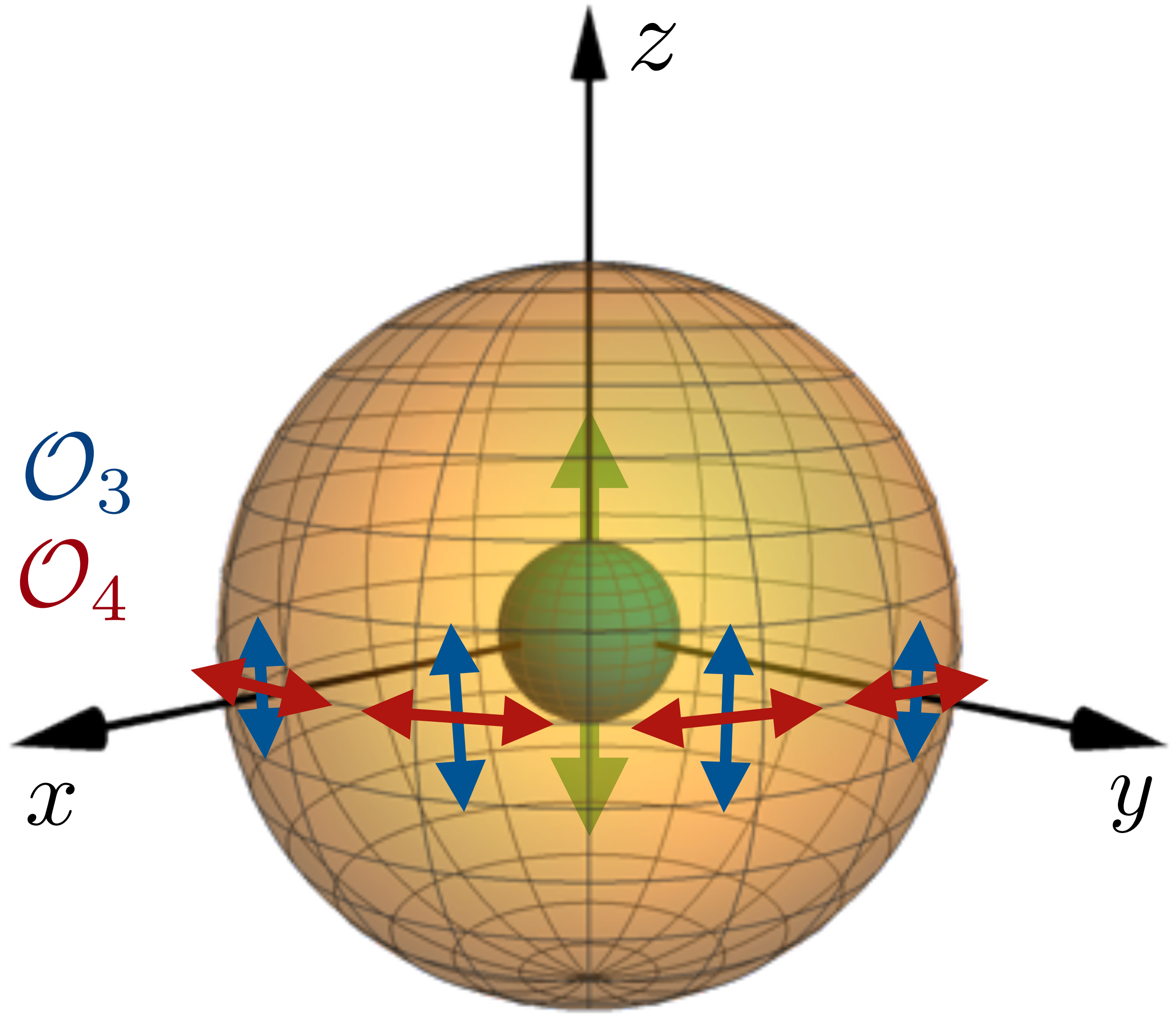


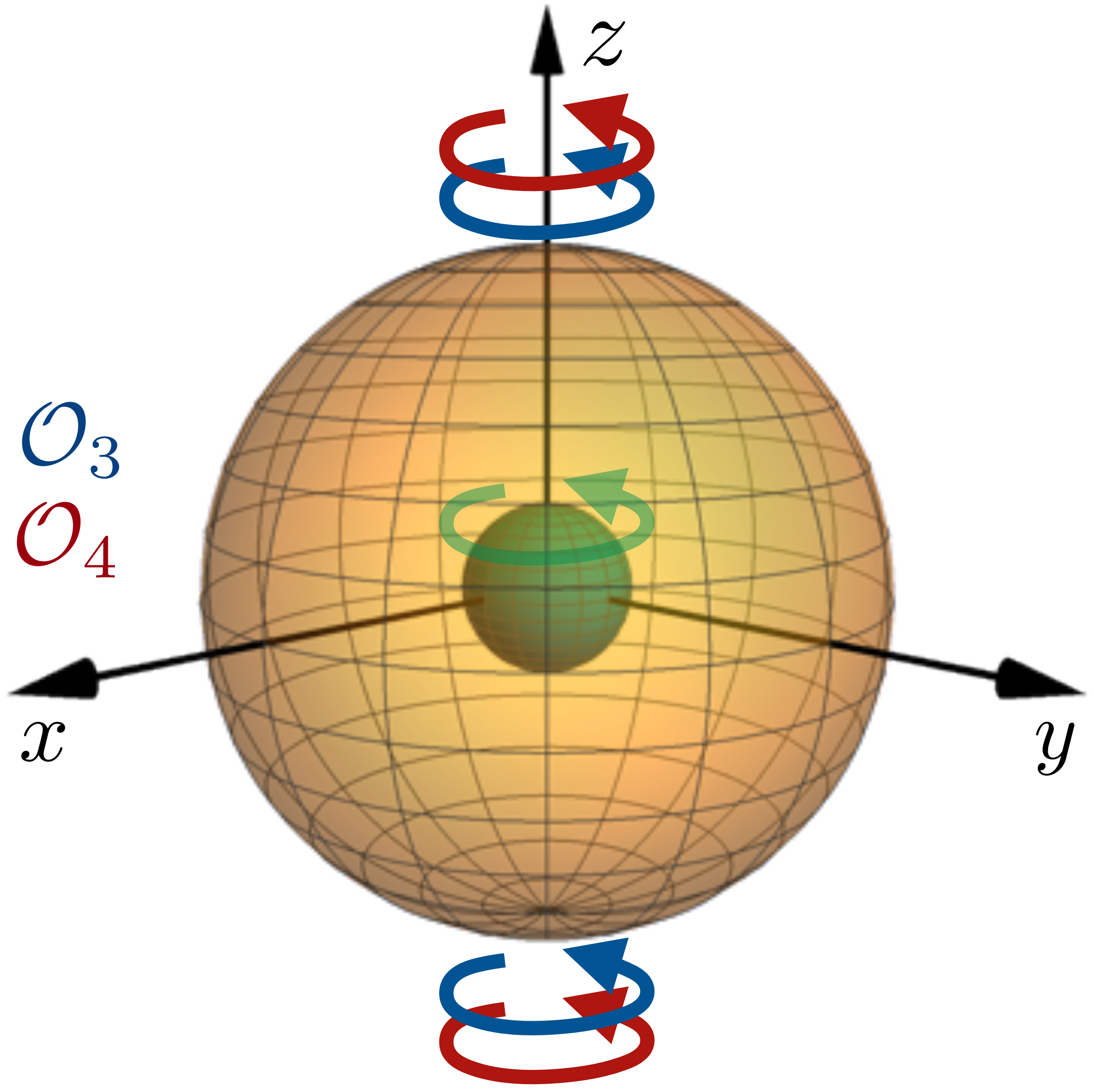
$$\mathcal{O}_1 = -\tfrac{1}{2} F_{\mu\nu}\tilde{F}^{\mu\nu}(X\cdot X)$$
$$\mathcal{O}_2 = -\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu}(X\cdot X)$$
$$\mathcal{O}_3 = F_{\mu\rho}F^{\nu\rho}X^{\mu}X_{\nu}$$
$$\mathcal{O}_4 = \tilde{F}_{\mu\rho}\tilde{F}^{\nu\rho}X^{\mu}X_{\nu}$$

explosive photon production (under certain conditions)

$$\mu R \gtrsim 1, \qquad \mu \sim g^2 X^2 m$$



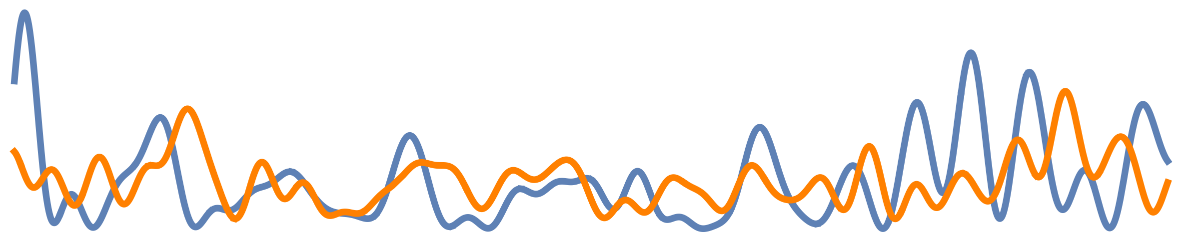


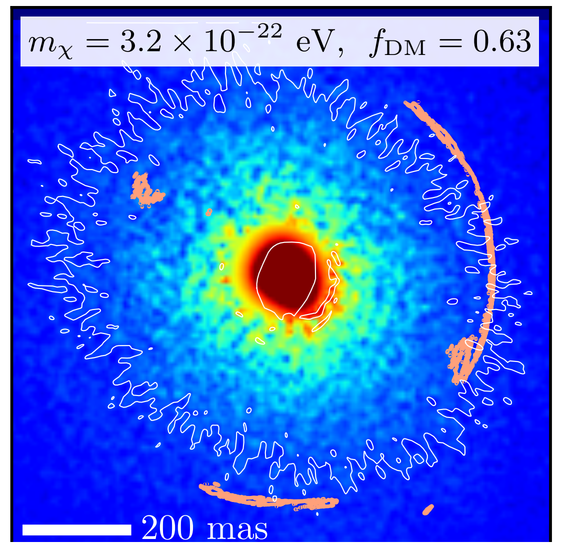


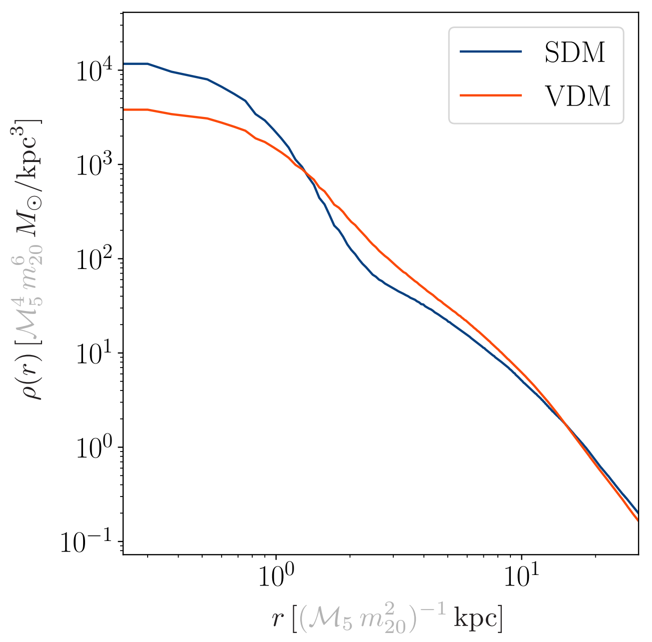
spin and dark matter sub-structure

Phenomenology

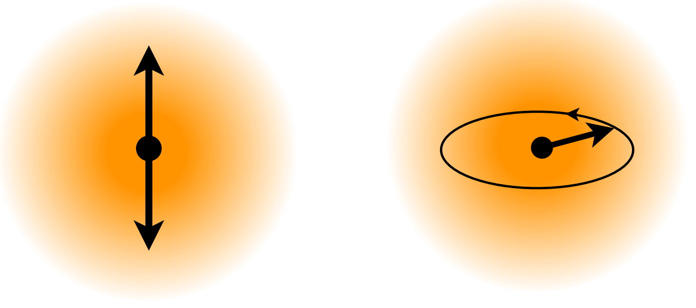
- reduced interference

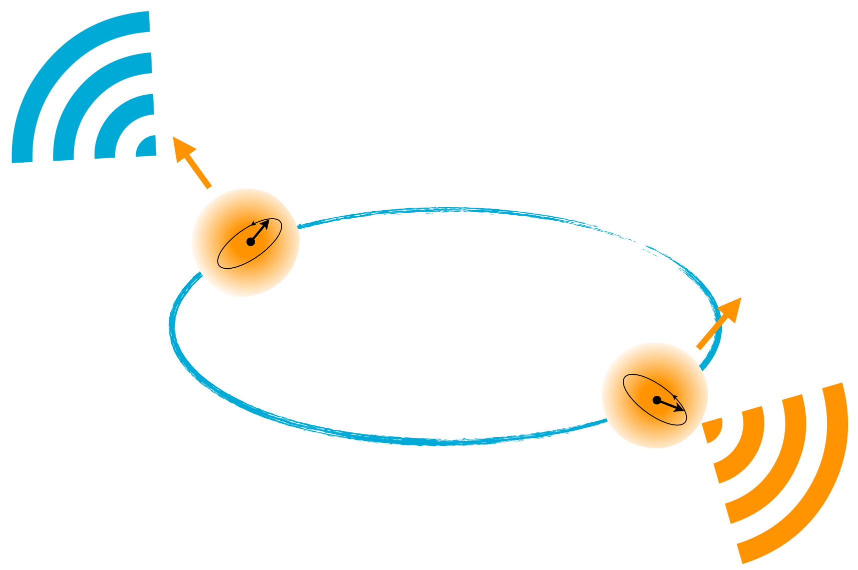




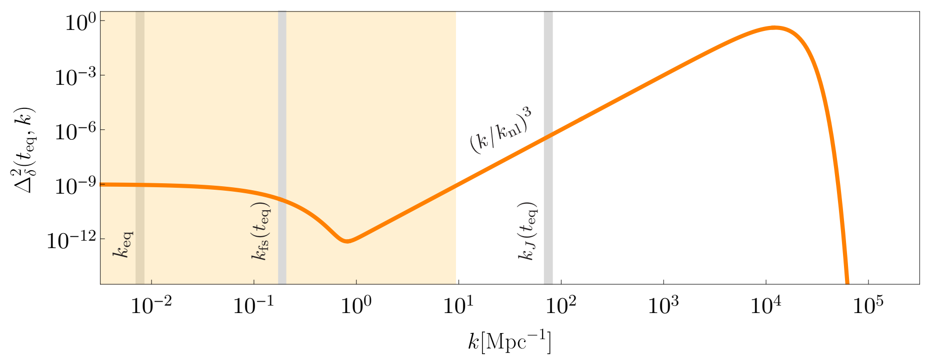


- polarized solitons, with macroscopic spin

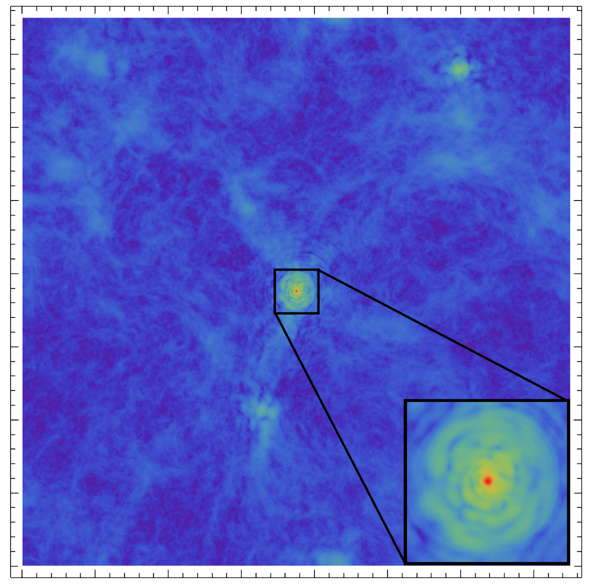


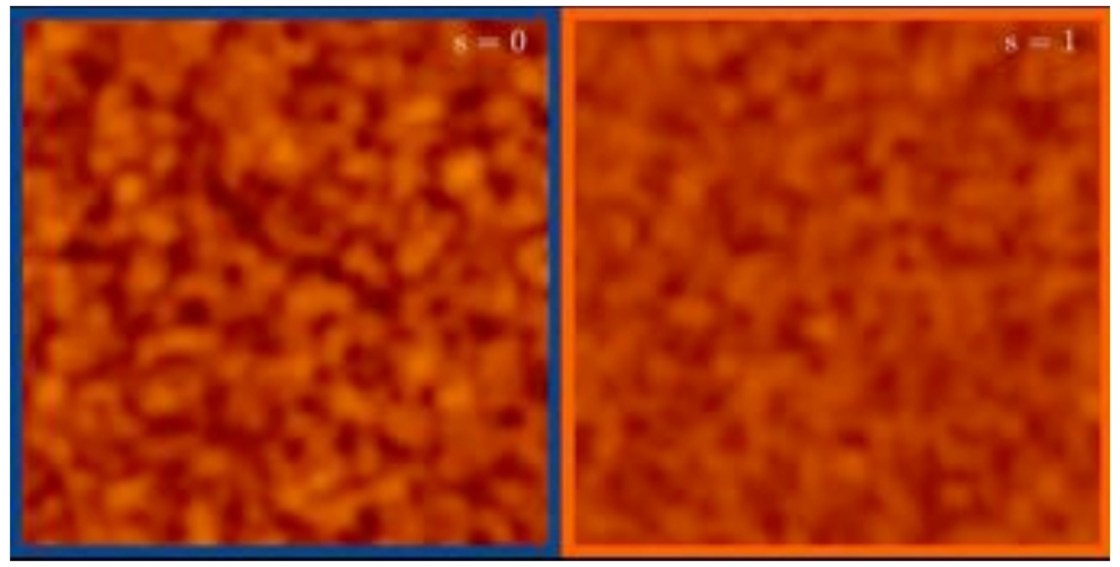


- growth of structure, nucleation time-scales



$\Longrightarrow$





i-SPin: An integrator for multicomponent Schrodinger-Poisson systems with self-interactions

arXiv: 2211.08433

Mudit Jain & Mustafa Amin

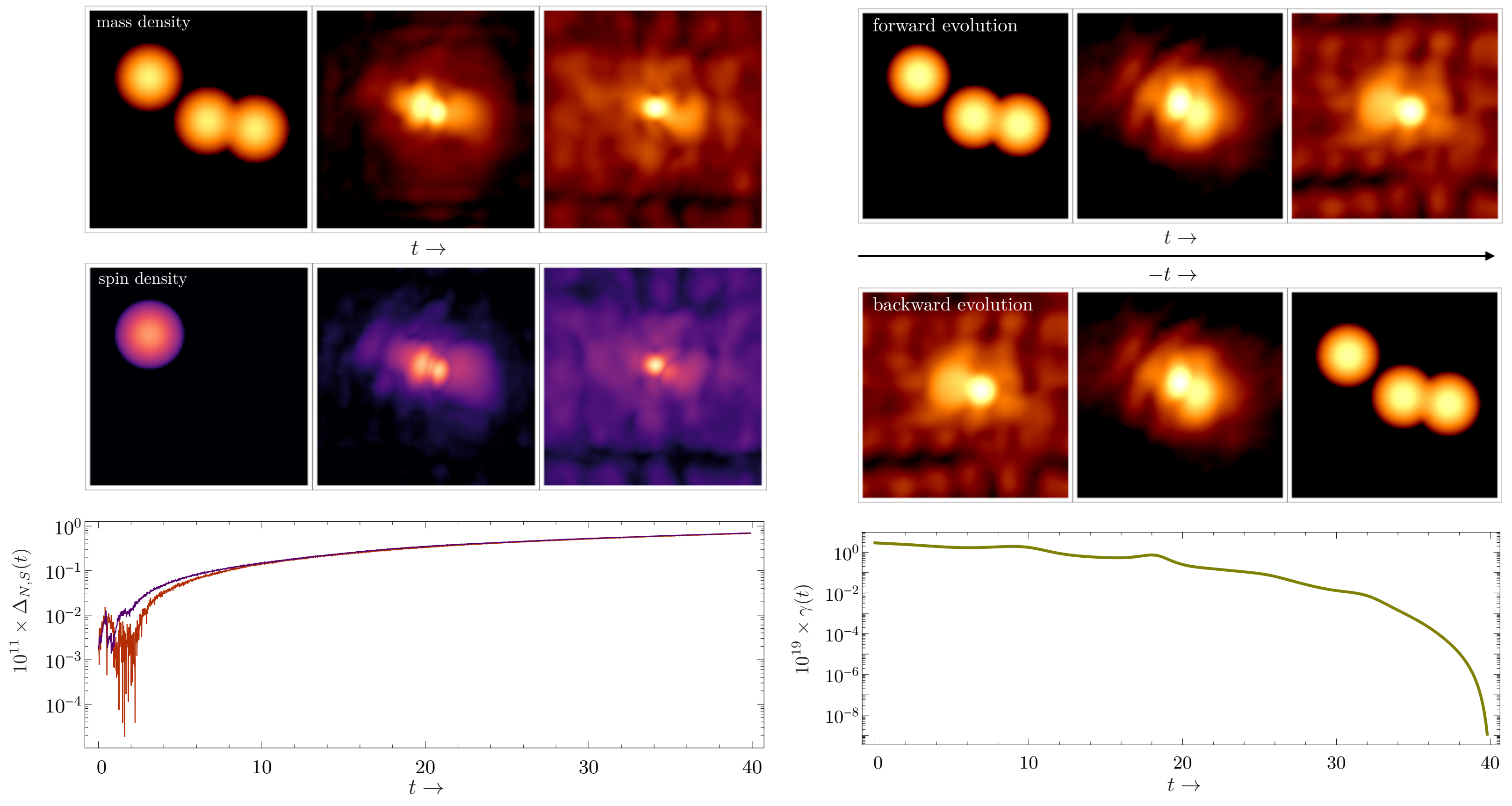
i-SPin: An algorithm (and publicly available code) to numerically evolve multicomponent Schrodinger-Poisson (SP) systems, including attractive/repulsive self-interactions + gravity

problem: If SP system represents the non-relativistic limit of a massive vector field, non-gravitational self-interactions (in particular, *spin-spin* type interactions) introduce new challenges related to mass and spin conservation which are not present in purely gravitational systems.

solution: Above challenges addressed with a novel analytical solution for the non-trivial 'kick' step in the algorithm (sec 4.3.2)

features: (i) second order accurate evolution (ii) spin and mass conserved to machine precision (iii) reversible

generalizations: n-component fields with SO(n) symmetry, an expanding universe relevant for cosmology, and the inclusion of external potentials relevant for laboratory settings



$$i\hbar\,\partial_t \mathbf{\Psi} = -\frac{\hbar^2}{2m}\nabla^2\mathbf{\Psi} + m\,\Phi\,\mathbf{\Psi} - \frac{\lambda(\hbar c)^3}{4(mc^2)^2}\left[(\mathbf{\Psi}\cdot\mathbf{\Psi})\,\mathbf{\Psi}^\dagger + 2\,(\mathbf{\Psi}^\dagger\cdot\mathbf{\Psi})\,\mathbf{\Psi}\right]$$

$$V_{\rm nrel}(\rho,\boldsymbol{\mathcal{S}}) = -\frac{\lambda(\hbar c)^3}{8(mc^2)^2}\left[3\rho^2 - \frac{(\boldsymbol{\mathcal{S}}\cdot\boldsymbol{\mathcal{S}})}{\hbar^2}\right]$$

number density $\rho = \mathbf{\Psi}^\dagger\mathbf{\Psi}$

spin density $\boldsymbol{\mathcal{S}} = i\hbar\mathbf{\Psi}\times\mathbf{\Psi}^\dagger$

i-SPin 2: An integrator for general spin-s Gross-Pitaevskii systems

arXiv: 2305.01675

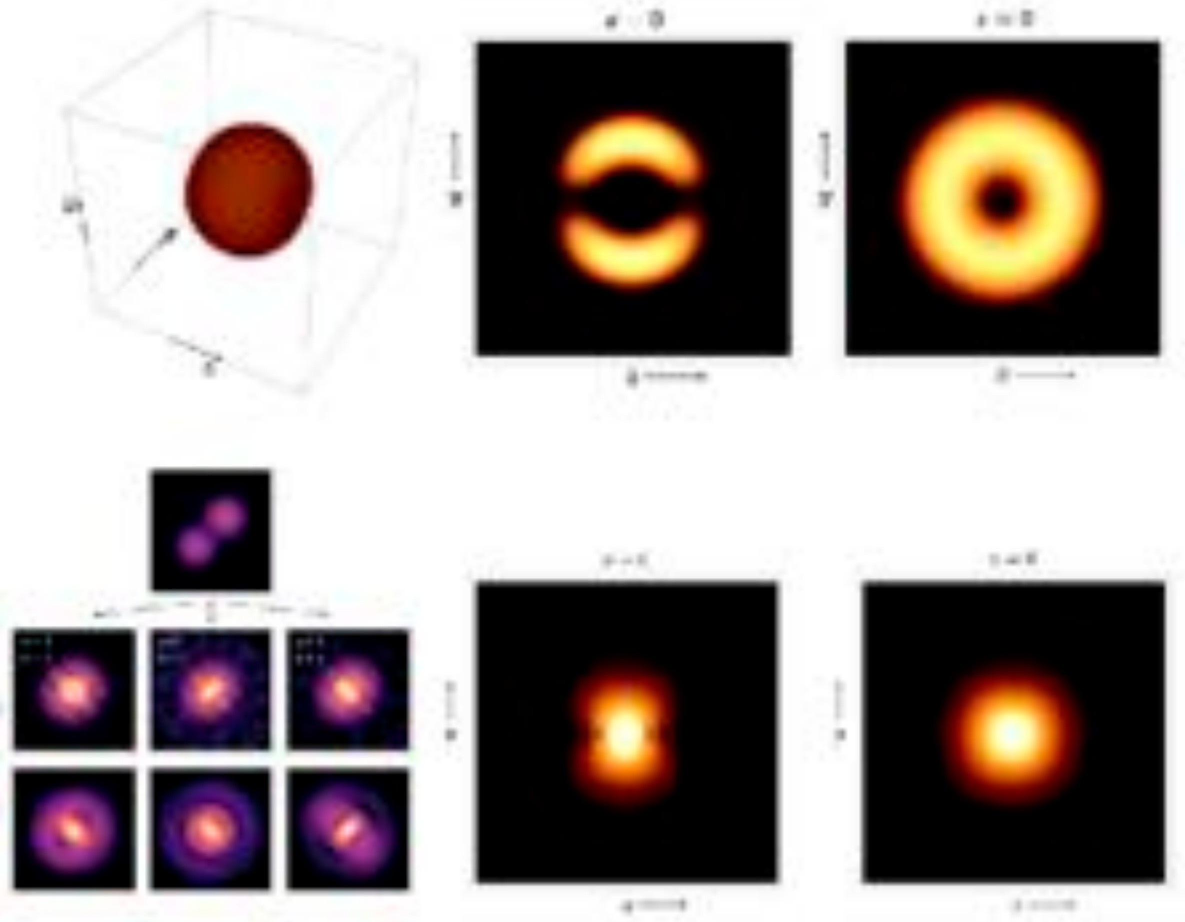
Mudit Jain, Mustafa Amin & H. Pu

i-Spin 2: An algorithm for evolving general spin-s Gross-Pitaevskii / non-linear Schrodinger systems carrying a variety of interactions, where the $2s+1$ components of the 'spinor' field represent the different spin-multiplicity states.

Allowed interactions: Nonrelativistic interactions up to quartic order in the Schrodinger field (both short and long-range, and spin-dependent and spin-independent interactions), including explicit spin-orbit couplings. The algorithm allows for spatially varying external and/or self-generated vector potentials that couple to the spin density of the field.

Applications: (a) Laboratory systems such as spinor Bose-Einstein condensates (BECs). (b) Cosmological/astrophysical systems such as self-interacting bosonic dark matter.

Numerical features: Our symplectic algorithm is second-order accurate in time, and is extensible to the known higher-order accurate methods.



$$\mathcal{S}_{\rm nr} = \int {\rm d}t\,{\rm d}^3x\left[\frac{i}{2}\psi_n^\dagger\dot{\psi}_n + {\rm c.c.} - \frac{1}{2\mu}\nabla\psi_n^\dagger\cdot\nabla\psi_n - \mu\rho V(\boldsymbol{x}) - \gamma\boldsymbol{\mathcal{S}}\cdot\tilde{\boldsymbol{B}}(\boldsymbol{x},t) - V_{\rm nrel}(\rho,\boldsymbol{\mathcal{S}}) - \frac{\xi}{2}\frac{1}{(2s+1)}|\psi_n\hat{A}_{nn'}\psi_{n'}|^2 + i\,g_{ij}\,\psi_n^*\,[\hat{S}_i]_{nn'}\,\nabla_j\,\psi_{n'}\right],$$

with $\tilde{\boldsymbol{B}}(\boldsymbol{x},t) = f(t)\boldsymbol{B}(\boldsymbol{x})$, and

$$V_{\rm nrel}(\rho,\boldsymbol{\mathcal{S}}) = -\frac{1}{2\mu^2}\left[\lambda\rho^2 - \alpha\,(\boldsymbol{\mathcal{S}}\cdot\boldsymbol{\mathcal{S}})\right].$$

number density $\rho = \psi_n^\dagger\psi_n$

spin density $\boldsymbol{\mathcal{S}} = \psi_n^*\,\hat{\boldsymbol{S}}_{nn'}\,\psi_{n'}$