An Introduction to Relativistic Quantum Field Theory
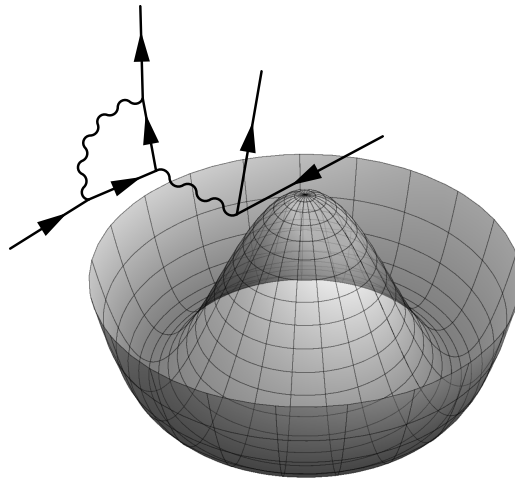
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Abstract

These are QFT Lecture Notes for Phys 622 at Rice University. At the moment, they constitute a work in progress, with half of the notes typeset, the other half (later half of chapter 5 onwards) hand-written. Please report any typos/errors to mustafa.a.amin@rice.edu.

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Preface

These are notes for the Phys 622 course, *An Introduction to QFT*, at Rice University. This is *not* a comprehensive textbook on QFT. Typically, students from high energy physics and condensed matter theory take this course at Rice. The condensed matter students go on to take many-body physics as a second course, whereas the high energy physicists take a more Standard Model focused course next. Along with high energy physics and condensed matter theory, at times some undergrads, cosmologists/astrophysicists, biophysicists and theoretical chemists also take this course. This diversity of students, and their academic paths have guided the preparation of these notes – though it is still focussed on relativistic field theory. My goal is to set up the foundations of QFT, give a flavor for calculations (with sufficient, simplified examples) but leave the inevitable complications of the real world to later, more specialized courses.

The course is roughly divided into two parts. In the first part, I only deal with scalar field QFT. It culminates in the derivation of Feynman rules, and a number of calculations of scattering at tree level. Spin is only introduced in the second half of the semester, where we focus on how symmetries guide and constrain the nature of different spin fields. In a one semester course, it was possible to introduce, but not do any calculations in non-Abelian Gauge theories. I include spontaneous symmetry breaking in the context of Anderson-Higgs mechanism for particle physics and also its relevance, for example, in superconductivity here as well. Dictated by student interest, I have also included some brief notes on solitons and topological considerations in field theories.

At the time of posting, only the first half of the course is typeset in LaTeX, the rest of the notes are handwritten. Next year, I will hopefully typeset the rest as well.

Acknowledgements

These notes do not contain much in terms of original material. QFT is a mature subject, with many comprehensive textbooks and pedagogical notes that are easily available. When I taught this course for the first time (2016), I benefited from Prof. Paul Stevenson's excellent notes for this class at Rice, which he taught for many years. His experience showed especially in a wonderfully economical path through the material. Prof. David Tong's (Cambridge) notes were the other set of notes I turned to repeatedly.

As far as textbooks go, I learnt QFT from Peskin and Schroeder, as well as by dipping in and out of A. Zee's *QFT in a Nutshell* for inspiration and anecdotes. In preparing these notes, I referred to them repeatedly. In addition, I thoroughly enjoyed *QFT for the Gifted Amateur* by Tom Lancaster and Stephen Blundell. The short chapters in both Zee and Lancaster & Blundell, the appropriately simplified sample calculations, the pedagogical (and at times whimsical) prose and diagrams made them a pleasure to read. Both books were also helpful in helping me learn (at a cursory level) condensed matter applications of QFT from their self-contained chapters.

CONTENTS

CHAPTER 1

AN INVITATION

In "Classical" Mechanics, particles are objects localized in space. Their position varies with time according to ordinary differential equations. In single particle quantum mechanics, position became an operator and the notion of a particle became "fuzzier" (think about localized probability density in terms of wavefunctions, uncertainty principle etc.)

Classical fields on the other hand are extended objects, with values at each point in space. Typically, partial differential equations govern the time evolution of fields (think of the electromagnetic fields: $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ governed by Maxwell's equations). How can we think about quantum mechanics of fields ? What is the a connection between fields and particles?

In this course, we will take the view point of the field and particles being part of the same physical entity. For example, photons are quantized excitations of the electromagnetic field. Equivalently we can think of the electromagnetic field as a collection of quantized excitations: photons. Similar statements hold for the electron and the electron field, quarks and a quark field ... you get the idea.

What is this course about?
This course is about learning the rules that govern the behavior of fields and their excitations (particles), insisting on consistency with Quantum Mechanics and Special Relativity. Symmetries will play an important role in determining these rules.[1]

For a heuristic conceptual roadmap to QFT, see Fig. 1.1. An important aspect of QFT in general is that we are forced to deal with an infinite number of degrees of freedom. This, as we will see, this will end up being connected to merging of special relativity and quantum mechanics. These infinite degrees of freedom will also lead to some glaring difficulties in calculating observables; these difficulties will require conceptual leaps to overcome them. As always, these rough and somewhat abstract statements will make more sense when we have done some concrete examples. I hope that the course will allow you to appreciate (and perhaps even make you uneasy) about some of the heuristic statements made in this introduction.

1.1 Some Highlights from QFT

- QFT is one of the most successful theoretical frameworks we have. In Quantum Electrodynamics (QED), agreement between theory and experiment for the *anamolous muon magnetic moment* ($g_\mu -$

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[1] Occasionally we will work in the non-relativistic limit, and sometimes only talk about classical fields also. The tools developed here will remain useful.
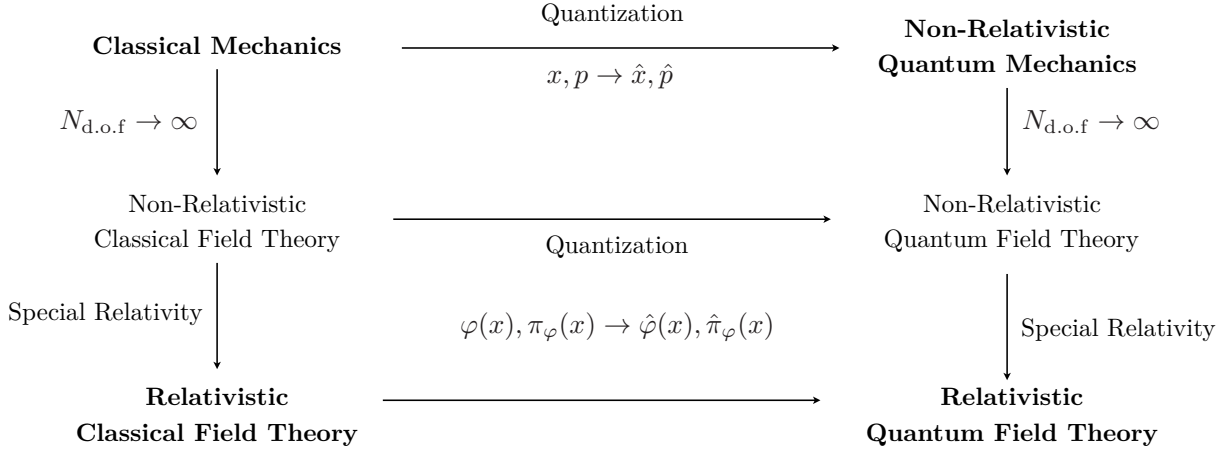
Figure 1.1: A roadmap to Relativistic Quantum Field Theory (adapted from Kuhlmein, "Quantum Field Theory", *The Stanford Encyclopedia of Philosophy*, 2015.)

2)/2 is within one part in 10^{10}.

$$\left(\frac{g_\mu - 2}{2}\right)_{\text{exp}} = 0.001\,159\,652\,180\,73(28)$$
$$\left(\frac{g_\mu - 2}{2}\right)_{\text{th}} = 0.001\,159\,652\,181\,78(77)$$

(1.1.1)

- All particles of a field are indistinguishable (just excitations of the same field, you cannot label them).

- "particles" = excitations, can be created and destroyed.

- Bose & Fermi Statistics emerge naturally.

- QFT plays a central role in condensed matter as well as atomic and molecular physics. Important especially for collective dynamics, phase transitions etc.

- Cosmology: QFT important in understanding the origin of density perturbations in our universe, as well as the hot big bang.

- Big Unsolved Questions: QFT of gravity? Vacuum energy?

Before diving into QFT, we will review (1) Special Relativity (2) Lagrangian and Hamiltonian Mechanics (3) Quantum Mechanics and finally (4) Classical Fields. Putting these together, we will get to QFT; and before all that, let us begin with units.

1.2   Units

- Note the dimensions of $[\hbar] = ML^2T^{-1}$ and $[c] = LT^{-1}$. In "everyday" units, $\hbar = 1.05 \times 10^{-34}\,\text{m}^2\,\text{kg}\,\text{s}^{-1}$, whereas speed of light $c = 3 \times 10^8\,\text{m}\,\text{s}^{-1}$. We work in units where $\hbar = c = 1$. If needed, we will also set $k_{\text{B}} = 1$. In "usual" units $k_B \approx 1.38 \times 10^{-23}\,\text{kg}\,\text{m}^2\,\text{s}^{-2}\,\text{K}^{-1}$.

- Mass, energy and momentum can be measured with the same unit. We will typically use a GeV as a unit for these quantities. Note that $1\,\text{GeV} \approx 1.60 \times 10^{-10}\,\text{kg}\,\text{m}^2\,\text{s}^{-2}$.

- Length and time intervals have the same units. We will typically use GeV^{-1} as a unit for them.

- examples: mass of proton $= 0.938\,\text{GeV} = 0.938\,\text{GeV}/c^2 \approx 10^{-27}\,\text{kg}$), size of a proton $\sim 5\,\text{GeV}^{-1} = 5\text{GeV}^{-1}\,\hbar c \approx 10^{-15}\,\text{m} = 1\,\text{fermi}$).

Exercise 1.2.1 : Our universe if filled with blackbody radiation (Cosmic Microwave Background) left over from the time when the first atoms formed. The present temperature of this radiation is $\sim 3K$. Using dimensional analysis, make an order of magnitude estimate of the number of density of photons in the CMB in units of (i) GeV^3 (ii) cm^{-3}.

Exercise 1.2.2 : (1) If you drop a tomato from a height of $\sim 1\,\text{meter}$, what is its kinetic energy just before it hits the ground in (i) Joules (ii) GeV? (2) At what height would you have to drop an ant ($\sim 2\,\text{mg}$) for it's kinetic to be of order a GeV when it hits the ground. (3) What is the typical center of mass energy of protons at the Large Hadron Collider? (4) What is the ionization energy of a Hydrogen atom ?

In this chapter I review relevant aspects of Einstein's Special Theory of Relativity, Lagrangian and Hamiltonian Mechanics as well as Quantum Mechanics for a countable (and finite) number of degrees of freedom. Though you are likely familiar with at least some of this material, some formal aspects such as Poisson Brackets and time-evolution operators introduced here might not have been covered in earlier courses.

For most of this course we will work in natural units with $\hbar = c = 1$ where $\hbar$ and c are the Planck's constant and speed of light respectively. In this review chapter, I set $c = 1$ so length and time have the same units, but do not set $\hbar = 1$ since it helps in seeing quantum aspects more clearly.

2.1 Special Relativity

In two sentences, here is Einstein's Special Theory of Relativity:

- The laws of physics are the same in all intertial frames.

- The speed of light (in vacuum) c is independent of the reference frame.

Interval and the metric: The inifinitesimal, frame-invariant spacetime interval:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu \qquad \text{with} \qquad \mu, \nu = 0, 1, 2, 3\,. \tag{2.1.1}$$

where "0" labels the time co-ordinate. We have adopted the Einstein summation convention, where repeated upstairs and downstairs indices are summed over. $g_{\mu\nu}$ are components of the metric tensor which (among other responsibilities), determines the interval between events. Think of $g_{\mu\nu}$ as entires of a matrix $\mathbf{g}$, that is $\mathbf{g}(\mu, \nu) \equiv g_{\mu\nu}$, where the first index labels the row, and the second the column with $\mu, \nu = 0, 1, 2, 3$. In cartesian co-ordinates (and with slight abuse of notation):

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.1.2}$$

Explicity $ds^2 = (dx^0)^2 - \delta_{ij}dx^i dx^j = dt^2 - d\mathbf{x} \cdot d\mathbf{x}$. If the interval corresponding to nearby events $ds^2 < 0 (> 0)$, then the events are said to be connected by a space-like (time-like) interval. $ds^2 = 0$ corresponds to a light-like interval. The same definitions carry over for finite intervals (obtained by joining together infinitesimal intervals $\sim \int ds$). One event can influence another only if the interval between them

is not spacelike. All the events along the trajectory of a massive (massless) particle are connected by a time-like (light-like) interval. This can be visualized as a "light-cone" (see Fig. 2.1).
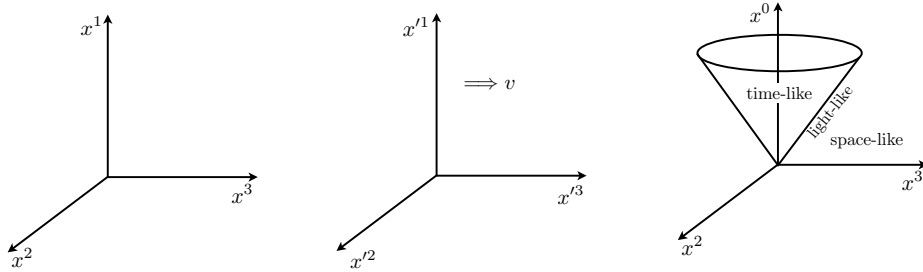
Figure 2.1

Lorentz tranformations: A Lorentz transformation allows us to change reference frames. A familiar example is the transformation of co-ordinates from one inertial frame to another moving at a velocity v along along one of the cartesian axes (see Fig. 2.1).

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \qquad \text{where} \qquad \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{2.1.3}$$

with $\gamma = (1 - v^2)^{-1/2}$ and recall that $c = 1$ in our units. More generally, the defining property of Lorentz transformations is

$$\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} g_{\mu\nu} = g_{\rho\sigma}. \tag{2.1.4}$$

There is a possibility of confusion here when one tries to write this in matrix form. So let me say a few more words. We have a matrix Λ whose entry in the μ row and ν column is given by $\Lambda(\mu, \nu) \equiv \Lambda^{\mu}{}_{\nu}$. The defining property of the Lorentz transformation in matrix form is $\Lambda^T g \Lambda = g$. To see this, note that in terms of matrix entries, the left hand side is $\sum_{\nu,\mu=0}^{3} \Lambda^T(\rho, \mu) g(\mu, \nu) \Lambda(\nu, \sigma) = \sum_{\nu,\mu=0}^{3} \Lambda(\mu, \rho) g(\mu, \nu) \Lambda(\nu, \sigma)$. Since we have defined $\Lambda(\mu, \nu) \equiv \Lambda^{\mu}{}_{\nu}$ and $g(\mu, \nu) \equiv g_{\mu\nu}$, we have $\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} g_{\mu\nu} = g_{\rho\sigma}$ where now we revert back to Einstein summation. It is worth noting that for the inverse transformation, $\Lambda^{-1}(\mu, \nu) = \Lambda_{\nu}{}^{\mu}$, whereas the inverse metric $g^{-1}(\mu, \nu) = g^{\mu\nu}$.

An immediate consequence of the defining property of the Lorentz transformation is that $|\det\Lambda| = 1$. Hence, 4-volume elements d^4x are Lorentz invariant: $d^4x' = d^4x |\det \Lambda| = d^4x$.

Exercise 2.1.1 : Verify that the Lorentz transformation in eq. (2.1.3) satisfies its defining property (2.1.4).

4-vectors and dot products: component form, a Lorentz four-vector is a 4-component object which transforms as $A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}$. Explicitly:

$$A^{\mu} = (A^0, \mathbf{A}) \qquad \text{and} \qquad A_{\mu} = g_{\mu\nu} A^{\nu} = (A^0, -\mathbf{A}), \tag{2.1.5}$$

where with $\mathbf{A} = \{A^1, A^2, A^3\}$. In the second equality, note that the metric allows us to raise and lower indices. We will often use the following shorthand(s)

$$x = x^{\mu} = (x^0, \mathbf{x}) = (t, \mathbf{x}),$$
$$k = k^{\mu} = (k^0, \mathbf{k}) = (E, \mathbf{k}), \tag{2.1.6}$$

where in the last line we are thinking of k as the four-momentum of a particle, with E being the energy. Their dot product is Lorentz invariant (ie. its value does not change under Lorentz transformations)

$$
\begin{aligned}
x \cdot k &= g_{\mu\nu} x^\mu k^\nu = x_\mu k^\mu = x^\mu k_\mu = Et - \mathbf{x} \cdot \mathbf{k}\,, \\
k^\mu k_\mu &= E^2 - |\mathbf{k}|^2 = m^2\,.
\end{aligned}
\tag{2.1.7}
$$

where m is the rest-mass of the particle.

Some useful differential operators are listed below:

$$
\begin{aligned}
\partial_\mu &= \frac{\partial}{\partial x^\mu} = (\partial_0, \nabla) && \text{and} && \partial^\mu = (\partial_0, -\nabla)\,, \\
\Box &= \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2 && \text{and} && \partial \cdot A = \partial_\mu A^\mu = \partial_0 A^0 + \nabla \cdot \mathbf{A}\,.
\end{aligned}
\tag{2.1.8}
$$

Exercise 2.1.2 : Show that $f(x) = e^{ik \cdot x}$ satisfies $(\Box + m^2) f(x) = 0$ only if $k^\mu k_\mu = m^2$.

Exercise 2.1.3 : Consider two reference frames related by the Lorentz transformation in eq. (2.1.3).

Let $x^\mu(P) = \left(x_P^0, 0, 0, x_P^3\right)$ and $x^\mu(Q) = \left(x_Q^0, 0, 0, x_Q^3\right)$ be the co-ordinates of two events P and Q in the "unprimed" frame. The co-ordinates of these events in the "primed" frame are given by $x'^\mu(P) = \left(x_P'^0, 0, 0, x_P'^3\right)$ and $x'^\mu(Q) = \left(x_Q'^0, 0, 0, x_Q'^3\right)$. Show that if the events are not simultaneous ($x_P^0 \neq x_Q^0$), and are space-like separated in the unprimed frame (i.e. $\Delta x^\mu \Delta x_\mu < 0$ where $\Delta x^\mu = x^\mu(P) - x^\mu(Q)$), then there exists a velocity v such that in the primed frame, the events are simultaneous $x_P'^0 = x_Q'^0$. Find this velocity v.

While simultaneity is frame-dependent, intervals are not. That is, $\Delta x'^\mu \Delta x'_\mu = \Delta x^\mu \Delta x_\mu$. Hence the space-like, time-like and light-like nature of intervals is invariant under Lorentz transformations.

2.2 Classical Mechanics

I am going to go through a quick, formal review of classical mechanics with an eye towards Quantum Mechanics.

2.2.1 Lagrangian Mechanics

Start with a (given) Lagrangian $L(q_\alpha, \dot{q}_\alpha, t)$ where q_α are the generalized co-ordinate of the system ($\alpha = 1, 2 \ldots N$). The equations of motion for q_α are obtained by extremizing the *Action*:

$$
S = \int_{t_i}^{t_f} dt\, L(q_\alpha, \dot{q}_\alpha, t)\,,
\tag{2.2.1}
$$

That is $q_\alpha(t)$ are such that for $q_\alpha(t) \to q_\alpha(t) + \delta q_\alpha(t)$ (where $\delta q_\alpha(t)$ are arbitrary apart from $\delta q(t_i) = \delta q(t_f) = 0$) we have $\delta S = 0$. For $\delta S = 0$, q_α must satisfy (see Appendix A.3):

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha} \qquad \text{Euler-Lagrange equations}
\tag{2.2.2}
$$

For example, for a collection of coupled harmonic oscillators with unit mass and *time-independent* couplings[1] $M_{\alpha\rho}$:

$$L(q_\alpha, \dot{q}_\alpha) = \sum_{\alpha=1}^{N} \left(\frac{1}{2}\dot{q}_\alpha^2 - \sum_{\rho=1}^{N} \frac{1}{2}M_{\alpha\rho}\, q_\alpha q_\rho \right) ,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) = \frac{\partial L}{\partial q_\alpha} \tag{2.2.3}$$

$$\implies \ddot{q}_\alpha + \sum_{\rho=1}^{N} M_{\alpha\rho} q_\rho = 0 \,.$$

2.2.2 Hamiltonian Mechanics

The Hamiltonian is a Legendre transform of the Lagrangian and contains the same information as the Lagrangian. In Quantum Mechanics and QFT, the Hamiltonian is often more convenient to work with, so lets do a quick review:

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha} \qquad\qquad \text{conjugate momentum}$$

$$H \equiv \sum_{\alpha=1}^{N} (p_\alpha \dot{q}_\alpha) - L \qquad\qquad \text{Hamiltonian} \tag{2.2.4}$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}\,, \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \qquad \text{Hamilton's equations}$$

For the coupled harmonic oscillators example,

$$p_\alpha = \dot{q}_\alpha \,,$$

$$H = \sum_{\alpha=1}^{N} \left(\frac{1}{2}p_\alpha^2 + \sum_{\rho=1}^{N} \frac{1}{2}M_{\alpha\rho}\, q_\alpha q_\rho \right) , \tag{2.2.5}$$

$$\dot{p}_\alpha = -\sum_{\rho=1}^{N} M_{\alpha\rho}\, q_\rho \,.$$

2.2.3 Poisson Brackets

The Poisson Bracket is defined as:

$$\{f(q_\alpha, p_\alpha), g(q_\alpha, p_\alpha)\} \equiv \sum_{\alpha=1}^{N} \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} \right) , \tag{2.2.6}$$

where f and g are arbitrary functions on the space of generalized co-ordinates and their conjugate momenta. The time evolution of $f(q_\alpha, p_\alpha, t)$ is given by

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \,. \tag{2.2.7}$$

which you can see immediately by noting that $df/dt = \sum_\alpha [(\partial f/\partial q_\alpha)\dot{q}_\alpha + (\partial f/\partial p_\alpha)\dot{p}_\alpha] + \partial f/\partial t$ and using Hamilton's equations of motion for $\dot{p}_\alpha$ and $\dot{q}_\alpha$. If the function f does not explicitly depend on time, then

$$\frac{df}{dt} = \{f, H\} \,. \tag{2.2.8}$$

---

[1]Typically, we consider $M_{\alpha\rho}$ that only couples nearest neighbors.

The time evolution of f is generated by H. In particular, for $f = q_\alpha$ and $g = p_\rho$, we have

$$\{q_\alpha, p_\rho\} = \delta_{\alpha\rho} \,. \tag{2.2.9}$$

Similarly, $\{q_\alpha, q_\rho\} = \{p_\alpha, p_\rho\} = 0$. The time evolution equations for q_α and p_ρ become

$$\frac{dq_\alpha}{dt} = \{q_\alpha, H\}\,, \qquad \text{and} \qquad \frac{dp_\alpha}{dt} = \{p_\alpha, H\}\,. \tag{2.2.10}$$

The last two equations are equivalent to Hamilton's equations of motion, and also to the Euler-Lagrange equations.

Exercise 2.2.1 : For the coupled harmonic oscillator example in eq. (2.2.5), evaluate the Poisson Brackets $\{p_\alpha, H\}$, to recover $\dot{p}_\alpha = -\sum_\rho M_{\alpha\rho} q_\rho$.

2.3   Quantum Mechanics

I am going to review relevant aspects of Quantum Mechanics; the results here are the most relevant part of this review chapter.

2.3.1   Canonical Quantization

One route to getting from classical to quantum mechanics is as follows (thanks to Dirac[2]):

- replace the co-ordinates and momenta by operators (think of them as matrices)

$$q_\alpha, p_\alpha \longrightarrow \hat{q}_\alpha, \hat{p}_\alpha \,. \tag{2.3.1}$$

  The functions f and g inherit the operator structure from p and q: $f, g \to \hat{f}, \hat{g}$.

- Replace the Poisson Bracket by the "Commutator"

$$\{f, g\} \to -\frac{i}{\hbar} \left[\hat{f}, \hat{g}\right]\,. \tag{2.3.2}$$

  Note the appearance of i and Planck's constant $\hbar$. The commutator is defined as

$$\left[\hat{f}, \hat{g}\right] \equiv \hat{f}\hat{g} - \hat{g}\hat{f}\,. \tag{2.3.3}$$

  Operators $\hat{f}$ and $\hat{g}$ do not necessarily commute. For $\hat{f} = \hat{q}_\alpha$ and $\hat{q} = \hat{p}_\rho$, we get

$$[\hat{q}_\alpha, \hat{p}_\rho] = i\hbar\delta_{\alpha\rho}\,. \tag{2.3.4}$$

  This should look familiar! Also note that $[\hat{q}_\alpha, \hat{q}_\rho] = [\hat{p}_\alpha, \hat{q}_\rho] = 0$. One could directly start from these commutation relations as well, without going through Poisson Brackets.

- The time evolution of $\hat{f}(\hat{q}_\alpha, \hat{p}_\alpha)$ (no explicit time dependence) is given by

$$\frac{d\hat{f}}{dt} = -\frac{i}{\hbar}\left[\hat{f}, \hat{H}\right]\,. \tag{2.3.5}$$

  For co-ordinates and momenta,

$$\frac{d\hat{q}_\alpha}{dt} = -\frac{i}{\hbar}\left[\hat{q}_\alpha, \hat{H}\right]\,, \qquad \text{and} \qquad \frac{d\hat{p}_\alpha}{dt} = -\frac{i}{\hbar}\left[\hat{p}_\alpha, \hat{H}\right]\,. \tag{2.3.6}$$

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[2]Dirac, *Principles of Quantum Mechanics*, Oxford University Press (1982)

Comments:

- This above quantization procedure procedure is not a guarantee of finding the correct quantum theory. Higher order terms in $\hbar$ might be relevant. The above procedure is motivated by recovering classical physics in the limit "$\hbar \to 0$". There also exists an ambiguity in the above procedure regarding the orderings of operators; which one is correct? $q^3 p^2 \to \hat{q}^3 \hat{p}^2$ or $q^3 p^2 \to \hat{p}^2 \hat{q}^3$? Ultimately, you have to check with nature whether you have the correct quantum Hamiltonian.

- **Heisenberg Picture** : Note that we are working in the "Heisenberg Picture" where the operators $\hat{f}(\hat{q}_\alpha, \hat{p}_\alpha)$ evolve with time according to[3]

$$\frac{d\hat{f}}{dt} = -\frac{i}{\hbar}\left[\hat{f}, \hat{H}\right], \qquad \text{or equivalently} \qquad \hat{f}(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{f}(t_0)e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}. \qquad (2.3.7)$$

  The states $|\psi\rangle$ of the system is time-independent. To find the expectation value of an observable corresponding to the operator $\hat{f}(t)$ in a given state $|\psi\rangle$, we have to calculate $\langle\psi|\hat{f}(t)|\psi\rangle$. Notice that it is a combination of operators sandwiched between states that appears in the expectation values.

- **Schrödinger Picture**: A mathematically equivalent way of thinking about time evolution of quantum systems is to think of states $|\psi(t)\rangle_{\rm s}$ evolving with time, and the operators $\hat{f}_{\rm s}$ being time-independent. States evolve according to the Schrödinger equation[4]:

$$\frac{d}{dt}|\psi(t)\rangle_{\rm s} = -\frac{i}{\hbar}\hat{H}|\psi(t)\rangle_{\rm s}, \qquad \text{or equivalently} \qquad |\psi(t)\rangle_{\rm s} = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\psi(t_0)\rangle_{\rm s}. \qquad (2.3.8)$$

  As you can check, by setting $|\psi(t_0)\rangle_{\rm s} = |\psi\rangle$ and $\hat{f}(t_0) = \hat{f}_{\rm s}$, the expectation values constructed in either picture will yield the same answer $_{\rm s}\langle\psi(t)|\hat{f}_{\rm s}|\psi(t)\rangle_{\rm s} = \langle\psi|\hat{f}(t)|\psi\rangle$. The same argument works for arbitrary matrix elements: $f_{ab} = {}_{\rm s}\langle a(t)|\hat{f}_{\rm s}|b(t)\rangle_{\rm s} = \langle a|\hat{f}(t)|b\rangle$ (think about transition probabilities), thus observables will be equal when calculated in either picture.

Exercise 2.3.1 : Verify that $\hat{f}(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{f}(t_0)e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ is a solution of $d\hat{f}/dt = -(i/\hbar)[\hat{f}, \hat{H}]$. Be careful about the fact that $\hat{H}$ is an operator, not a number. You should interpret $e^{\hat{A}} = \sum_{n=0}^{\infty}(1/n!)\hat{A}^n$.

2.3.2 Worked Example: Harmonic Oscillators

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction – Sidney Coleman

Let us revert once more to coupled harmonic oscillators with unit mass; see equations (2.2.3) and (2.2.5). For time-independent couplings $M_{\alpha\rho}$ there exists $C_{\alpha\rho}$ such that if $\tilde{q}_\alpha = \sum_\rho C_{\alpha\rho}q_\rho$, then

$$H = \sum_{\alpha=1}^{N}\left(\frac{1}{2}\tilde{p}_\alpha^2 + \frac{1}{2}\omega_\alpha^2\,\tilde{q}_\alpha^2\right). \qquad (2.3.9)$$

The "tilde" co-ordinates are the normal-modes of the system. For example, for a collection of 2-masses and three springs (see Fig. 2.2), the two normal modes would be the modes where the masses oscillate together (in phase) or with an opposite phase. Normal modes are exceptionally convenient, because they

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[3]We assume that $\hat{H}$ has no explicit time-dependence for simplicity.

[4]The familiar wave-function in position space corresponding to the state $|a(t)\rangle_{\rm s}$ is obtained via $\Psi_a(t, \mathbf{x}) = {}_{\rm s}\langle\mathbf{x}|a(t)\rangle_{\rm s}$.
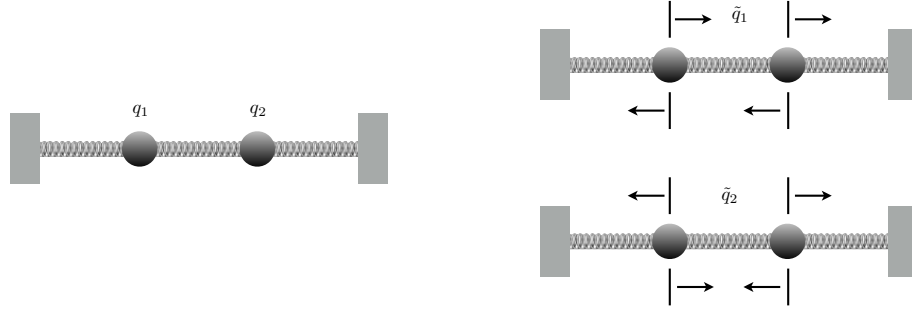
Figure 2.2: Normal modes for two masses connected by springs.

evolve independently from each other! For our system (dropping the "tilde" now), we have Hamilton's equations

$$\dot{q}_\alpha = p_\alpha, \qquad \text{and} \qquad \dot{p}_\alpha = -\omega_\alpha^2 q_\alpha. \tag{2.3.10}$$

Equivalently, the Euler-Lagrange equations are:

$$\ddot{q}_\alpha + \omega_\alpha^2 q_\alpha = 0, \tag{2.3.11}$$

with the solutions $q_\alpha \propto e^{\pm i\omega_\alpha t}$.

Canonical Quantization

Let us turn the crank of quantizing our theory. For the system at hand $q_\alpha, p_\alpha \longrightarrow \hat{q}_\alpha, \hat{p}_\alpha$ with $[\hat{q}_\alpha, \hat{p}_\rho] = i\hbar\delta_{\alpha\rho}$. The Hamiltonian and equations of motion are

$$\hat{H} = \sum_{\alpha=1}^{N} \left(\frac{1}{2}\hat{p}_\alpha^2 + \frac{1}{2}\omega_\alpha^2\,\hat{q}_\alpha^2 \right), \qquad \frac{d\hat{q}_\alpha}{dt} = \hat{p}_\alpha, \qquad \text{and} \qquad \frac{d\hat{p}_\alpha}{dt} = -\omega_\alpha^2\hat{q}_\alpha. \tag{2.3.12}$$

Exercise 2.3.2 : For the above Hamiltonian, evaluate the commutator $[\hat{p}_\alpha, \hat{H}]$ using the commutators for $\hat{q}_\alpha$ and $\hat{p}_\rho$. Then use the time evolution equation $(d/dt)\hat{p}_\alpha = -(i/\hbar)[\hat{p}_\alpha, \hat{H}]$ to recover $(d/dt)\hat{p}_\alpha = -\omega_\alpha^2\hat{q}_\alpha$. It is often useful to remember the following identity for commutators: $[\hat{a}\hat{b}, \hat{c}] = \hat{a}[\hat{b}, \hat{c}] + [\hat{a}, \hat{c}]\hat{b}$.

Next, we introduce the formalism of creation and annihilation operators, which will turn out to be quite useful when we deal with fields in the next chapter.

Creation and Annihilation Operators

It is convenient to define the "creation" and "annihilation" operators

$$\hat{a}_\alpha(t) = \sqrt{\frac{\omega_\alpha}{2\hbar}} \left(\hat{q}_\alpha(t) + i\frac{\hat{p}_\alpha(t)}{\omega_\alpha} \right) \qquad \text{and} \qquad \hat{a}_\alpha^\dagger(t) = \sqrt{\frac{\omega_\alpha}{2\hbar}} \left(\hat{q}_\alpha(t) - i\frac{\hat{p}_\alpha(t)}{\omega_\alpha} \right). \tag{2.3.13}$$

Recall that $\hat{q}_\alpha$ and $\hat{p}_\alpha$ are Hermitian because they correspond to observables (i.e. they must have real eigenvalues). These can be inverted to yield

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_\alpha}} \left(\hat{a}_\alpha(t) + \hat{a}_\alpha^\dagger(t) \right) \qquad \text{and} \qquad \hat{p}_\alpha(t) = -i\sqrt{\frac{\hbar\omega_\alpha}{2}} \left(\hat{a}_\alpha(t) - \hat{a}_\alpha^\dagger(t) \right). \tag{2.3.14}$$

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The time dependence of $\hat{a}_\alpha(t)$ and $\hat{a}_\alpha^\dagger(t)$ can be obtained by using our knowledge of $d\hat{q}_\alpha/dt$ and $d\hat{p}_\alpha/dt$, which yields $(d/dt)\hat{a}_\alpha(t) = -i\omega_\alpha\hat{a}_\alpha(t)$ and $(d/dt)\hat{a}_\alpha^\dagger(t) = i\omega_\alpha\hat{a}_\alpha^\dagger(t)$. The solutions are

$$\hat{a}_\alpha(t) = \hat{a}_\alpha(0)e^{-i\omega_\alpha t} \qquad \text{and} \qquad \hat{a}_\alpha^\dagger(t) = \hat{a}_\alpha^\dagger(0)e^{+i\omega_\alpha t}\,. \tag{2.3.15}$$

Exercise 2.3.3 : Derive $(d/dt)\hat{a}_\alpha(t) = -i\omega_\alpha\hat{a}_\alpha(t)$ and $(d/dt)\hat{a}_\alpha^\dagger(t) = i\omega_\alpha\hat{a}_\alpha^\dagger(t)$.

Using eq. (2.3.14), we have the "mode-expansion" of $\hat{q}_\alpha$:

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_\alpha}}\left(\hat{a}_\alpha(0)e^{-i\omega_\alpha t} + \hat{a}_\alpha^\dagger(0)e^{i\omega_\alpha t}\right)\,. \tag{2.3.16}$$

To reduce clutter, We drop the (0) part in the time-independent creation and annihilation operators, and simply write

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_\alpha}}\left(\hat{a}_\alpha e^{-i\omega_\alpha t} + \hat{a}_\alpha^\dagger e^{i\omega_\alpha t}\right)\,. \tag{2.3.17}$$

From now on, when we refer to $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$, we will always mean the time-independent ones. The corresponding expression for p_α is

$$\hat{p}_\alpha(t) = -i\sqrt{\frac{\hbar\omega_\alpha}{2}}\left(\hat{a}_\alpha e^{-i\omega_\alpha t} - \hat{a}_\alpha^\dagger e^{i\omega_\alpha t}\right)\,. \tag{2.3.18}$$

Why are $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ useful? It is worth reminding ourselves of the important properties of $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$:

$$\begin{aligned}
&\left[\hat{a}_\alpha, \hat{a}_\rho^\dagger\right] = \delta_{\alpha\rho}\,,\\
&[\hat{a}_\alpha, \hat{a}_\rho] = \left[\hat{a}_\alpha^\dagger, \hat{a}_\rho^\dagger\right] = 0\,,\\
&\hat{H} = \sum_\alpha \left(\hat{a}_\alpha^\dagger\hat{a}_\alpha + \frac{1}{2}\right)\hbar\omega_\alpha\,.
\end{aligned} \tag{2.3.19}$$

Exercise 2.3.4 : Derive the above expressions using the commutators for the co-ordinates and momenta, the Hamiltonian in eq. (2.3.12) as well as the mode expansions above.

Why the name "creation" and "annihilation" operators? Note another property (that you should verify):

$$\left[\hat{H}, \hat{a}_\alpha\right] = -\hbar\omega_\alpha\hat{a}_\alpha\,, \qquad \text{and} \qquad \left[\hat{H}, \hat{a}_\alpha^\dagger\right] = \hbar\omega_\alpha\hat{a}_\alpha^\dagger\,. \tag{2.3.20}$$

The last line tells us about creation and annihilation of quanta. To see this, note that if $|\psi\rangle$ is an eigenstate of the $\hat{H}$ with energy E, then $\hat{a}_\alpha^\dagger|\psi\rangle$ has an energy $E + \hbar\omega_\alpha$:

$$\hat{H}\hat{a}_\alpha^\dagger|\psi\rangle = (E + \hbar\omega_\alpha)\hat{a}_\alpha^\dagger|\psi\rangle\,. \tag{2.3.21}$$

That is, $\hat{a}_\alpha^\dagger$ raises the energy by $\hbar\omega_\alpha$; it "creates" a quantum $\hbar\omega_\alpha$. Similarly,

$$\hat{H}\hat{a}_\alpha|\psi\rangle = (E - \hbar\omega_\alpha)\hat{a}_\alpha|\psi\rangle\,. \tag{2.3.22}$$

That is, $\hat{a}_\alpha$ lowers the energy by $\hbar\omega_\alpha$; it "annihilates" a quantum $\hbar\omega_\alpha$. The *vacuum state* is defined as the state that is annihilated by any $\hat{a}_\alpha$:

$$|0\rangle \equiv |0, 0 \ldots 0\rangle, \qquad \text{where} \qquad \hat{a}_\alpha|0\rangle = 0\,. \tag{2.3.23}$$

Each "slot" in $|0, 0 \ldots 0\rangle$ corresponds to different $\alpha = 1, \ldots N$. We can build normalized eigenstates of energy by repeatedly acting on the vacuum state using our creation operators

$$|n_1, n_2 \ldots n_N\rangle = \prod_{\alpha=1}^{N} \frac{(\hat{a}_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha!}} |0\rangle \,, \tag{2.3.24}$$

where n_α are the number of quanta (occupation number) with energy $\hbar\omega_\alpha$ is this state. Note that $\hat{a}_\alpha^\dagger \hat{a}_\alpha |n_1, n_2 \ldots n_N\rangle = n_\alpha |n_1, n_2 \ldots n_N\rangle$. The total energy of this state

$$E_{\text{tot}} = \sum_{\alpha=1}^{N} E_\alpha = \sum_{\alpha=1}^{N} \left(n_\alpha + \frac{1}{2} \right) \hbar\omega_\alpha \,. \tag{2.3.25}$$

Even if there are no quanta $n_\alpha = 0$ for all α, we still have vacuum energy $E_{\text{vac}} = \sum_{\alpha=1}^{N} (1/2) \hbar\omega_\alpha$. This is important when absolute values of energy matter (e.g. when gravity is involved ("Cosmological Constant" problem)) or when dealing with non-trivial boundary conditions ("Casimir effect"). In this course we can ignore this vacuum contribution; for us only differences in energy matter.

Exercise 2.3.5 : Evaluate $\langle 0|(\hat{a}_\alpha + \hat{a}_\alpha^\dagger)^4|0\rangle$ using the commutation relations for $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ as well as their action on the vacuum.

2.4 Special Relativity and Single-Particle Quantum Mechanics

For simplicity of expressions, let $\hbar = c = 1$. Consider a relativistic particle with a Hamiltonian such that $\hat{H}|\mathbf{k}\rangle = \sqrt{\mathbf{k}^2 + m^2}|\mathbf{k}\rangle$ for a particle with mass m in its momentum eigenstate. Let us calculate the "amplitude" $\mathcal{A}$ (where the probability $\propto |\mathcal{A}|^2$) of a localized particle moving from $(t_0 = 0, \mathbf{x}_0)$ to $(t, \mathbf{x})$. That is, we take a localized state $|\mathbf{x}_0\rangle$, time evolve it: $e^{-i\hat{H}(t-t_0)}|\mathbf{x}_0\rangle$, and ask whether this state has an overlap with $\langle \mathbf{x}|$. To be consistent with special relativity, we must get $\mathcal{A} = 0$ if $(t_0, \mathbf{x}_0)$ and $(t, \mathbf{x})$ are space-like separated. Upon an explicit calculation[5]

$$\mathcal{A} = \langle \mathbf{x}|e^{-i\hat{H}(t-t_0)}|\mathbf{x}_0\rangle \sim e^{-m|\mathbf{x}-\mathbf{x}_0|} \neq 0 \tag{2.4.1}$$

where we have assumed that $|\mathbf{x} - \mathbf{x}_0|^2 \gg (t - t_0)^2$. This is small, but non-zero. Thus single-particle Quantum Mechanics is inconsistent with Special Relativity. As we will soon see next, a radical new approach is needed where particle number is not conserved. In anticipation of what is to follow in the next chapter, and to avoid clutter in this subsection, I have set $\hbar = 1$ and will continue to do so from now onwards.

Comment: QM can be consistent with SR. We have only shown that single-particle QM is inconsistent with SR. Note that in QFT, the Schrödinger equation: $i(d/dt)|\psi\rangle_{\text{s}} = \hat{H}|\psi\rangle_{\text{s}}$ is still valid. You have to make sure you use the correct Hamiltonian made out of operator valued fields, whose evolution is consistent with SR, and also use appropriate states. Using the Schrödinger equation proves to be rather unwieldy when dealing with fields, so the formalism (while mathematically equivalent), will look quite different from the Schrödinger equation.

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[5]see for example, section 8.3 in Lancaster and Blundell, or section 2.1 in Peskin and Schroeder.

QUANTIZING SCALAR FIELDS

We are now ready to talk about fields. In this chapter we will limit ourselves to *scalar* fields. We start with classical, relativistic scalar fields and then quantize them. Most of this chapter is devoted to *free* scalar fields. Interactions will be taken up in the next chapter. We get a taste of what we mean by particles as well as how causality is built into our field theory.

3.1   Classical Scalar Fields

We now turn to fields, specifically real, relativistic scalar fields $\varphi(t, \mathbf{x}) = \varphi(x)$.[1] The Lagrangian density

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) \,, \tag{3.1.1}$$

where $V(\varphi) = \frac{1}{2} m^2 \varphi^2 + (1/3!) \lambda_3 \varphi^3 \pm (1/4!) \lambda \varphi^4 \dots$ is the potential, which contains the "mass" m, and "self-interaction" terms with strengths λ_3, λ etc. Note that $(1/2) \partial^\mu \varphi \partial_\mu \varphi = (1/2) \dot{\varphi}^2 - (1/2)(\nabla \varphi)^2$ which includes the "kinetic" and "gradient" terms. Why does the Lagrangian density have this particular form? Wait till the second half of this semester to get answers to this very relevant question. Hint: Symmetries, and Lorentz invariance will play a role. Of course, as always, these only provide guidance, with experiments having the final say. For the moment take this Lagrangian density as it is.

The Lagrangian and Lagrangian density are related by

$$L = \int d^3 x \mathcal{L} \,. \tag{3.1.2}$$

We will get sloppy and call $\mathcal{L}$ the Lagrangian as well. By extremizing the action $S = \int d^4 x \mathcal{L}$, we arrive at the Euler-Lagrange equations[2], that is, the equations of motion for the field φ:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi} \implies \partial_\mu \partial^\mu \varphi + \partial_\varphi V(\varphi) = 0 \,. \tag{3.1.3}$$

It is initially useful to think about $\mathbf{x}$ as a label. That is $\varphi(t, \mathbf{x}) = \varphi_{\mathbf{x}}(t)$, we can now think of $q_\alpha(t) \leftrightarrow \varphi_{\mathbf{x}}(t)$. We can define a conjugate momentum *density*, and a Hamiltonian density

$$
\begin{aligned}
\pi &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} && \text{conjugate momentum density} \\
\mathcal{H} &= \pi \dot{\varphi} - \mathcal{L} && \text{Hamiltonian density}
\end{aligned}
\tag{3.1.4}
$$

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[1] An example of such a field would be the (as yet to be detected) axion field, which might constitute dark matter.
[2] See Appendix A.3

More explicitly

$$\pi = \dot{\varphi}, \qquad \text{and} \qquad \mathcal{H} = \frac{\pi^2}{2} + \frac{(\nabla \varphi)^2}{2} + V(\varphi). \tag{3.1.5}$$

The Hamiltonian density is the energy density, and each term has a meaning:

$$\frac{\pi^2}{2} = \text{kinetic energy density; time variation costs energy},$$
$$\frac{(\nabla \varphi)^2}{2} = \text{gradient energy density; spatial variation costs energy}, \tag{3.1.6}$$
$$V(\varphi) = \text{potential energy density}.$$

The Hamiltonian H and the Hamiltonian density $\mathcal{H}$ are related by $H = \int d^3x \mathcal{H}$ which is also consistent with $H = \int d^3x \pi \dot{\varphi} - L$ as it should be. Think about $\int d^3x$ as the sum over all degrees of freedom. Again, it gets cumbersome to say momentum density and Hamiltonian density, so we will get sloppy and call them the conjugate momentum and density instead. Hamilton's equations:

$$\dot{\varphi} = \frac{\delta H}{\delta \pi} = \pi \qquad \text{and} \qquad \dot{\pi} = -\frac{\delta H}{\delta \varphi} = \nabla^2 \varphi - V'(\varphi). \tag{3.1.7}$$

where $\delta H/\delta \pi$ and $\delta H/\delta \varphi$ are *functional* derivatives[3]. Again, see Appendix A.3.

Exercise 3.1.1 : Show that $\delta H/\delta \pi = \pi$ and $-\delta H/\delta \varphi = \nabla^2 \varphi - V'(\varphi)$. You might want to consult Appendix A.3 for the definition of functional derivatives. Think of H as a functional of π and φ. This exercise should also convince you that the definition of conjugate momentum density $\pi = \partial \mathcal{L}/\partial \dot{\varphi} = \delta L/\delta \dot{\varphi}$ where you can treat L as a functional of $\dot{\varphi}$ and φ.

3.1.1 Poisson Brackets

The Poisson Bracket of two functionals (with all quantities evaluated at the same time): $F = \int d^3x \mathcal{F}(\varphi, \pi)$, $G = \int d^3x \mathcal{G}(\varphi, \pi)$, is defined as

$$\{F, G\} \equiv \int d^3x \left(\frac{\delta F}{\delta \varphi} \frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi} \frac{\delta G}{\delta \varphi} \right), \tag{3.1.8}$$

whereas the time evolution (using Hamilton's equations of motion) is given by

$$\frac{dF}{dt} = \{F, H\}. \tag{3.1.9}$$

Consider the special case where $F = \varphi(t, \mathbf{y})$, and $G = \pi(t, \mathbf{z})$. In this case, the Poisson bracket and time evolution are given by

$$\{\varphi(t, \mathbf{y}), \pi(t, \mathbf{z})\} = \delta^{(3)}(\mathbf{y} - \mathbf{z}),$$
$$\dot{\varphi} = \{\varphi, H\} \qquad \text{and} \qquad \dot{\pi} = \{\pi, H\}. \tag{3.1.10}$$

Exercise 3.1.2 : Show that $dF/dt = \{F, H\}$ using Hamilton's equations. Then show that $\delta \varphi(t, \mathbf{y})/\delta \varphi(t, \mathbf{x}) = \delta^{(3)}(\mathbf{y} - \mathbf{x})$ (similarly for π), which together yields the desired equations (3.1.10).

---

[3]A functional can be a function of an entire function, not just its value at a given point. A function is special case of functionals.

3.2 Canonical Quantization of Scalar Fields

We follow our nose and put hats on our field and its conjugate momenta, and specify the fundamental commutation relation between them

$$\varphi(t,\mathbf{x}), \pi(t,\mathbf{x}) \longrightarrow \hat{\varphi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x}),$$
$$[\hat{\varphi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}).$$

(3.2.1)

We have set $\hbar$ to unity, and have followed $\{\cdot, \cdot\} \rightarrow -i[\cdot, \cdot]$. Note that this is an *equal time* commutation relation. The commutator at different times can be obtained from the equations of motion. The time evolution of these operators follows from Hamilton's equations

$$\frac{d\hat{\varphi}}{dt} = -i\left[\hat{\varphi}, \hat{H}\right] = \hat{\pi}, \qquad \text{and} \qquad \frac{d\hat{\pi}}{dt} = -i\left[\hat{\pi}, \hat{H}\right] = \nabla^2\hat{\varphi} - V'(\hat{\varphi}).$$

(3.2.2)

Combining the above two equations, we have

$$\frac{d^2\hat{\varphi}}{dt^2} - \nabla^2\hat{\varphi} + V'(\hat{\varphi}) = 0.$$

(3.2.3)

This is the equation of motion for our operator valued scalar field.

3.2.1 Free Quantum Scalar Field

Life is easy when we are free –

Let us consider the special case where $V(\varphi) = (1/2)m^2\varphi^2$. The equation of motion

$$\frac{d^2\hat{\varphi}}{dt^2} - \nabla^2\hat{\varphi} + m^2\hat{\varphi} = 0.$$

(3.2.4)

For this special case, the equation of motion is *linear* in the field. This invites us to move to Fourier Space. Let us imagine that our field is restricted to a box of volume $V = L^3$ and satisifies periodic boundary conditions. Then the Fourier transform of the field (and its inverse) are given by

$$\hat{\varphi}(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\varphi}_{\mathbf{k}}, \qquad \hat{\varphi}_{\mathbf{k}} = \frac{1}{\sqrt{V}}\int_V d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{\varphi}(\mathbf{x}), \qquad \text{with} \qquad \mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \quad (3.2.5)$$

where n_i are integers, and similar expressions hold for $\hat{\pi}$ and $\hat{\pi}_{\mathbf{k}}$. Plugging the Fourier transform into the equation of motion we get

$$\frac{d^2\hat{\varphi}_{\mathbf{k}}}{dt^2} + \omega_{\mathbf{k}}^2\hat{\varphi}_{\mathbf{k}} = 0, \qquad \text{with} \qquad \omega_{\mathbf{k}}^2 = |\mathbf{k}|^2 + m^2.$$

(3.2.6)

where we note that $\omega_{\mathbf{k}}$ is a function of k = the magnitude of $\mathbf{k}$. The Fourier modes $\hat{\varphi}_{\mathbf{k}}$ are decoupled from each other and each one satisfies the same equation as a harmonic oscillator! This would not have been possible if the $V(\varphi)$ had terms beyond the mass term. What about the commutation relations? We inherit it from the commutation relation in position space

$$[\hat{\varphi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \implies [\hat{\varphi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{q}}(t)] = i\delta_{\mathbf{k},-\mathbf{q}}.$$

(3.2.7)

Note that the delta function in Fourier space is a Kronekor delta, a consequence putting the field in a box with periodic boundary conditions, which resulted in discrete $\mathbf{k}$. Also notice the minus sign in $\delta_{\mathbf{k},-\mathbf{q}}$.

The Hamiltonian can be written as

$$\hat{H} = \int_V d^3x \left(\frac{\hat{\pi}^2}{2} + \frac{(\nabla\hat{\varphi})^2}{2} + \frac{m^2}{2}\hat{\varphi}^2\right) = \sum_{\mathbf{k}}\left(\frac{1}{2}\hat{\pi}_{\mathbf{k}}\hat{\pi}_{-\mathbf{k}} + \frac{\omega_{\mathbf{k}}^2}{2}\hat{\varphi}_{\mathbf{k}}\hat{\varphi}_{-\mathbf{k}}\right),$$

(3.2.8)

where we have used the fact that $\hat{\varphi}(x)$ is Hermitian, which implies $\hat{\varphi}_{\mathbf{k}}^{\dagger} = \hat{\varphi}_{-\mathbf{k}}$. The time evolution of $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$

$$\frac{d\hat{\varphi}_{\mathbf{k}}}{dt} = \hat{\pi}_{\mathbf{k}}\,, \qquad \frac{d\hat{\pi}_{\mathbf{k}}}{dt} = -\omega_{\mathrm{k}}^2 \hat{\varphi}_{\mathbf{k}}\,. \tag{3.2.9}$$

Exercise 3.2.1 : Derive the form of the Hamiltonian in Fourier space (second equality in eq. (3.2.8)), as well as the time evolution equations (3.2.9).

You should compare the Hamiltonian and the time evolution equations to our corresponding equations for harmonic oscillators (normal modes). In Fourier space, the free scalar field is just a collection of harmonic oscillators.

In complete analogy with the harmonic oscillators, it is convenient to introduce creation and annihilation operators, and write the mode expansion of the field in terms of the (time-independent) creation and annihilation operators

$$\hat{\varphi}_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-i\omega_{\mathrm{k}}t} + \hat{a}_{-\mathbf{k}}^{\dagger} e^{i\omega_{\mathrm{k}}t} \right)\,. \tag{3.2.10}$$

Notice the minus sign in $\hat{a}_{-\mathbf{k}}^{\dagger}$.

Exercise 3.2.2 : Following the derivation in the harmonic oscillator example from earlier in the notes, define the time-dependent creation and annihilation operators in terms of $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$. Be careful about $\hat{\varphi}_{\mathbf{k}}^{\dagger} = \hat{\varphi}_{-\mathbf{k}}$. Invert the relations to get $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ in terms of $\hat{a}_{\mathbf{k}}(t)$ and $\hat{a}_{-\mathbf{k}}^{\dagger}(t)$. Find the time evolution equations for $\hat{a}_{\mathbf{k}}(t)$ and $\hat{a}_{-\mathbf{k}}^{\dagger}(t)$. Use these to finally arrive at our mode expansion above.

Let us move back to position space:

$$\hat{\varphi}(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-i\omega_{\mathrm{k}}t + i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^{\dagger} e^{i\omega_{\mathrm{k}}t - i\mathbf{k}\cdot\mathbf{x}} \right)\,. \tag{3.2.11}$$

Notice that I have flipped the sign of the subscript of $\hat{a}_{\mathbf{k}}^{\dagger}$ term, as well as that of $i\mathbf{k}\cdot\mathbf{x}$ in the exponent multiplying $\hat{a}_{\mathbf{k}}^{\dagger}$. We will do this repeatedly.

Exercise 3.2.3 : Quickly verify that $\sum_{\mathbf{k}} f(\mathrm{k}) g(\mathbf{k}) = \sum_{\mathbf{k}} f(\mathrm{k}) g(-\mathbf{k})$. This justifies our sign flip in the term containing $\hat{a}^{\dagger}$ above.

We now take advantage of our nice relativistic notation $ik\cdot x = i\omega_{\mathrm{k}}t - i\mathbf{k}\cdot\mathbf{x}$ and $-ik\cdot x = -i\omega_{\mathrm{k}}t + i\mathbf{k}\cdot\mathbf{x}$ to write the most useful form of our fields and conjugate momenta

$$\hat{\varphi}(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}_{\mathbf{k}}^{\dagger} e^{ik\cdot x} \right)\,,$$

$$\hat{\pi}(x) = -\frac{i}{\sqrt{V}} \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathrm{k}}}{2}} \left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} - \hat{a}_{\mathbf{k}}^{\dagger} e^{ik\cdot x} \right)\,. \tag{3.2.12}$$

Let us recall the important properties of $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^{\dagger}$. You should verify as many of these as possible.

$$[\hat{\varphi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{q}}(t)] = i\delta_{\mathbf{k},-\mathbf{q}}$$
$$\implies \left[\hat{a}_{\mathbf{k}}(t), \hat{a}_{\mathbf{q}}^{\dagger}(t)\right] = \delta_{\mathbf{k},\mathbf{q}}\,. \tag{3.2.13}$$

Note that there is no pesky minus sign or i in the $\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger$ commutation relation. The Hamiltonian can be written as

$$\hat{H} = \sum_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2} \right) \omega_{\mathbf{k}}. \tag{3.2.14}$$

Note that if $|\psi\rangle$ is an eigenstate of the $\hat{H}$ with energy E, then $\hat{a}_{\mathbf{k}}^\dagger |\psi\rangle$ has an energy $E + \omega_{\mathbf{k}}$:

$$\hat{H}\hat{a}_{\mathbf{k}}^\dagger |\psi\rangle = (E + \omega_{\mathbf{k}})\hat{a}_{\mathbf{k}}^\dagger |\psi\rangle. \tag{3.2.15}$$

That is, $\hat{a}_{\mathbf{k}}^\dagger$ raises the energy by $\omega_{\mathbf{k}}$; it creates a "particle" of energy $\omega_{\mathbf{k}}$. Similarly, $\hat{a}_{\mathbf{k}}$ lowers the energy by $\omega_{\mathbf{k}}$; it annihilates a particle of energy $\omega_{\mathbf{k}}$. These "particles" are completely delocalized (have infinite spatial extent), since they have a fixed momentum. A localized particle can be created by creating a wave-packet from the superposition of such definite momentum "particles".

A general, normalized eigenstate of the Hamiltonian is

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2} \ldots \rangle = \prod_{\mathrm{s}} \frac{\left(\hat{a}_{\mathbf{k}_{\mathrm{s}}}^\dagger \right)^{n_{\mathbf{k}_{\mathrm{s}}}}}{\sqrt{n_{\mathbf{k}_{\mathrm{s}}}!}} |0\rangle, \tag{3.2.16}$$

with energy

$$E_{\mathrm{tot}} = \sum_{\mathrm{s}} E_{n_{\mathbf{k}_{\mathrm{s}}}} = \sum_{\mathbf{k}} \left(n_{\mathbf{k}} + \frac{1}{2} \right) \omega_{\mathbf{k}}. \tag{3.2.17}$$

Note that $n_{\mathbf{k}}$ are the number of particles with momentum $\mathbf{k}$. Notice that even when $n_{\mathbf{k}} = 0$ for all $\mathbf{k}$, we have an infinite vacuum energy: $E_{\mathrm{vac}} = \sum_{\mathbf{k}}(1/2)\omega_{\mathbf{k}}$.

A single particle state with momentum $\mathbf{k}_1$ is described by $n_{\mathbf{k}_1} = 1$, $n_{\mathbf{k}_{\mathrm{s}}} = 0$ for $s \neq 1$:

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2} \ldots \rangle = |1, 0 \ldots \rangle \equiv |\mathbf{k}_1\rangle = \hat{a}_{\mathbf{k}_1}^\dagger |0\rangle, \tag{3.2.18}$$

An example of a two-particle state is described by $n_{\mathbf{k}_1} = 1$, $n_{\mathbf{k}_2} = 1$ and $n_{\mathbf{k}_{\mathrm{s}}} = 0$ for $s \neq 1, 2$:

$$\begin{aligned} |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3} \ldots \rangle = |1, 1, 0 \ldots \rangle &\equiv |\mathbf{k}_1, \mathbf{k}_2\rangle \\ &= \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2}^\dagger |0\rangle, \\ &= \hat{a}_{\mathbf{k}_2}^\dagger \hat{a}_{\mathbf{k}_1}^\dagger |0\rangle, \\ &= |\mathbf{k}_2, \mathbf{k}_1\rangle. \end{aligned} \tag{3.2.19}$$

Bose Statistics emerge naturally! Above, we used the fact that $\hat{a}_{\mathbf{k}_1}^\dagger$ and $\hat{a}_{\mathbf{k}_2}^\dagger$ commute with each other.

Exercise 3.2.4 : Write down a normalized eigenstate of the Hamiltonian with two $\mathbf{k}_1$ momentum particles, one $\mathbf{k}_2$ momentum particles and five $\mathbf{k}_3$ momentum particles (in terms of the creation operators acting on the vacuum).

Exercise 3.2.5 : Let us put our free field in contact with a thermal bath at a temperature $T = \beta^{-1}$. The density matrix $\hat{\rho} = e^{-\beta\hat{H}}/\mathcal{Z}$ where $\mathcal{Z} = \mathrm{Tr}[e^{-\beta\hat{H}}]$ is the partition function. Take the trace over *Fock states* of the form $|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3} \ldots \rangle$, where each $n_{\mathbf{k}_{\mathrm{s}}}$ can have any value from 0 to ∞. The thermal expectation value of an operator is then $\langle \mathcal{O} \rangle_{\mathrm{th}} = \mathrm{Tr}[\hat{\rho}\mathcal{O}]$. Find an expression for the expectation value of the number density of quanta (number of quanta per fixed volume) with a fixed momentum $\mathbf{k}$.

So far, we have restricted the field to a box of volume V. Let us now allow $V \to \infty$. The field in position and momentum space are related by

$$\hat{\varphi}(t, \mathbf{x}) = \int d^3\bar{k}\, \hat{\varphi}(t, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\hat{\varphi}(t, \mathbf{k}) = \int d^3x\, \hat{\varphi}(t, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{3.2.20}$$

where $d^3\bar{k} = d^3k/(2\pi)^3$. Notice, that I am writing $\hat{\varphi}(t, \mathbf{k})$ instead of $\hat{\varphi}_{\mathbf{k}}(t)$ just for the sake of distinguishing it from the finite box case. Similar expressions hold for the conjugate momentum as well. We can get from the finite to the infinite box expressions through $\sum_{\mathbf{k}} \to V \int d^3\bar{k}$ and $\hat{\varphi}_{\mathbf{k}}(t) \to V^{-1/2}\hat{\varphi}(t, \mathbf{k})$.

The fundamental commutation relations become

$$[\hat{\varphi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \implies [\hat{\varphi}(t, \mathbf{k}), \hat{\pi}(t, \mathbf{q})] = i\bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}). \tag{3.2.21}$$

where $\bar{\delta}^{(3)}(\mathbf{k}) = (2\pi)^3\delta^{(3)}(\mathbf{k})$ is the (scaled) Dirac delta function. There are two ways of deriving the Fourier space commutation relation. First, one can simply use our finite V to infinite V conversion, and recognize

$$\lim_{V\to\infty} V\delta_{\mathbf{k},-\mathbf{q}} = \bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}). \tag{3.2.22}$$

Another way is to substitute the inverse Fourier transform:

$$[\hat{\varphi}(t, \mathbf{k}), \hat{\pi}(t, \mathbf{q})] = \int d^3x d^3y\, e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [\hat{\varphi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})],$$
$$= i \int d^3x d^3y\, e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \delta^{(3)}(\mathbf{x} - \mathbf{y}),$$
$$= i \int d^3x\, e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}, \tag{3.2.23}$$
$$= i\bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}).$$

where in the last line we used following representation of the Dirac delta function: $\bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}) = \int d^3x e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}$.

We can now continue with our now familiar program of defining creation annihilation operators and such. However, before we do so, let us digress a bit to note that $d^3\bar{k}$ is not a Lorentz invariant measure. A Lorentz invariant measure would be

$$(dk) \equiv \frac{d^3\bar{k}}{(2\omega_{\mathbf{k}})}. \tag{3.2.24}$$

Exercise 3.2.6 : Here, I guide you through the proof that shows that (dk) is indeed Lorentz invariant.

First show that $\int (dk)F(k^\mu) = \int d^4\bar{k}\delta(k^2 - m^2)\Theta(k^0)F(k^\mu)$ for arbitrary F. Here, $k^2 = k^\mu k_\mu$, $\Theta(x) = 0$ for $x < 0$ and 1 for $x > 0$. Now realize that $\int d^4\bar{k}\delta(k^2 - m^2)\Theta(k^0)$ is Lorentz invariant because $d^4\bar{k}\delta(k^2 - m^2)$ is manifestly Lorentz invariant, and so is $\Theta(k^0)$ since a Lorentz transformation cannot change the sign of k^0. That completes the proof.

It is nicer to have Lorentz invariant measures, which motivates us to make the following (scaled) definitions for the creation and annihilation operators in the continuum case[4]

$$\hat{a}_{\mathbf{k}} \to V^{-1/2}(2\omega_{\mathbf{k}})^{-1/2}\hat{a}(\mathbf{k}). \tag{3.2.25}$$

---

[4]Note that this scaling relation is different from the scaling relation between an arbitrary function $f_{\mathbf{k}}$ and $f(\mathbf{k})$ we have been using, because of our insistence on using a Lorentz invariant measure this time around.

With this scaling, the mode expansion of the field in terms of creation and annihilation operators becomes

$$\hat{\varphi}(x) = \int (dk) \left(\hat{a}(\mathbf{k}) e^{-ik \cdot x} + \hat{a}^{\dagger}(\mathbf{k}) e^{ik \cdot x} \right) , \tag{3.2.26}$$

and the commutation relation for the creation and annihilation operators becomes

$$\left[\hat{a}(\mathbf{k}), \hat{a}^{\dagger}(\mathbf{q}) \right] = 2\omega_{\mathrm{k}} \delta^{(3)}(\mathbf{k} - \mathbf{q}) . \tag{3.2.27}$$

The Hamiltonian in terms of the creation and annihilation operators

$$\hat{H} = \int (dk) \hat{a}^{\dagger}(\mathbf{k}) \hat{a}(\mathbf{k}) \omega_{\mathrm{k}} + V \int (dk) \omega_{\mathrm{k}}^2 . \tag{3.2.28}$$

The second term is the vacuum energy. Note that its infinity arises not just from the infinite volume up front. The energy density

$$\frac{E_{\mathrm{vac}}}{V} = \int (dk) \omega_{\mathrm{k}}^2 = \int \frac{d^3 k}{(2\pi)^3} \omega_{\mathrm{k}} . \tag{3.2.29}$$

is also formally infinite. In reality, we should expect that our theory is only valid up to some large momentum, which provides an upper bound for the integral. Nevertheless, such a large vacuum energy plays no role for us (no gravity in this course) if we promise to only care about differences above this energy.

We can build up eigenstates of the Hamiltonian with $\hat{a}^{\dagger}(\mathbf{k})$. For example, the single particle and two particle states

$$|\mathbf{k}\rangle = \hat{a}^{\dagger}(\mathbf{k})|0\rangle \qquad \text{and} \qquad |\mathbf{k}_1, \mathbf{k}_2\rangle = \hat{a}^{\dagger}(\mathbf{k}_1) \hat{a}^{\dagger}(\mathbf{k}_2)|0\rangle . \tag{3.2.30}$$

The vacuum is defined by $\hat{a}(\mathbf{k})|0\rangle = 0$, and we normalize it $\langle 0|0 \rangle = 1$. Note, that apart from vacuum our eigenstates are not normalized. In particular, the single particle state

$$\langle \mathbf{k}|\mathbf{k}\rangle = 2\omega_{\mathrm{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}) \langle 0|0\rangle \to \infty . \tag{3.2.31}$$

The Dirac-delta function up front comes from a volume factor. The infinity is a result of our completely delocalized state. This is an inconvenience that we can live with. One can always construct normalized states by taking a superposition of many such eigenstates.

We have defined multi-particle states of definite momenta. What about our usual "localized particle" ? Consider the action of $\hat{\varphi}(x)$ on the vacuum $|0\rangle$. To reduce clutter, let $x^0 = 0$. Using eq. (3.2.26), we have

$$\hat{\varphi}(\mathbf{x})|0\rangle = \int (dk) e^{-i\mathbf{k} \cdot \mathbf{x}} \hat{a}^{\dagger}(\mathbf{k})|0\rangle = \int (dk) e^{-i\mathbf{k} \cdot \mathbf{x}} |\mathbf{k}\rangle = \frac{1}{2m} \int_{|\mathbf{k}| \ll m} \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} |\mathbf{k}\rangle + \ldots \propto |\mathbf{x}\rangle + \ldots \tag{3.2.32}$$

where $|\mathbf{x}\rangle$ is what we would have referred to as describing a localized single particle at $\mathbf{x}$ from our old-school quantum mechanics. Hence we interpret $\hat{\varphi}(\mathbf{x})|0\rangle$ as creating (or destroying) a localized particle at the position $\mathbf{x}$.[5]

Exercise 3.2.7 : Here, I guide you through a faster route to getting the mode expansion in (3.2.26). The Klein-Gordon equation, $(\partial^2 + m^2)\hat{\varphi} = 0$, has plane wave solutions of the form $\hat{f}(k^{\mu}) e^{-ik \cdot x}$ as long as $k^2 = m^2$. Since the equation is linear, and plane waves form a complete basis, the general solution has the form

$$\hat{\varphi}(x) = \int d^4 k \, \delta(k^2 - m^2) \hat{f}(k^{\mu}) e^{ik \cdot x} . \tag{3.2.33}$$

---

[5]Admittedly this seems a bit unsatisfactory, since we cut-off the higher momentum bits. We have to be extra careful about what we even mean by a localized state (ie. the meaning of $|\mathbf{x}\rangle$! This heuristic picture is further complicated when we move to interacting fields.

Show using the properties of the delta function that the above expression becomes

$$\hat{\varphi}(x) = \int (dk) \left[\hat{f}(\omega_k, \mathbf{k}) e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{x})} + \hat{f}(-\omega_k, \mathbf{k}) e^{i(\omega_k t + \mathbf{k} \cdot \mathbf{x})} \right] . \tag{3.2.34}$$

with $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$. Now, flip $\mathbf{k} \to -\mathbf{k}$ in the second term (and do the same in the integration measure and limits). Using the fact that $\hat{\varphi}$ is Hermitian, and defining $\hat{f}(\omega_k, \mathbf{k}) \equiv \hat{a}(\mathbf{k})$ (note that ω_k is a function of $\mathbf{k}$), we arrive arrive at eq. (3.2.26). Note that in the mode expansion, we use the shorthand $k \cdot x = \omega_k t - \mathbf{k} \cdot \mathbf{x}$.

3.2.2 Propagation Amplitudes and Positive and Negative Frequency Solutions

For future convenience, let us split our mode expansion given in eq. (3.2.26) into positive and negative frequency parts

$$\hat{\varphi}(x) = \hat{\varphi}^+(x) + \hat{\varphi}^-(x), \tag{3.2.35}$$

where

$$\hat{\varphi}^+(x) \equiv \int (dk) \hat{a}(\mathbf{k}) e^{-ik \cdot x} \qquad \text{positive frequency part}, $$
$$\hat{\varphi}^-(x) \equiv \int (dk) \hat{a}^\dagger(\mathbf{k}) e^{ik \cdot x} \qquad \text{negative frequency part}. \tag{3.2.36}$$

It is $\hat{\varphi}^-$ that is responsible for creating a particle out of the vacuum. Let us consider the amplitude for a particle to propagate from y to x:[6]

$$
\begin{aligned}
\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle &= \langle 0|\hat{\varphi}^+(x)\hat{\varphi}^-(y)|0\rangle, \\
&= \int (dk)(dq) e^{-ik \cdot x + iq \cdot y} \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{q})|0\rangle, \\
&= \int (dq) e^{-iq \cdot (x-y)}, \qquad \text{because } \left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{q}) \right] = 2\omega_k \delta^{(3)}(\mathbf{k} - \mathbf{q}), \\
&\equiv \Delta^+(x - y).
\end{aligned}
\tag{3.2.37}
$$

Similarly,

$$\langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle = \langle 0|\hat{\varphi}^+(y)\hat{\varphi}^-(x)|0\rangle = \int (dq) e^{-iq \cdot (y-x)} = \Delta^+(y - x) \tag{3.2.38}$$

Note that $\Delta^+(x-y)$ behaves like a number, not as an operator. It multiplies the identity operator. Putting these amplitudes together, the commutator

$$
\begin{aligned}
\langle 0| \left[\hat{\varphi}(x), \hat{\varphi}(y) \right] |0\rangle &= \Delta^+(x - y) - \Delta^+(y - x), \\
&= \int (dq) \left(e^{-iq \cdot (x-y)} - e^{iq \cdot (x-y)} \right).
\end{aligned}
\tag{3.2.39}
$$

Note that instead of amplitude, we could have directly evaluated following commutators $[\hat{\varphi}^+(x), \hat{\varphi}^-(y)] = \Delta^+(x - y)$ and $[\hat{\varphi}^-(x), \hat{\varphi}^+(y)] = -\Delta^+(x - y)$, and put them together to get

$$[\hat{\varphi}(x), \hat{\varphi}(y)] = \Delta^+(x - y) - \Delta^+(y - x) = \int (dq) \left(e^{-iq \cdot (x-y)} - e^{iq \cdot (x-y)} \right), \tag{3.2.40}$$

since the commutator yields a number, and is the same as $\langle 0| [\hat{\varphi}(x), \hat{\varphi}(y)] |0\rangle$ if the vacuum is normalized $\langle 0|0\rangle = 1$. Hence the commutator can be though of as the amplitude of a particle to propagate from y to x minus the amplitude of a particle to travel from x to y. We will have use for these expression of the commutator in the discussion on causality and when we later consider interacting fields.

---

[6]A word of caution: The language used to describe the mathematical objects here is imprecise. Do not take them as cannon, read different references to get a more complete picture.

3.2.3 Green's functions

Let us consider the case where $x^0 > y^0$ and define[7] $i\Delta_R(x-y) \equiv \theta(x^0 - y^0)\langle 0| \left[\hat{\varphi}(x), \hat{\varphi}(y)\right] |0\rangle$. Then, we can show that

$$(\partial_\mu \partial^\mu + m^2)\Delta_R(x-y) = -\delta^{(4)}(x-y). \tag{3.2.41}$$

where $\partial_\mu = \partial/\partial x^\mu$. Thus $-\Delta_R(x-y)$ is a Green's function for the free Klein-Gordon equation! It is called the *Retarded* Green's function because it vanishes for $x^0 < y^0$. The lesson here is that our commutator is related to the propagation of signals from one point to another.

We will come across the *Feynman* Green's function later in the course, when we carry out scattering calculations. For the moment, let me just write down the definition:

$$
\begin{aligned}
i\Delta_F(x-y) &= \theta(x^0 - y^0)\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle, \\
&= \theta(x^0 - y^0)\Delta^+(x-y) + \theta(y^0 - x^0)\Delta^+(y-x).
\end{aligned}
\tag{3.2.42}
$$

As you can check, it is also a Green's function for the Klein-Gordon equation.

Exercise 3.2.8 : Verify eq. (3.2.41). It might be useful to first show that $(\partial^\mu \partial_\mu + m^2)\Delta^+(x-y) = 0$.

3.2.4 Causality

The fact that a "measurement" at x cannot influence a measurement at y for space-like separations $(x-y)^2 < 0$ should be built into our theory. Else, signals are propagating faster than light, and that is bad for a theory we claimed to be consistent with Special Relativity. In QFT, this statement of *Causality* is formally written as

$$\left[\hat{\mathcal{O}}_1(x), \hat{\mathcal{O}}_2(y)\right]_{(x-y)^2<0} = 0. \tag{3.2.43}$$

where $\hat{\mathcal{O}}_1(x)$ and $\hat{\mathcal{O}}_2(x)$ are Hermitian operators corresponding to some observables. For our scalar field theory, since the $\hat{\varphi}$ field is all there is, such operators are constructed from functions of $\hat{\varphi}$ (and their conjugate momenta). The simplest example of such operators is the field $\hat{\varphi}(x)$ itself. Hence, it better be true that:

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_{(x-y)^2<0} = 0. \tag{3.2.44}$$

To verify this in our free field theory, recall that any space-like separated events can be made simultaneous by a Lorentz transformation. Since the commutator above (see eq. (3.2.40)) is manifestly Lorentz invariant, it is sufficient to show that $[\hat{\varphi}(x), \hat{\varphi}(y)]_{x^0=y^0} = 0$ (at equal times). Writing the commutator in eq. (3.2.40) for the equal-time case, we immediately have

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_{x^0=y^0} = \int (dq) \left(e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\right) = 0. \tag{3.2.45}$$

The last equality follows from the fact that the integral is odd in $\mathbf{q}$. Having proved that the commutator vanishes for space-like separations, you should convince yourself that in general, it does not vanish for time-like separations.

Recall that in the previous chapter, we found that $\mathcal{A} = \langle \mathbf{x}|e^{-i\sqrt{\mathbf{P}^2+m^2}t}|\mathbf{x}_0\rangle \neq 0$ for spacelike separations, which we took to be the death of single particle quantum mechanics. What about an equivalent expression in field theory? As you can check, $\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle = \Delta^+(x-y) \neq 0$ for spacelike separations again! So what have we really gained by moving to field theory? Let us delve a little bit deeper. In single particle

---

[7]$\theta(x) = 1$ for $x > 0$ and 0 for $x < 0$

quantum mechanics, we state that we have one particle throughout, and no other excitations. So a non-zero overlap of states over spacelike intervals does violate causality. However, in a non-single particle theory (field theory): $\langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle \neq 0$ could mean that there are correlations between different excitations not necessarily related to propagation of any signals from one point to another. The appropriate thing to calculate is whether a *measurement* at spacelike separated points can affect each other. That operation is indeed the commutator, which fortunately is zero.[8] A connection between causality and existence of antiparticles can be better appreciated after we discuss complex fields.

3.2.5 Free Complex Scalar Fields

So far we have been dealing with real valued scalar fields. Let us consider a Lagrangian for a free, classical complex field $\varphi(x)$:[9]

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi^* \varphi \tag{3.2.46}$$

where φ^* is the complex conjugate of φ. We will think about φ and φ^* as being independent fields (we can also chose the real and imaginary parts of φ). In this case the conjugate momentum (density) corresponding to these fields is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}^*(x) \qquad \text{and} \qquad \pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^*} = \dot{\varphi}(x). \tag{3.2.47}$$

The Hamiltonian is

$$H = \int d^3x \left(\pi\dot{\varphi} + \pi^*\dot{\varphi}^* - \mathcal{L} \right) = \int d^3x \left(\pi^*\pi + \nabla\varphi^* \cdot \nabla\varphi + m^2\varphi^*\varphi \right). \tag{3.2.48}$$

It is possible to define a "charge" $Q = i \int d^3x[(\pi\varphi)^* - \varphi\pi]$, such that $dQ/dt = \{Q, H\} = 0$, that is the charge is conserved. Here, I seem to have pulled Q out of the hat. When we learn about Noether's theorem in the second half of the course, this definition of Q will seem natural. The conserved Q is a consequence of the fact that $\varphi(x) \to e^{i\alpha}\varphi(x)$ leaves the Lagrangian invariant. We note that this result would hold even when we have an arbitrary potential of the form $V(\varphi^*\varphi)$ (instead of the just the free field case studied here).

Canonical Quantization

Let us postulate the usual commutation relations:

$$[\hat{\varphi}(x), \hat{\pi}(y)]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \qquad \text{and} \qquad \left[\hat{\varphi}^\dagger(x), \hat{\pi}^\dagger(y)\right]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{3.2.49}$$

with $[\hat{\varphi}(x), \hat{\varphi}(y)] = \left[\hat{\varphi}^\dagger(x), \hat{\varphi}^\dagger(y)\right] = \ldots = 0$, and the " $\dagger$ " denotes the Hermitian conjugate. The Hamiltonian becomes

$$\hat{H} = \int d^3x \left(\hat{\pi}^\dagger\hat{\pi} + \nabla\hat{\varphi}^\dagger \cdot \nabla\hat{\varphi} + m^2\hat{\varphi}^\dagger\hat{\varphi} \right), \tag{3.2.50}$$

with the equations of motion given by

$$\begin{aligned}
\frac{d\hat{\varphi}}{dt} &= -i\left[\hat{\varphi}, \hat{H}\right] = \hat{\pi}^\dagger &\qquad \text{and} \qquad& \frac{d\hat{\pi}}{dt} = -i\left[\hat{\pi}, \hat{H}\right] = \nabla^2\hat{\varphi}^\dagger - m^2\hat{\varphi}^\dagger, \\
\frac{d\hat{\varphi}^\dagger}{dt} &= -i\left[\hat{\varphi}^\dagger, \hat{H}\right] = \hat{\pi} &\qquad \text{and} \qquad& \frac{d\hat{\pi}^\dagger}{dt} = -i\left[\hat{\pi}^\dagger, \hat{H}\right] = \nabla^2\hat{\varphi} - m^2\hat{\varphi}.
\end{aligned} \tag{3.2.51}$$

---

[8]Thanks to Daniel Green for a discussion on this, though we are both still a little uneasy about the details of the interpretation.

[9]While we consider free scalar fields, the manipulations on this page can be easily generalized to more general potentials $V(\varphi^*\varphi) = \sum_{n=1}^{N} (1/n!)\lambda_n(\varphi^*\varphi)^n$.

Take note of the location of the † s in the above equations. In terms of second-order in time equations, we have

$$\frac{d^2\hat{\varphi}}{dt^2} - \nabla^2\hat{\varphi} + m^2\hat{\varphi} = 0 \,, \tag{3.2.52}$$

and its Hermitian conjugate. As in the classical field case, we have also have a conserved charge:

$$\hat{Q} \equiv i \int d^3x \left(\hat{\varphi}^\dagger \hat{\pi}^\dagger - \hat{\varphi}\hat{\pi} \right) \,, \qquad \text{such that} \qquad \frac{d\hat{Q}}{dt} = -i \left[\hat{Q}, \hat{H} \right] = 0 \,. \tag{3.2.53}$$

Exercise 3.2.9 : Derive the rightmost sides of eq. (3.2.51). Then, using the definition of $\hat{Q}$ in eq. (3.2.53), show that $[\hat{Q}, \hat{H}] = 0$. (Hint: You will need to use integration by parts over the spatial volume; assumed fields die sufficiently fast at spatial infinity.)

Mode expansion

I claim that the mode expansions for our fields are given by

$$\hat{\varphi}(x) = \int (dk) \left(\hat{b}(\mathbf{k})e^{-ik\cdot x} + \hat{d}^\dagger(\mathbf{k})e^{ik\cdot x} \right) \qquad \text{and} \qquad \hat{\varphi}^\dagger(x) = \int (dk) \left(\hat{d}(\mathbf{k})e^{-ik\cdot x} + \hat{b}^\dagger(\mathbf{k})e^{ik\cdot x} \right) \,. \tag{3.2.54}$$

Note that this is reasonable. For a real scalar field, $\hat{\varphi}(x) = \hat{\varphi}^\dagger(x) = \int (dk) \left(\hat{a}(\mathbf{k})e^{-ik\cdot x} + \hat{a}^\dagger(\mathbf{k})e^{ik\cdot x} \right)$, and hence the second operator in the mode expansion $\hat{a}^\dagger$ ended up being a hermitian conjugate of the first operator ($\hat{a}$). But for a complex field, $\hat{\varphi}^\dagger(x) \neq \hat{\varphi}(x)$, hence we need two sets of creation and annihilation operators, thus indicating the existence of two distinct particles. The creation and annihilation operators are written in a manner so that upon taking the Hermitian conjugate of the right hand side in the mode expansion of $\hat{\varphi}$, we get the right hand side in the mode expansion of $\hat{\varphi}^\dagger$ above. Similarly, the mode expansions of conjugate momentum densities are:

$$\pi(x) = -i \int (dk)\omega_\mathrm{k} \left(\hat{d}(\mathbf{k})e^{-ik\cdot x} - \hat{b}^\dagger(\mathbf{k})e^{ik\cdot x} \right) \qquad \text{and} \qquad \pi^\dagger(x) = -i \int (dk)\omega_\mathrm{k} \left(\hat{b}(\mathbf{k})e^{-ik\cdot x} - \hat{d}^\dagger(\mathbf{k})e^{ik\cdot x} \right) \,. \tag{3.2.55}$$

The commutation relations satisfied by the creation and annihilation operators are

$$\left[\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{q}) \right] = 2\omega_\mathrm{k}\, \delta^{(3)}(\mathbf{k} - \mathbf{q}) \,, \qquad \text{and} \qquad \left[\hat{d}(\mathbf{k}), \hat{d}^\dagger(\mathbf{q}) \right] = 2\omega_\mathrm{k}\, \delta^{(3)}(\mathbf{k} - \mathbf{q}) \,, \tag{3.2.56}$$

with all others being zero.

Charge

The Hamiltonian and the charge $\hat{Q}$ in terms of the creation and annihilation operators become

$$\hat{H} = \int (dk) \left(\hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k}) + \hat{d}^\dagger(\mathbf{k})\hat{d}(\mathbf{k}) \right) \omega_\mathrm{k} + \text{const.}$$
$$\hat{Q} = \int (dk) \left(\hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k}) - \hat{d}^\dagger(\mathbf{k})\hat{d}(\mathbf{k}) \right) \,. \tag{3.2.57}$$

I will derive these expressions so that we can get some practice with the manipulations of creation and annihilation operators, delta functions and facility with changing dummy variables. But before we do

that, let us digress to understand the physical interpretation of these expressions. This meaning is clearer when we put these fields in a box with finite volume. In this case

$$\hat{H} = \sum_{\mathbf{k}} \left(\hat{b}_{\mathbf{k}}^{\dagger} \hat{b}_{\mathbf{k}} + \hat{d}_{\mathbf{k}}^{\dagger} \hat{d}_{\mathbf{k}} \right) \omega_{\mathrm{k}} + \mathrm{const.}$$

$$\hat{Q} = \sum_{\mathbf{k}} \left(\hat{b}_{\mathbf{k}}^{\dagger} \hat{b}_{\mathbf{k}} - \hat{d}_{\mathbf{k}}^{\dagger} \hat{d}_{\mathbf{k}} \right) . \tag{3.2.58}$$

From the expression for the Hamiltonian we see that, we have two sets of particles, each has the same mass, and they contribute equally to the Hamiltonian. Since $\hat{b}_{\mathbf{k}}^{\dagger} \hat{b}_{\mathbf{k}}$ and $\hat{d}_{\mathbf{k}}^{\dagger} \hat{d}_{\mathbf{k}}$ simply count the number of b and d particles with momentum $\mathbf{k}$, the conservation of $\hat{Q}$ is a statement about conservation of a difference between the number of these particles. The sign difference corresponding to the number operator allows us to interpret one set of particles as "positively" charged and the other as "negatively" charged with all else being equal. Think of these as particles and their anti-particles. [10] [11]

As promised, let us derive the expressions for $\hat{Q}$ in eq. (3.2.57). By using the mode expansions of $\hat{\varphi}$ and $\hat{\varphi}^{\dagger}$ in eq. (3.2.54) and $\hat{\pi} = d\hat{\varphi}^{\dagger}/dt$ and $\hat{\pi}^{\dagger} = d\hat{\varphi}/dt$ in eq. (3.2.55) we get

$$\hat{Q} = \underbrace{i \int d^3x \, \hat{\varphi}^{\dagger} \hat{\pi}^{\dagger}}_{\text{①}} - \underbrace{i \int d^3x \, \hat{\varphi} \hat{\pi}}_{\text{②}} ,$$

$$\text{①} = \int d^3x (dk)(dq)\omega_{\mathrm{q}} \left(\hat{d}(\mathbf{k})e^{-ik\cdot x} + \hat{b}^{\dagger}(\mathbf{k})e^{ik\cdot x} \right) \left(\hat{b}(\mathbf{q})e^{-iq\cdot x} - \hat{d}^{\dagger}(\mathbf{q})e^{iq\cdot x} \right) , \tag{3.2.59}$$

$$\text{②} = \int d^3x (dk)(dq)\omega_{\mathrm{q}} \left(\hat{b}(\mathbf{q})e^{-iq\cdot x} + \hat{d}^{\dagger}(\mathbf{q})e^{iq\cdot x} \right) \left(\hat{d}(\mathbf{k})e^{-ik\cdot x} - \hat{b}^{\dagger}(\mathbf{k})e^{ik\cdot x} \right) .$$

For ② we have interchanged $\mathbf{k}$ and $\mathbf{q}$ since they are dummy variables. You might have noticed that I did not change the label for ω_{k}. To understand this, note that $i(k \pm q) \cdot x = (\omega_{\mathrm{k}} \pm \omega_{\mathrm{q}})t - (\mathbf{k} \pm \mathbf{q}) \cdot \mathbf{x}$ and, $\int d^3x \, e^{-i(\mathbf{k}\pm\mathbf{q})\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{k} \pm \mathbf{q})$, which always sets $\omega_{\mathrm{k}} = \omega_{\mathrm{q}}$.

For the difference, ① − ②, first focus on the db and bd terms. Since b and d commute these terms cancel each other. The same is true for $b^{\dagger}d^{\dagger}$ and $d^{\dagger}b^{\dagger}$ terms. Hence we are left with

$$\hat{Q} = \frac{1}{2} \int (dk) \left(\hat{b}^{\dagger}(\mathbf{k})\hat{b}(\mathbf{k}) + \hat{b}(\mathbf{k})\hat{b}^{\dagger}(\mathbf{k}) - \hat{d}^{\dagger}(\mathbf{k})\hat{d}(\mathbf{k}) - \hat{d}(\mathbf{k})\hat{d}^{\dagger}(\mathbf{k}) \right) , \tag{3.2.60}$$

where we used $(dq)\omega_{\mathbf{q}} = d^3q/(2(2\pi)^3)$ along with the $\delta(\mathbf{k} - \mathbf{q})$. Finally, using the commutation relations (see eq. (3.2.56)), we arrive at

$$\hat{Q} = \int (dk) \left(\hat{b}^{\dagger}(\mathbf{k})\hat{b}(\mathbf{k}) - \hat{d}^{\dagger}(\mathbf{k})\hat{d}(\mathbf{k}) \right) . \tag{3.2.61}$$

Exercise 3.2.10 : Derive the expression for $\hat{Q}$ in eq. (3.2.58) using a finite box size. This should be follows immediately from our now familiar rules for going between the finite and infinite volume cases: $(2\omega_k V)\int(dk) \leftrightarrow \sum_{\mathbf{k}}$ and $V^{-1/2}(2\omega_{\mathrm{k}})^{-1/2}\hat{a}(\mathbf{k}) \leftrightarrow \hat{a}_{\mathbf{k}}$. For practice, you should also (i) re-write the mode functions and definition for $\hat{Q}$ in the finite box case, and (ii) work through the manipulations at the end of this subsection on *Charge* to re-derive the expression for $\hat{Q}$ in terms of the creation and annihilation operators.

---

[10]Note these particles are not electrons/positrons etc. which are quanta of spin 1/2 fields, not scalar fields. A reasonable (but approximate) real life example of particles described by scalar fields with our usual electric charge would be pions. The charge here need not be electric charge.

[11]There was some ambiguity in how we chose to order operators in the Hamiltonian and the charge in terms of fields and their conjugate momenta. Ultimately, we always write these as a sum of a finite part and an infinite constant (related to the vacuum), which renders this ambiguity inconsequential. Also the sign of the charge is convention dependent.

Causality, again

Given the above mode expansions in eq. (3.2.54), you can again check that

$$[\hat{\varphi}(x), \hat{\varphi}^\dagger(y)] = \Delta^+(x-y) - \Delta^+(y-x), \qquad \text{and} \qquad [\hat{\varphi}(x), \hat{\varphi}^\dagger(y)]_{(x-y)^2<0} = 0. \qquad (3.2.62)$$

Recall that we have conserved charge and a notion of positive and negative charged particles. Picking a convention for the sign of the charge, we can say that $\varphi^\dagger|0\rangle$ creates negatively charged particles out of vacuum (and annihilates positively charged ones), whereas $\hat{\varphi}|0\rangle$ creates positively charged particles (and annihilates negatively charged ones).

The vanishing of the commutator outside on space-like separations can be interpreted as follows. The $\Delta^+(x-y)$ represents the amplitude of propagation of a negatively charged particle from y to x whereas $\Delta^+(y-x)$ represents the propagation of a positively charged particle from x to y. Each individually has non-zero contributions outside the light-cone. However, for the commutator to vanish, they must be equal to each other outside the light-cone. This of course is only possible because they have the same mass. Thus in a way, causality requires the existence of antiparticles (opposite charge, same mass!). In the case of the real scalar field earlier, particles are their own antiparticles.

For further discussion, see for example, section 2.1 and 2.4 in Peskin and Schroeder. Another short discussion can be found in section 2.1 of *Modern QFT, A Concise Introduction* by Banks. For a detailed discussion of conceptual issues related to causality, also see *The Conceptual Framework for QFT*, by Duncan.[12]

Non-relativistic Fields

Before we move on to interactions, let me make a brief digression to cold-atom systems. Bose-Einstein condensates of cold atoms are well described by a non-linear Schrödinger equation. We can get to this equation by considering a multi-particle wavefunction with interactions, or by taking the non-relativistic limit of the our relativistic Klein-Gordon equation. While we do not have the time to go through this in detail, you can work through the problem below to get a bit of the flavor.

Exercise 3.2.11 : Consider the Lagrangian density $\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 |\varphi|^2 - \lambda |\varphi|^4$ where φ is a complex

scalar field. Derive the Euler-Lagrange equation for φ. Then change variables $\varphi(t, \mathbf{x}) = \exp[-imt]\psi(t, \mathbf{x})$, and derive an equation for ψ (a complex scalar field as well) assuming the time-scale and length-scale of variation in ψ is much larger and longer that m^{-1}. This is the non-relativistic limit of our theory. You should arrive at the (non-linear) Schrodinger equation. But be careful here: ψ should not be interpreted as a single-particle wave function. Define a conserved charge for this system.

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[12]Thanks to D. Baumann for this reference.

WEAKLY INTERACTING FIELDS

In sections 3.2.1 and 3.2.5 we dealt with free scalar fields. For such cases, the classical and quantum field equations are linear in the fields. This means that each Fourier mode evolves independently, and the problem essentially reduces to that of quantizing a harmonic oscillator for each Fourier mode. The dynamics is simple, but also boring. There are no interactions – no scattering and no decays.[1] In this chapter we introduce interactions, which will allow for non-trivial scattering and decays. We will concern overselves with perturbative calculations, where the interactions introduced are in some sense weak. Non-perturbative field theory is fascinating, but beyond the scope of this course (for the most part). The formalism we develop in this chapter will lead us to Feynman Diagrams.

4.1   Adding Interactions

To make our scalar field theory more interesting, and somewhat more realistic, we need to introduce interactions. At the level of the Lagrangian density, this means adding nonlinear terms in the fields, or coupling different fields. Let us look at a couple of examples:

Massive φ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \,, \tag{4.1.1}$$

where $(\partial\varphi)^2 = g_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi$ and we will drop the "hats" from the fields. We are dealing with quantum fields from now onwards. The φ^4 makes the equations of motion nonlinear. We will also learn that it allows for processes like two φ quanta scattering of each other.

Similarly, we can write down a Lagrangian density with both a real and complex scalar field. We will now denote the complex field as ψ:

Scalar version of Quantum Electrodynamics:

$$\mathcal{L} = |\partial\psi|^2 - M^2\psi^\dagger\psi + \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - g\varphi\psi^\dagger\psi \,. \tag{4.1.2}$$

The interaction term $g\varphi\psi^\dagger\psi$ allows for processes like the decay of a φ quantum, into a ψ particle and anti-particle. It also allows for scattering of ψ particles and anti-particles via an exchange of φ quanta and so on. In this toy example, you can think of φ as representing a photon for $m \to 0$ and ψ and $\psi^\dagger$ for electrons.[2]

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[1] Classically speaking, ripples in the field just pass through each other, without any changes.

[2] Note photons are quanta of gauge fields, and electrons of fermionic fields. They are definitely not represented by scalar fields.

4.1.1 Perturbative Control

Free theory, without the interaction term was simple and solvable. We want to make use of it as much as possible, but include effects from the interaction terms so that interesting processes become allowed. We humbly start by thinking about including the effects of the interaction perturbatively. It is reasonable that this imposes a restriction on the *coupling constants* λ and g. However, saying that they are small, and that their effects will be small is not as trivial as it seems.

Note that λ must be dimensionless, whereas g must have dimensions. To see this, recall that in our $c = \hbar = 1$ system of units, energy, mass, momentum can be measured with the same units. It is convenient to define a "mass-dimension" denoted by $[...]$ such that $[\text{mass}] = [\text{energy}] = [\text{momentum}] = 1$ and correspondingly, $[\text{length}] = [\text{time}] = -1$. The action always has mass-dimension 0. Now, since $[d^4x] = -4$, we must have $[\mathcal{L}] = 4$. By looking at the $(\partial\varphi)^2$ or $|\partial\psi|^2$ terms, $[\mathcal{L}] = 4$ and $[\partial] = 1$ implies that $[\varphi] = [\psi] = 1$. Moving back to the interaction terms, we can see that $[\lambda] = 0$ and $[g] = 1$. Since λ is indeed dimensionless, saying that $|\lambda| \ll 1$ is reasonable. But saying that g is small is not possible without constructing a dimensionless ratio with some other mass or energy scale.

To understand the kind of problems that might arise, first suppose we want to calculate the amplitude of some process, say the scattering of two φ particles in the φ^4 theory. For $|\lambda| \ll 1$, a perturbative calculation of the Amplitude (in terms of λ) can be expected to have the form

$$\mathcal{A}(\varphi\varphi \to \varphi\varphi) = \sum_{n=0} \lambda^n f_n(\{p^\mu\}) = 1 + \lambda f_1(\{p^\mu\}) + \lambda^2 f_2(\{p^\mu\}) + \dots \tag{4.1.3}$$

with a good chance that first few terms yield a reliable answer. Here $\{p^\mu\}$ stand for the momenta of the incoming and out-going particles, and f_n are dimensionless functions of the external momenta of the incoming and outgoing particles.

Now, suppose we want to calculate the amplitude the scattering of ψ particles: $\mathcal{M}(\psi\psi \to \psi\psi)$. This time, we follow our nose, and write

$$\mathcal{A}(\psi\psi \to \psi\psi) = \sum_{n} \left(\frac{g}{E}\right)^n f_n(\{p^\mu\}) . \tag{4.1.4}$$

For some g, if $E \gg g$, then we can get away with calculating the first few terms. However, if our experiment is very low energy ($E \ll g$), the above expansion is useless. Can $E \ll g$ be avoided? For this problem, yes! Since our particles have mass M, our energy scale $E \gtrsim M$. Hence, we can get away with the "small g" expansion if $g \ll M$.

To summarize, if we want to do perturbative calculations using λ or g to organize our expansion (for arbitrary E), then we should at least make sure that $\lambda \ll 1$ and $g \ll M$ (if $M \to 0$, at least at this heuristic level, we cannot use this small g expansion at low energies).

There are some general lessons to be learnt. For the real scalar field example, consider general interaction terms of the form $(\lambda_n/n!)\varphi^n$ (where $n > 2$). Then for perturbative control we need $|\lambda_n|/E^{4-n} \ll 1$. For massive fields, we expect $E \gtrsim m$. Hence, it is sufficient to have $|\lambda_n| \ll m^{4-n}$. For fields with different mass particles, we have to make sure that the coupling constants g_n are smaller than the appropriate powers of the lightest mass involved: m_l.

These considerations are meant as a guidance, not proof of what actually happens in explicit calculations. Some $f'_n s$ might be zero or formally infinite (say at resonances, or from higher order contributions), complicating our simple arguments. Moreover, by insisting on "renormalizability" of the theory, we can severely limit the kind of interaction terms that can be added. More on this, later.[3]

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[3]You might also want to read pg 47-50 of David Tong's lecture notes to get a broader picture of the structure of lagrangians. I also recommend reading section 4.1 of Peskin and Schroeder to get an overview of the what principles we typically follow in writing down interaction terms.

4.2 Time Evolution in the Interaction Picture

In the previous chapter we discussed the *Heisenberg* and *Schrödinger Pictures* as being equivalent way of capturing time evolution. For weakly interacting fields, yet another picture is useful: the *Interaction Picture* which is hybrid on the Heisenberg and Schrödinger pictures. Consider a Hamiltonian[4]

$$H = H_0 + H_{\text{int}}\,,\tag{4.2.1}$$

where H_0 is the free Hamiltonian and $H_{\text{int}} = \int d^3x \mathcal{H}_{\text{int}} = -\int d^3x \mathcal{L}_{\text{int}}$. We will be thinking about H_{int} as being a small correction to H. In terms of examples of interactions mentioned earlier, $\mathcal{H}_{\text{int}} = (\lambda/4!)\varphi^4$ or $g\varphi\psi^\dagger\psi$.

Recall that in the Heisenberg picture, the operators evolve according to $f(t) = e^{iH(t-t_0)}f(t_0)e^{-iH(t-t_0)}$ but states $|\alpha\rangle$ do not, whereas in the Schrödinger picture, states evolve according to $|\alpha(t)\rangle_{\text{s}} = e^{-iH(t-t_0)}|\alpha(t_0)\rangle_{\text{s}}$, whereas operators f_{s} do not. Operators and states in different pictures agree at $t = t_0$: $f(t_0) = f_{\text{s}}$ and $|\alpha\rangle = |\alpha(t)\rangle_{\text{s}}$. Note that the evolution is determined by the full Hamiltonian H. The expectation value $\langle\alpha|f(t)|\alpha\rangle = {}_{\text{s}}\langle\alpha(t)|f_{\text{s}}|\alpha(t)\rangle_{\text{s}}$ have to be the same in either picture since it is an observable.

In the Interaction picture, we will evolve operators using the free part of the Hamiltonian H_0: $f_{\text{I}}(t) = e^{iH_0(t-t_0)}f(t_0)e^{-iH_0(t-t_0)}$. How must the states $|\alpha(t)\rangle_{\text{I}}$ evolve in this picture ? We know that the expectation values must agree with those in the Schrödinger pictures. Hence,

$$\begin{aligned}
{}_{\text{I}}\langle\alpha(t)|f_{\text{I}}(t)|\alpha(t)\rangle_{\text{I}} &= {}_{\text{s}}\langle\alpha(t)|f_{\text{s}}|\alpha(t)\rangle_{\text{s}}\,,\\
{}_{\text{I}}\langle\alpha(t)|e^{iH_0(t-t_0)}f_{\text{s}}e^{-iH_0(t-t_0)}|\alpha(t)\rangle_{\text{I}} &= {}_{\text{I}}\langle\alpha(t_0)|e^{iH(t-t_0)}f_{\text{s}}e^{-iH(t-t_0)}|\alpha(t_0)\rangle_{\text{I}}\,.
\end{aligned}\tag{4.2.2}$$

where we used $f_{\text{I}}(t_0) = f_{\text{s}}$ on the left-hand-side and $|\alpha(t_0)\rangle_{\text{s}} = |\alpha(t_0)\rangle_{\text{I}}$ on the right-hand-side. Comparing the two sides, we have

$$|\alpha(t)\rangle_{\text{I}} = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}|\alpha(t_0)\rangle_{\text{I}} \equiv U(t,t_0)|\alpha(t_0)\rangle_{\text{I}}\tag{4.2.3}$$

where we have defined the time evolution operator

$$U(t,t_0) \equiv e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\,,\tag{4.2.4}$$

for evolving states in the Interaction picture. Note that $e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \neq e^{-i(H-H_0)(t-t_0)}$ since the operators do not necessarily commute. Let us understand how the time evolution operator itself evolves with time:

$$\begin{aligned}
\frac{d}{dt}U(t,t_0) &= i\left[e^{iH_0(t-t_0)}H_0 e^{-iH(t-t_0)} - e^{iH_0(t-t_0)}He^{-iH(t-t_0)}\right]\,,\\
&= i\left[e^{iH_0(t-t_0)}H_0 \underbrace{e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}}_{1}e^{-iH(t-t_0)} - e^{iH_0(t-t_0)}H\underbrace{e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}}_{1}e^{-iH(t-t_0)}\right]\,,\\
&= i\left[e^{iH_0(t-t_0)}H_0 e^{-iH_0(t-t_0)}\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{U} - e^{iH_0(t-t_0)}He^{-iH_0(t-t_0)}\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{U}\right]\,,\\
&= i\left[e^{iH_0(t-t_0)}\left(H_0 - H\right)e^{-iH_0(t-t_0)}U(t,t_0)\right]\,,\\
&= -i\left[e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}U(t,t_0)\right]\,,\\
&= -iH_{\text{I}}(t)U(t,t_0)\,,
\end{aligned}$$

where, in the last line we defined the interaction picture version of the operator H_{int}:

$$H_{\text{I}}(t) = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}\,.\tag{4.2.5}$$

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[4]Split in the Schrödinger picture.

If we were being truly consistent with notation, we should have written $H_{I,\text{int}}$. But that is cumbersome, so we will strick with H_I. Note that H_I has explicit time dependence now even if H_{int} did not. If H_I was a number rather than an operator, the $(d/dt)U = -iH_\text{I}U$ has a formal solution $U(t,t_0) = e^{-i\int_{t_0}^t d\tau H_\text{I}(\tau)}$. However, since H_I is an operator, and it does not commute with itself at different times, we need to do a bit more work.

It is possible to write down a compact, formal solution for the time-evolution operator:

$$U(t,t_0) = T\left\{\exp\left(-i\int_{t_0}^t H_\text{I}(\tau)d\tau\right)\right\} = \sum_{n=0}^\infty \frac{1}{n!}(-i)^n \int_{t_0}^t d\tau_1 \dots \int_{t_0}^t d\tau_n T\left\{H_\text{I}(\tau_1)H_\text{I}(\tau_2)\dots H_\text{I}(\tau_n)\right\}.$$

(4.2.6)

This is the *Dyson*-series expansion. The symbol $T\{\dots\}$ is for *time ordering*; it means

$$T\left\{H_\text{I}(\tau_1)H_\text{I}(\tau_2)\right\} = \begin{cases} H_\text{I}(\tau_1)H_\text{I}(\tau_2) & \tau_1 > \tau_2\,, \\ H_\text{I}(\tau_2)H_\text{I}(\tau_1) & \tau_2 > \tau_1\,. \end{cases}$$

(4.2.7)

Let us verify that U above satisfies $dU/dt = -iH_\text{I}U$. To this end, let us write out the first few terms of the expansion for U:

$$\begin{aligned} U(t,t_0) &= 1 + (-i)\int_{t_0}^t d\tau_1 H_\text{I}(\tau_1) + \frac{1}{2!}(-i)^2 \int_{t_0}^t d\tau_1 \int_{t_0}^t d\tau_2 T\{H_\text{I}(\tau_1)H_\text{I}(\tau_2)\} + \dots\,, \\ &= 1 + (-i)\int_{t_0}^t d\tau_1 H_\text{I}(\tau_1) + (-i)^2 \int_{t_0}^t d\tau_2 H_\text{I}(\tau_2) \int_{t_0}^{\tau_2} d\tau_1 H_\text{I}(\tau_1) + \dots\,, \end{aligned}$$

(4.2.8)

Let us understand the changes from the first to second line for the third term on the right-hand side. Using the definition of the time-ordering symbol:

$$\frac{1}{2!}\int_{t_0}^t d\tau_1 \int_{t_0}^t d\tau_2 T\{H_\text{I}(\tau_1)H_\text{I}(\tau_2)\} = \frac{1}{2!}\left(\int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 H_\text{I}(\tau_1)H_\text{I}(\tau_2) + \int_{t_0}^t d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 H_\text{I}(\tau_2)H_\text{I}(\tau_1)\right),$$

(4.2.9)

where you should pay attention to the change in limit of integration based on whether $\tau_1 < \tau_2$ or not. We can now interchange the dummy integration variables in the first term on the right hand side to yield $2 \times \int d\tau_2 \dots$. This 2 precisely cancels the 2! in front of the brackets. This behavior carries over to the higher order terms: the $n!$ gets cancelled each time when we open up the time ordering symbol and change the limits of integration.

Now, let us differentiate both side of eq. (4.2.8):

$$\begin{aligned} \frac{d}{dt}U(t,t_0) &= 0 + (-i)\,H_\text{I}(t) + (-i)^2\,H_\text{I}(t)\int_{t_0}^t d\tau_1 H_\text{I}(\tau_1) + \dots \\ &= (-i)\,H_\text{I}(t)\left[1 + (-i)\int_{t_0}^t d\tau_1 H_\text{I}(\tau_1) + \dots\right], \\ &= -iH_\text{I}(t)U(t,t_0)\,, \end{aligned}$$

(4.2.10)

which is what we needed to show. We only considered the first couple of terms, you might want to convince yourself of the general result using induction.

Let us generalize the definition of $U(t,t_0)$ from our particular time t_0 (where the different pictures agreed) to an arbitrary time $t' \leq t$ as

$$U(t,t') \equiv T\left\{\exp\left(-i\int_{t'}^t H_\text{I}(\tau)d\tau\right)\right\}.$$

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As you can check, this general $U(t, t')$ satisfies $\partial_t U(t, t') = -iH_I(t)U(t, t')$. Moreover, you can prove[5] based on the integral expression above that $U(t, t') = U(t, t'')U(t'', t')$ where $t \geq t'' \geq t'$. In particular, $U(t, t_0) = U(t, t')U(t', t_0)$ for $t \geq t' \geq t_0$. This immediately yields

$$U(t, t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)} \, .$$

which is manifestly Unitary. This expression of course agrees with $U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ when $t' \to t_0$. On the other hand, note that we have $U(t_0, t) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}$ which follows from defining $U(t_0, t)$ as the time evolution operator evolving states from t to t_0 and using the fact that $U(t, t_0)$ is unitary.

It is worthwhile knowing the following identities for the time-evolution operator

1. $U(t, t) = 1$.

2. $U^\dagger(t, t')U(t, t') = U(t, t')U^\dagger(t, t') = 1$. This is the property of *Unitarity*.

3. $U(t, t') = U(t, t'')U(t'', t')$ with $t \geq t'' \geq t'$.

4. $U(t, t') = U^{-1}(t', t)$.

Exercise 4.2.1 : Verify the above identities of the time evolution operator. You are allowed to use $U(t, t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}$ if you want or the integral expression.

4.2.1   The S-matrix

We will often be interested in evolving a state from far back in time $t' = -\infty$ to the far future $t = \infty$. This motivates the definition of the *S-matrix*:

$$S \equiv U(\infty, -\infty) \, ,$$
$$= T\left\{\exp\left[-i\int_{-\infty}^\infty d\tau H_I(\tau)\right]\right\} = T\left\{\exp\left[-i\int_{-\infty}^\infty d^4x \mathcal{H}_I\right]\right\} = T\left\{\exp\left[i\int_{-\infty}^\infty d^4x \mathcal{L}_I\right]\right\} \, . \tag{4.2.11}$$

where, for example, $\mathcal{L}_I = -(\lambda/4!)\varphi_I^4$, with $\varphi_I(t, \mathbf{x}) = e^{iH_0(t-t_0)}\varphi(t_0, \mathbf{x})e^{-iH_0(t-t_0)}$. Note the following properties of the S-matrix:

1. S is Lorentz invariant.

2. S is Unitary.

3. $S = 1 - i\bar{\delta}^{(4)}(p_i - p_f)\mathcal{M}$ where p_i and p_f are the 4-momenta of initial and final state respectively, 1 indicates "nothing-happened" between the initial and final states, and $\bar{\delta}^{(4)}(p_i - p_f)$ guarantees energy-momentum conservation.

Exercise 4.2.2 : Verify that S is Lorentz invariant. Assume that the interaction Hamiltonian density is constructed out of some polynomial in the fields. Hint: This needs no calculation. One of the main things you need to argue is why time ordering does not spoil Lorentz invariance.

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[5]Proving this will make you think about what time-ordering does to commutators

Why use the Interaction Picture?

First, note that S matrix is written in terms of the interaction part of the interaction term only. If the interaction term is controlled by a small coupling constant, we can expand S in that coupling constant systematically (this is essentially what the Dyson expansion does).

Second, the φ_I appearing S are solutions to the free-field Hamiltonian H_0. To see this, note that at some fixed time t_0, we can always expand an arbitrary field in terms of our creation and annihilation operators associated with the free field

$$\varphi(t_0, \mathbf{x}) = \int (dk) \left(a(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right) . \tag{4.2.12}$$

The time evolution of the field in the interaction picture

$$\varphi_I(t, \mathbf{x}) = e^{iH_0(t-t_0)} \varphi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} = \int (dk) \left(a(\mathbf{k}) e^{-ik \cdot x} + a^\dagger(\mathbf{k}) e^{ik \cdot x} \right) , \tag{4.2.13}$$

which is just the free field! Note that we used the fact that the time evolution of the creation and annihilation operators for the free theory: $e^{iH_0(t-t_0)} a(\mathbf{k}) e^{-iH_0(t-t_0)} = a(\mathbf{k}) e^{-i\omega_k t}$. What this means is that the S-matrix (or more generally, U) is made up of free fields, in particular, it is just a string of free-field creation and annihilation operators.

What about single or multi-particle states? Can we produce them by acting with our free-field creation operators on the free field vacuum? The answer, in general, is no. Interactions do not "turn off" in the asymptotic past/future and the vacuum of the interacting theory is not the same as that of the free theory. However, we will trick ourselves for the moment into thinking that the answer is "yes". We will eventually pay a price for this trickery and have to come back to address it.

There are a couple of important results (stated here without proof) that should convince us that working in the interaction picture is all we need. It will also come in handy later in the course. To state the results easily, I need to introduce (n) point Green's functions. You can ignore these results on a first reading.

Green's Functions

The n-point Green's function of fields in the full interacting theory:

$$G^{(n)}(x_1, x_2 \ldots x_n) \equiv \langle \Omega | T\{\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\} | \Omega \rangle \tag{4.2.14}$$

where φ are the Heisenberg fields in the interacting theory, and $|\Omega\rangle$ is the vacuum of the interacting theory. This n-point correlation function is a fundamental quantity in QFT's, and can be an observable itself or be directly related to observable. For example, this Green's function can be directly related to scattering amplitudes (S matrix elements), through the LSZ reduction formula discussed below (see Peskin & Schroeder for a detailed derivation). This connection is easiest is written most easily in Fourier space. Let $G^{(n)}(k_1, k_2 \ldots, k_n) = \int \left(\prod_{i=1}^{n} d^4 x_i e^{ik_i \cdot x_i} \right) G^{(n)}(x_1, x_2 \ldots x_n)$.

LSZ Reduction Formula

Let $|i\rangle = |\mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_n\rangle$ be an initial state at $t \to -\infty$. Think about this as collection of particles of the theory with definite momenta at early times. Similarly, consider a final state $|f\rangle = |\mathbf{p}_1, \mathbf{p}_2 \ldots \mathbf{p}_r\rangle$ at $t \to \infty$. Then the transition or scattering amplitude is related to the Green's function via (assuming a real scalar field theory)

$$\langle \mathbf{p}_r, \ldots \mathbf{p}_2, \mathbf{p}_1 | S | \mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_n \rangle = (-iZ^{-1/2})^{n+r} \prod_{i=1}^{n} \prod_{j=1}^{r} \left(k_i^2 - m^2 \right) \left(p_j^2 - m^2 \right) G^{(n+r)}(k_1, \ldots k_n; p_1 \ldots p_r) , \tag{4.2.15}$$

where m is the mass of the field. The factor Z is related to renormalization (to be discussed later). We will find in our calculations that the Green's function has factors of the form $(p_j^2 - m^2)(k_i^2 - m^2)$ which precisely cancel the ones in the above expression to yield finite results for the scattering amplitude.

The above expressions are in terms of fields and vacuua in the full theory. Let us now connect them to interacting (free fields) and the free vacuum. This is done through the Gell-Mann-Low theorem.

Gell-Mann-Low Theorem

The connection between the expectation value of the time ordered product of Heisenberg picture fields ($\varphi(x_i)$) in the full interacting theory vacuum ($|\Omega\rangle$), and the corresponding expression for the same quantity using the interaction picture fields ($\varphi_I(x_i)$) and the free field vacuum $|0\rangle$:

$$\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\}|\Omega\rangle = \frac{\langle0|T\{\varphi_I(x_1)\varphi_I(x_2)\ldots\varphi_I(x_n)S\}|0\rangle}{\langle0|S|0\rangle} \tag{4.2.16}$$

where S is constructed form fields in the interaction picture (free fields), see eq. (4.2.11). For a proof, see for example, Peskin & Schroeder.

4.2.2 Normal Ordering and Wick's theorem

In the asymptotic past and future, let us imagine that the initial and final states are given by, for example, two particle states $|i\rangle_{\text{free}} = a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle$ and $|f\rangle_{\text{free}} = |\mathbf{k}_3, \mathbf{k}_4\rangle$. Then the matrix element:

$$\begin{aligned} S_{if} &= {}_{\text{free}}\langle f|U(\infty, -\infty)|i\rangle_{\text{free}} \\ &= \langle0|a(\mathbf{k}_3)a(\mathbf{k}_4)T\{1 - i\int d^4x\mathcal{H}_I + \ldots\}\}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle \\ &= \langle0|T\{a(\mathbf{k}_3)a(\mathbf{k}_4)\{1 - i\int d^4x\mathcal{H}_I + \ldots\}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\}|0\rangle \end{aligned} \tag{4.2.17}$$

where in the second line, $T\{\ldots\}$ is also made up of a string of creation and annihilation operators of the free field. We enveloped the initial and final state creation and annihilation operators inside the $T\{\ldots\}$ as well, since they are in the asymptotic past and future, hence already time ordered. In this way, the matrix elements are completely calculated by creation and annihilation operators acting on the free-field vacuum.

We know that annihilation operators annihilate the vacuum to the right, whereas creation operators annihilate the vacuum to the left. Wouldn't it be nice if somehow we could move all the creation operators to the left and all the annihilation operators to the right, while picking up delta functions from the commutation relations for the creation and annihilation operators. The formal procedure for doing this is through *Wick's Theorem*. Before getting to Wick's theorem, let us start with some preliminaries.

Normal Ordering

A *Normal Ordered* of a product operators in $O_1O_2\ldots O_n$ (each operator is constructed from strings of creation and annihilation operators), denoted by the same operators between two colons : $\ldots$:, is such that : $O_1O_2\ldots O_n$: has all the creation operators to the left, and annihilation operators to the right. For example

$$: a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2) : = a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)\,. \tag{4.2.18}$$

An important property of Normal ordered products is that they have a zero vacuum expectation value:

$$\langle0| : O_1O_2\ldots O_n : |0\rangle = 0\,. \tag{4.2.19}$$

In the above example, this is manifest:

$$\langle0| : a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2) : |0\rangle = \langle0|a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)|0\rangle = 0\,. \tag{4.2.20}$$

Contractions and Wick's Theorems

Consider a collection of fields $\varphi_a(x), \varphi_b(y) \ldots$ in the interaction picture (hence free) constructed out of creation and annihilation operators. Recall that

$$
\begin{aligned}
\varphi_a(x) &= \int (dk) \left(a(\mathbf{k}) e^{-ik\cdot x} + a^\dagger(\mathbf{k}) e^{ik\cdot x} \right), \\
&= \varphi_a^+(x) + \varphi_a^-(x), \\
&\sim a + a^\dagger,
\end{aligned}
\tag{4.2.21}
$$

where in the second line we used the definition of the positive and negative frequency parts of the field (see section 3.2.2) $\varphi_a^+(x) = \int (dk) a(\mathbf{k}) e^{-ik\cdot x}$ and $\varphi_a^-(x) = \int (dk) a^\dagger(\mathbf{k}) e^{ik\cdot x}$. The third line is just short hand to remind us of the relevant operator structure. For example $\varphi_b \sim b + b^\dagger$ and so on. Keep in mind that $\varphi_b^+ \sim b$ whereas $\varphi_b^- \sim b^\dagger$ and so on for each field. Finally, from the section 3.2.2, recall that $[\varphi_a^+(x), \varphi_b^-(y)] = \delta_{ab} \int (dk) e^{-ik\cdot(x-y)} = \delta_{ab} \Delta^+(x-y)$.

As a warm-up, let us first consider the the product of two fields:

$$
\begin{aligned}
\varphi_a(x)\varphi_b(y) &= \left(\varphi_a^+(x) + \varphi_a^-(x) \right) \left(\varphi_b^+(y) + \varphi_b^-(y) \right), \\
&= \underbrace{\varphi_a^+(x)\varphi_b^+(y) + \varphi_a^-(x)\varphi_b^-(y) + \varphi_a^-(x)\varphi_b^+(y)}_{\text{normal ordered}} + \underbrace{\varphi_a^+(x)\varphi_b^-(y)}_{\text{not normal ordered}}, \\
&= \underbrace{: \varphi_a(x)\varphi_b(y) :}_{\text{normal ordered}} + \underbrace{[\varphi_a^+(x), \varphi_b^-(y)]}_{\text{a c-number}}, \\
&= : \varphi_a(x)\varphi_b(y) : + \delta_{ab}\Delta^+(x-y).
\end{aligned}
\tag{4.2.22}
$$

We define the *Wick Contraction* of two fields:

$$
\contraction{}{\varphi_a}{(x)}{\varphi_b} \varphi_a(x)\varphi_b(y) \equiv \delta_{ab}\Delta^+(x-y) = [\varphi_a^+(x), \varphi_b^-(y)].
\tag{4.2.23}
$$

The Wick contraction of fields satisfies the following useful properties: It is a *c*-number,

$$
\contraction{}{\varphi_a}{(\ldots)}{\varphi_b} \varphi_a(\ldots)\varphi_b = \contraction{}{\varphi_a}{}{\varphi_b} \varphi_a\varphi_b(\ldots), \qquad \text{and} \qquad \contraction{}{\varphi_a}{(x)}{\varphi_b} \varphi_a(x)\varphi_b(y) \neq \contraction{}{\varphi_b}{(y)}{\varphi_a} \varphi_b(y)\varphi_a(x).
\tag{4.2.24}
$$

Thus, for two fields we can write an ordinary product in terms of a Normal product and a Wick contraction.

$$
\varphi_a(x)\varphi_b(y) = : \varphi_a(x)\varphi_b(y) : + \contraction{}{\varphi_a}{(x)}{\varphi_b} \varphi_a(x)\varphi_b(y).
\tag{4.2.25}
$$

3-fields: Let us move two a three field example $\varphi_a(x)\varphi_b(y)\varphi_c(z)$. This will be a bit of work, but once we do it, the pattern will become obvious. To derive an expression for the ordinary product of three fields in terms of normal ordered product and Wick contractions, start with the two field result.

$$
\varphi_a\varphi_b = : \varphi_a\varphi_b : + \contraction{}{\varphi_a}{}{\varphi_b} \varphi_a\varphi_b.
\tag{4.2.26}
$$

Multiplying *both* sides of eq. (4.2.26) by φ_c^+ on the right, we get

$$
\varphi_a\varphi_b\varphi_c^+ = : \varphi_a\varphi_b : \varphi_c^+ + \contraction{}{\varphi_a}{}{\varphi_b} \varphi_a\varphi_b\varphi_c^+.
\tag{4.2.27}
$$

Now multiply both sides of eq. (4.2.26) by φ_c^- on the left, to get

$$
\varphi_c^- \varphi_a\varphi_b = \varphi_c^- : \varphi_a\varphi_b : + \varphi_c^- \contraction{}{\varphi_a}{}{\varphi_b} \varphi_a\varphi_b.
\tag{4.2.28}
$$

Now, rewrite the left hand side of eq. (4.2.28) $\varphi_c^- \varphi_a\varphi_b$, by commuting φ_c^- past φ_a to get

$$
\varphi_c^- \varphi_a\varphi_b = \varphi_a\varphi_c^- \varphi_b + [\varphi_c^-, \varphi_a]\varphi_b = \varphi_a\varphi_c^- \varphi_b - \contraction{}{\varphi_a}{}{\varphi_c} \varphi_a\varphi_c\varphi_b.
\tag{4.2.29}
$$

In the last equality we used $[\varphi_c^-, \varphi_a] = [\varphi_c^-, \varphi_a^+] = -[\varphi_a^+, \varphi_c^-] = -\underbrace{\varphi_a\varphi_c}$. Now, commute φ_c^- past φ_b on the rightmost expression in eq. (4.2.29), to get

$$\varphi_c^-\,\varphi_a\varphi_b = \varphi_a\varphi_b\varphi_c^- - \varphi_a\underbrace{\varphi_b\varphi_c} - \underbrace{\varphi_a\varphi_c}\varphi_b\,. \tag{4.2.30}$$

Substituting eq. (4.2.30) in eq. (4.2.28) and then adding the resulting equation to eq. (4.2.27), we get

$$\underbrace{\varphi_a\varphi_b\varphi_c^+ + \varphi_a\varphi_b\varphi_c^-}_{\varphi_a\varphi_b\varphi_c} - \varphi_a\underbrace{\varphi_b\varphi_c} - \underbrace{\varphi_a\varphi_c}\varphi_b = \,:\varphi_a\varphi_b:\varphi_c^+ + \underbrace{\varphi_a\varphi_b}\varphi_c^+ + \varphi_c^-:\varphi_a\varphi_b: + \varphi_c^-\underbrace{\varphi_a\varphi_b}\,. \tag{4.2.31}$$

On the right hand side, combine terms the two terms with normal ordering, and the two terms with Wick contractions separately, to yield

$$\varphi_a\varphi_b\varphi_c - \varphi_a\underbrace{\varphi_b\varphi_c} - \underbrace{\varphi_a\varphi_c}\varphi_b = \,:\varphi_a\varphi_b\varphi_c: + \underbrace{\varphi_a\varphi_b}\varphi_c\,, \tag{4.2.32}$$

which upon re-arranging yield the desired result:

$$\varphi_a\varphi_b\varphi_c = \,:\varphi_a\varphi_b\varphi_c: + \underbrace{\varphi_a\varphi_b}\varphi_c + \underbrace{\varphi_a\varphi_b\varphi_c} + \varphi_a\underbrace{\varphi_b\varphi_c}\,. \tag{4.2.33}$$

Exercise 4.2.3 : Derive the following result for four fields starting with the three field result.

$$\varphi_a\varphi_b\varphi_c\varphi_d = \,:\varphi_a\varphi_b\varphi_c\varphi_d:$$
$$+ \,:\underbrace{\varphi_a\varphi_b}\varphi_c\varphi_d: + \,:\underbrace{\varphi_a\varphi_b\varphi_c}\varphi_d: + \,:\underbrace{\varphi_a\varphi_b\varphi_c\varphi_d}: + \,:\varphi_a\underbrace{\varphi_b\varphi_c}\varphi_d: + \,:\varphi_a\underbrace{\varphi_b\varphi_c\varphi_d}: + \,:\varphi_a\varphi_b\underbrace{\varphi_c\varphi_d}:$$
$$+ \underbrace{\varphi_a\varphi_b}\underbrace{\varphi_c\varphi_d} + \underbrace{\varphi_a\varphi_b\varphi_c\varphi_d} + \varphi_a\underbrace{\varphi_b\varphi_c}\varphi_d$$

There is a pattern here. Wick's Theorem is just the generalized version of the above examples, and can be proved by induction.

Wick's Theorem for Ordinary Products

$$\varphi_a\varphi_b\cdots\varphi_z = \,:\varphi_a\varphi_b\varphi_c\cdots\varphi_z:$$
$$+ \sum_{\text{single Wick cont.}} :\underbrace{\varphi_a\varphi_b}\cdots\varphi_z: + \ldots$$
$$+ \sum_{\text{double Wick cont.}} :\underbrace{\varphi_a\varphi_b}\underbrace{\varphi_c\cdots\varphi_z}: + \ldots \tag{4.2.34}$$
$$+ \sum_{\text{triple Wick cont.}} \ldots$$
$$\vdots$$

In words, Wick's theorem for ordinary products states that an ordinary product can be written the sum of all possible pairings $\underbrace{\varphi\varphi}$ within normal products (including no pairings). To preserve your sanity, always write the fields in the same order as the ordinary product.

Why is this going to be useful? Note that since the normal ordered product always has annihilation operators on the right and creation operators on the left, the vacuum expectation value of all the normal ordered products is zero, unless there are no fields left after the Wick contractions (this means we need an even number of fields). For the example with three fields, the vacuum expectation value $\langle 0|\varphi_a\varphi_b\varphi_c|0\rangle = 0$,

whereas for the two field and four field examples, the fully contracted expressions are the only terms that contribute to the vev (vacuum expectation value). Explicitly,

$$\langle 0|\varphi_a\varphi_b|0\rangle = \contraction{}{\varphi}{_a}{\varphi}\varphi_a\varphi_b\,, \qquad \langle 0|\varphi_a\varphi_b\varphi_c|0\rangle = 0\,, \qquad \text{and} \qquad \langle 0|\varphi_a\varphi_b\varphi_c\varphi_d|0\rangle = \varphi_a\varphi_b\varphi_c\varphi_d + \varphi_a\varphi_b\varphi_c\varphi_d + \varphi_a\varphi_b\varphi_c\varphi_d\,,$$

and so on. This is nice, but remember that we actually needed vevs for time ordered products of fields. So, we need a Wick's theorem for time ordered products.

Wick's Theorem for Time-Ordered Products

In words, Wick's theorem for time-ordered products states that a time-ordered product can be written the sum of all possible pairings $\contraction{}{\varphi}{}{\varphi}\varphi\varphi$ within normal products (including no pairings). Explicitly

$$\begin{aligned}
T\{\varphi_a\varphi_b\ldots\varphi_z\} &= \;:\varphi_a\varphi_b\varphi_c\ldots\varphi_z:\\
&+ \sum_{\text{single Feynman cont.}} :\varphi_a\varphi_b\ldots\varphi_z: + \ldots\\
&+ \sum_{\text{double Feynman cont.}} :\varphi_a\varphi_b\varphi_c\ldots\varphi_z: + \ldots\\
&+ \sum_{\text{triple Feynman cont.}} \ldots\\
&\qquad\qquad \vdots
\end{aligned} \tag{4.2.35}$$

where the *Feynman* contraction is defined in terms of the Wick contraction as follows:

$$\contraction{}{\varphi}{_a(x)}{\varphi}\varphi_a(x)\varphi_b(y) = \begin{cases} \contraction{}{\varphi}{_a(x)}{\varphi}\varphi_a(x)\varphi_b(y) & x^0 > y^0\,,\\ \contraction{}{\varphi}{_b(y)}{\varphi}\varphi_b(y)\varphi_a(x) & y^0 > x^0\,. \end{cases} \tag{4.2.36}$$

This definition follows immediately from writing down $T\{\varphi_a(x),\varphi_b(y)\}$ in terms of Normal ordered products and Wick contractions for the $x^0 > y^0$ and the $x^0 < y^0$ cases separately. Unlike the Wick contraction case where $\contraction{}{\varphi}{_a(x)}{\varphi}\varphi_a(x)\varphi_b(y) \neq \varphi_b(y)\varphi_a(x)$, for the Feynman contraction, $\contraction{}{\varphi}{_a(x)}{\varphi}\varphi_a(x)\varphi_b(y) = \contraction{}{\varphi}{_b(y)}{\varphi}\varphi_b(y)\varphi_a(x)$.

4.2.3 Feynman Propagator

Note that $\contraction{}{\varphi}{_a(x)}{\varphi}\varphi_a(x)\varphi_b(y) = \delta_{ab}\Delta^+(x-y)$ and $\contraction{}{\varphi}{_b(y)}{\varphi}\varphi_b(y)\varphi_b(x) = \delta_{ab}\Delta^+(y-x) = \delta_{ab}\Delta^-(x-y)$ where $\Delta^\pm(x) \equiv \int (dk)e^{\mp ik\cdot x}$. With this in mind, let us consider the vev of the time-ordered product of two identical fields (drop the a and b labels):

$$\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = \contraction{}{\varphi}{(x)}{\varphi}\varphi(x)\varphi(y) = \theta(x^0 - y^0)\Delta^+(x-y) + \theta(y^0 - x^0)\Delta^-(x-y)\,, \tag{4.2.37}$$

where we have combined the two "branches" of the Feynman contraction using Heavyside functions, and used the fact that vevs of normal ordered products are zero. The *Feynman Propagator* is defined in terms of this time ordered product

$$i\Delta_F(x-y) \equiv \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = \theta(x^0 - y^0)\Delta^+(x-y) + \theta(y^0 - x^0)\Delta^-(x-y)\,. \tag{4.2.38}$$

The Feynman propagator is a Green's function of the Klein-Gordon equation, that is

$$(\partial^2 + m^2)\Delta_F(x-y) = -\delta^{(4)}(x-y)\,. \tag{4.2.39}$$

where $\partial^2 = \partial_\mu\partial^\mu$, with $\partial_\mu = \partial/\partial x^\mu$ and $\delta^{(4)}(x-y)$ is a four dimensional Dirac-delta function.[6] Note that $i\Delta_F(x-y) = i\Delta_F(y-x)$.

---

[6] Recall that in section 3.2.3, we had come across the Retarded Green's function $i\Delta_R(x-y) = \theta(x^0 - y^0)\langle 0|[\varphi(x),\varphi(y)]|0\rangle$ which also satisfied the above equation. Think about what the difference between Δ_F and Δ_R is.

$$\Im(k^0)$$

$-\omega_{\mathrm{k}} + i\dfrac{\epsilon}{2\omega_{\mathrm{k}}}$

$\Re(k^0)$

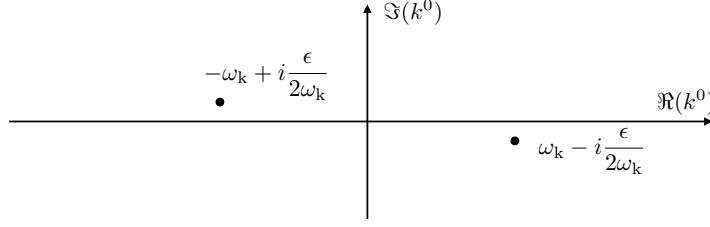$\omega_{\mathrm{k}} - i\dfrac{\epsilon}{2\omega_{\mathrm{k}}}$

Figure 4.1

Feynman Propagator in Momentum Space

The Feynman Propagator will play an essential role in our calculations of scattering/decay processes. The 4-dimensional Fourier transform of the Feynman propagator $i\Delta_F(k)$ is simpler to deal with than the $i\Delta_F(x)$. Let us calculate an explicit form for $i\Delta_F(k)$.

Let us remind ourselves of the Fourier transform definitions, and the definition of $\Delta_F(x)$:

$$\Delta_F(x) = \int d^4\bar{k}\, e^{-ik\cdot x}\Delta_F(k) \qquad \text{and} \qquad \Delta_F(k) = \int d^4x\, e^{ik\cdot x}\Delta_F(x)\,, \tag{4.2.40}$$

$$\Delta_F(x) = -i\left\{\theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^-(x)\right\}\,,$$

where recall that $d^n\bar{k} = d^nk/(2\pi)^n$. My claim is that

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon} \qquad\qquad \epsilon \to 0^+\,. \tag{4.2.41}$$

To prove that this is the correct result, we will check that its Fourier transform yields $\Delta_F(x)$. To this end

$$
\int d^4\bar{k}\,\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} = \int d^3\bar{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \int d\bar{k}^0\,\frac{e^{-ik^0 x^0}}{(k^0)^2 - \omega_{\mathrm{k}}^2 + i\epsilon}\,,
$$
$$
= \int d^3\bar{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \int d\bar{k}^0\,\frac{e^{-ik^0 x^0}}{\left[k^0 - \left(\omega_{\mathrm{k}} - i\frac{\epsilon}{2\omega_{\mathrm{k}}}\right)\right]\left[k^0 + \left(\omega_{\mathrm{k}} - i\frac{\epsilon}{2\omega_{\mathrm{k}}}\right)\right]}\,, \tag{4.2.42}
$$

where we have used $\epsilon \to 0^+$ while factoring the denominator. The integral over k^0 has poles at $k^0 = \pm\left(\omega_{\mathrm{k}} - i\frac{\epsilon}{2\omega_{\mathrm{k}}}\right)$ in the k^0-complex plane. See Fig. 4.1

If $x^0 > 0$, then we can close the contour in the lower half of the complex plane (since $e^{-ik^0 x^0} \to 0$ as the radius of the contour goes to infinity). The only pole within this closed contour is $k^0 = \omega_{\mathrm{k}} - i\epsilon/2\omega_{\mathrm{k}}$. Using the Residue theorem,

$$
\int d^4\bar{k}\,\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} = \int d^3\bar{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{1}{2\pi}(-2\pi i) \times \left(\text{ Res at } k^0 = \omega_{\mathrm{k}}\right)\,,
$$
$$
= \int d^3\bar{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{-i}{2\omega_{\mathrm{k}}}e^{-i\omega_{\mathrm{k}} x^0}\,, \tag{4.2.43}
$$
$$
= -i\int (dk)\, e^{-ik\cdot x}\,,
$$
$$
= -i\Delta^+(x)\,.
$$

The minus sign in the application of the Residue theorem arose because of the clockwise contour. The extra 2π in the denominator came from $d\bar{k}^0 = dk^0/2\pi$.

For the case of $x^0 < 0$, we have to close the contour in the upper half plane, and we pick up the pole at $k^0 = -(\omega_k - i\epsilon/2\omega_k)$. This yields

$$
\int d^4\bar{k}\,\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} = -i\Delta^-(x)\,. \tag{4.2.44}
$$

Putting the $x^0 > 0$ and $x^0 < 0$ results together, we have

$$\int d^4k \frac{e^{-ik \cdot x}}{k^2 - m^2 + i\epsilon} = -i\{\theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^-(x)\} = \Delta_F(x). \qquad (4.2.45)$$

This completes our proof. Note that the point of the $i\epsilon$ was to guide us in the choice of poles to yield the correct Fourier transform of the Feynman propagator.

Exercise 4.2.4 Consider a general(G) Green's function of the Klein-Gordon equation: $(\partial^2 + m^2)\Delta_G(x - y) = -\delta^{(4)}(x - y)$. Using the 4-d Fourier transform, show that $\Delta_G(k) = 1/(k^2 - m^2)$. Now, start with $\Delta_G(k) = 1/(k^2 - m^2)$, and try to get an explicit expression for $\Delta_G(x - y)$ using the inverse Fourier transform. You will be faced with a choice on how to evaluate the contour integral. The contour/pole prescription we chose in Fig. 4.1 yields the Feynman Green's function $\Delta_F(x-y)$. What is the contour/pole-prescription that is needed to recover the Retarded Green's function $\Delta_R(x - y)$?

4.3 Perturbative Calculations in A Toy Model

Let us now put all of this technology to work in a toy example:

$$\mathcal{L} = |\partial\psi|^2 - M^2\psi^\dagger\psi + \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - g\varphi\psi^\dagger\psi. \qquad (4.3.1)$$

where $\mathcal{H}_{\text{int}} = g\varphi\psi^\dagger\psi$, φ is a Hermitian field, and ψ a non-Hermitian one. Recall that for perturbative calculations, we want $g \ll m, M$. For convenience, I am going to write down the mode expansions for these fields in the interaction picture (we have dropped the "I" denoting the interaction picture.)

$$\varphi(x) = \int (dk) \left(a(\mathbf{k})e^{-ik \cdot x} + a^\dagger(\mathbf{k})e^{ik \cdot x}\right),$$

$$\psi(x) = \int (dk) \left(b(\mathbf{k})e^{-ik \cdot x} + d^\dagger(\mathbf{k})e^{ik \cdot x}\right), \qquad (4.3.2)$$

$$\psi^\dagger(x) = \int (dk) \left(d(\mathbf{k})e^{-ik \cdot x} + b^\dagger(\mathbf{k})e^{ik \cdot x}\right).$$

While far from reality, if it helps, you can think of φ particles as toy photons (γ) (they are really "toys", we will even allow them to have mass m), and ψ particles as toy electrons (e^-) and positrons (e^+). Again, I want to stress that this is a toy example. Real world photons are quanta of spin 1, massless gauge fields and electrons/positrons are quanta of spin 1/2 fermionic fields. Nevertheless, the essentials of perturbative calculations will be present in our toy example without distractions (and more constraining structure) from the higher spin fields.

Recall the following shorthand of the mode expansions, and their take-away:

- $\varphi \sim a + a^\dagger$: $\quad$ $a^\dagger$ creates a γ, and a annihilates it.

- $\psi \sim b + d^\dagger$, $\quad$ $\psi^\dagger \sim d + b^\dagger$: $\quad$ $d^\dagger$ creates an e^+, and d annihilates it. Whereas $b^\dagger$ creates a e^-, and b annihilates it.

Let us consider the amplitude for the following processes: (1) $\gamma \to e^- + e^+$ (2) $e^- + e^- \to e^- + e^-$ (3) $e^- + e^+ \to e^- + e^+$ (4) $e^- + \gamma \to e^- + \gamma$.

4.3.1 Decay: "γ" $\to e^- + e^+$

We will go through this calculation of the decay amplitude step by step, in excruciating detail. But having done so once, the rest will hopefully be quicker.
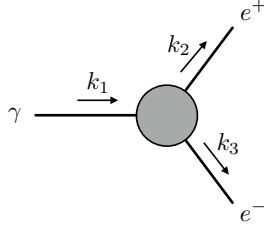
Figure 4.2: Decay of our massive scalar "photon" into a scalar "electron-positron" pair.

Initial and Final States

The first step is to write down the initial and final states.

$$|i\rangle = a^\dagger(\mathbf{k}_1)|0\rangle = |\underbrace{\mathbf{k}_1}_{\gamma}\rangle, \qquad \text{and} \qquad |f\rangle = b^\dagger(\mathbf{k}_3)d^\dagger(\mathbf{k}_2)|0\rangle = |\underbrace{\mathbf{k}_3}_{e^-}\underbrace{\mathbf{k}_2}_{e^+}\rangle. \tag{4.3.3}$$

Here we are considering initial and final states as momentum eigenstates (of the free theory). You can superpose a bunch of these to get a wavepacket if you want.

Dyson Expansion

We wish to calculate

$$\langle f|S|i\rangle = \langle f|T\{\exp\left(-i\int d^4x \mathcal{H}_I(x)\right)\}|i\rangle,$$

$$= \langle f|T\{1 + (-i)\int d^4x \mathcal{H}_I(x) + \frac{(-i)^2}{2!}\int d^4x d^4y \mathcal{H}_I(x)\mathcal{H}_I(y) + \ldots\}|i\rangle, \tag{4.3.4}$$

$$= \langle f|1 + (-ig)T\{\int d^4x(\psi^\dagger\psi\varphi)_x + \frac{(-ig)^2}{2!}T\{\int d^4x d^4y(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y\} + \ldots|i\rangle.$$

Let us look at this term by term, organized by orders in g.

- 0 order in g: On physical grounds, $\langle f|1|i\rangle$ should be zero because if there are no interactions, how can γ decay into e^- and e^+? For the sake of practice, let us compute this overlap and confirm:

$$\langle f|1|i\rangle = \langle 0|b(\mathbf{k}_3)d(\mathbf{k}_2)a^\dagger(\mathbf{k}_1)|0\rangle = \langle 0|a^\dagger(\mathbf{k}_1)b(\mathbf{k}_3)d(\mathbf{k}_2)|0\rangle = 0. \tag{4.3.5}$$

  In the second equality, since $a^\dagger$ commutes with bd, there is nothing preventing up from moving it through to the left. As a result either b or d can act on the right vacuum to give 0 or we can have $a^\dagger$ acting on the left vacuum state to yield zero.

- 1st order in g. The term we wish to know is

$$(-ig)\langle f|T\{\int d^4x(\psi^\dagger\psi\varphi)_x\}|i\rangle = (-ig)\int d^4x\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_2)(\psi^\dagger\psi\varphi)_x a^\dagger(\mathbf{k}_1)\}|0\rangle. \tag{4.3.6}$$

  Note that the time ordering symbol is innocuous for this term, which is already time ordered. The interaction in the "middle" in terms of time. Moreover the interaction part are all evaluated at the same x.[7]

---

[7]For next order in g terms, time ordering will play a more significant role.

Contractions

Let us evaluate the following time-ordered vev using Wick's theorem:

$$\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1)\}|0\rangle = \overline{b(\mathbf{k}_3)d(\mathbf{k}_2)\overline{\psi^\dagger(x)\psi(x)}\overline{\varphi(x)a^\dagger(\mathbf{k}_1)}} + \overline{b(\mathbf{k}_3)d(\mathbf{k}_2)\overline{\psi^\dagger(x)}\psi(x)\overline{\varphi(x)a^\dagger(\mathbf{k}_1)}}$$
$$+ \text{ all possible contractions (with no non-contracted fields)}.$$

where we have used Wick's theorem for time ordered fields, and the fact that vevs of normal ordered fields are 0. The types of contractions we have to deal with include

$$
\begin{aligned}
\overline{\varphi(x)a^\dagger}(\mathbf{k}_1) &= \langle 0|T\{\varphi(x)a^\dagger(\mathbf{k}_1)\}|0\rangle , \\
&= \langle 0|\varphi(x)a^\dagger(\mathbf{k}_1)|0\rangle, \\
&= \int (dq)e^{-iq\cdot x}\langle 0|a(\mathbf{q})a^\dagger(\mathbf{k}_1)|0\rangle , \\
&= \int (dq)e^{-iq\cdot x}\langle 0|a^\dagger(\mathbf{k}_1)a(\mathbf{q}) + [a(\mathbf{q}), a^\dagger(\mathbf{k}_1)]|0\rangle , \\
&= e^{-ik_1\cdot x} .
\end{aligned}
\tag{4.3.7}
$$

The second line is the innocuousness of time ordering for this term. The third line results from writing down the mode expansion for $\varphi(x)$; $\langle 0|a^\dagger(\mathbf{q}) = 0$ is why $a^\dagger(\mathbf{q})$ does not appear in the expansion. In the fourth line, we commuted a to the right past the $a^\dagger$, paying the price of doing so with the commutator. Finally, in the fifth line, we used the commutation relation $[a(\mathbf{q}), a^\dagger(\mathbf{k}_1)] = 2\omega_{k_1}\bar{\delta}^{(3)}(\mathbf{q}-\mathbf{k}_1)$ and integrated w.r.t $\int (dq)$. Note that final result is in terms of the initial momentum k_1 of the incoming particle.

Now note that contractions $\langle 0|T\{\varphi_a\varphi_b\}|0\rangle = \overline{\varphi_a\varphi_b} \propto \delta_{ab}$. Hence, if a $\neq$ b (i.e different field), we get no contribution. This implies that all contractions : $\overline{ba}$, $\overline{b\varphi}$, $\overline{d\varphi}$, $\overline{da},\overline{\varphi\psi}$ and $\overline{\psi^\dagger\varphi}$ vanish. Moreover, since $\psi \sim b + d^\dagger$ and $\psi^\dagger \sim d + b^\dagger$, only $\overline{b\psi^\dagger}$ and $\overline{d\psi}$ survive. Thus the only surviving contributions are

$$
\begin{aligned}
\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1)\}|0\rangle &= \overline{b(\mathbf{k}_3)d(\mathbf{k}_2)\overline{\psi^\dagger(x)\psi(x)}\overline{\varphi(x)a^\dagger(\mathbf{k}_1)}} , \\
&= \langle 0|b(\mathbf{k}_3)\psi^\dagger(x)|0\rangle \times \langle 0|d(\mathbf{k}_2)\psi(x)|0\rangle \times \langle 0|\varphi(x)a^\dagger(\mathbf{k}_1)|0\rangle , \\
&= e^{ik_3\cdot x}e^{ik_2\cdot x}e^{-ik_1\cdot x} .
\end{aligned}
\tag{4.3.8}
$$

Notice the sign difference between the incoming and outgoing momenta: k_2, k_3 are outgoing, whereas k_1 is incoming.

The Matrix Elements

Thus we have arrived at

$$(-ig)\langle f|T \int d^4x(\psi^\dagger\psi\varphi)_x|i\rangle = (-ig) \int d^4x e^{-i(k_3+k_2-k_1)\cdot x} = (-ig)\bar{\delta}^{(4)}(k_1 - k_2 - k_3) . \tag{4.3.9}$$

Let us stop at this first nontrivial result which appears at first order in g (for dragons lurk at higher orders). Combining this result with the 0th order result, we arrive at

$$\langle f|S - 1|i\rangle = (-ig)\bar{\delta}^{(4)}(k_1 - k_2 - k_3) + \mathcal{O}[g^2] . \tag{4.3.10}$$

Recall from section 4.2.1 where we discussed properties of the S-matrix, that $S = 1 - i\bar{\delta}^{(4)}(p_i - p_f)\mathcal{M}$ where p_i and p_f are the 4-momenta of the initial and final states. The 4-dimensional delta function which

was meant to impose energy-momentum conservation in the process is precisely what we found in our explicit calculation: $\bar{\delta}^{(4)}(k_1 - k_2 - k_3)$. Thus, at leading order in g, we have

$$\langle f|\mathcal{M}|i\rangle = g + \mathcal{O}[g^2]. \tag{4.3.11}$$

An exceptionally simple result! The factors of i and signs were all put in (with hindsight) to make the final results look nice.

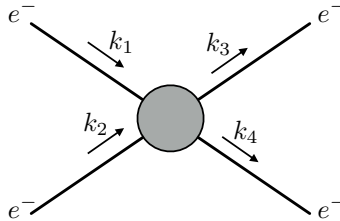## 4.3.2   Scattering: $e^- + e^- \to e^- + e^-$

Figure 4.3: Scattering of two "electrons" off of each other.

Let us now calculate the amplitude for the following scattering process at the leading non-trivial order in g. As with our decay calculation, we will proceed systematically, but now without loitering around for all the details.

Initial and Final States

$$|i\rangle = b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)|0\rangle = |\mathbf{k}_1\mathbf{k}_2\rangle \qquad \text{and} \qquad |f\rangle = b^\dagger(\mathbf{k}_3)b^\dagger(\mathbf{k}_4)|0\rangle = |\mathbf{k}_3\mathbf{k}_4\rangle, \tag{4.3.12}$$

where we will assume that $\mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{k}_3, \mathbf{k}_4$.

Dyson Expansion

Since $\mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{k}_3, \mathbf{k}_4$, we immediately have $\langle \mathbf{k}_3\mathbf{k}_4|\mathbf{k}_1\mathbf{k}_2\rangle = 0$. Hence we can directly write down the parts of the scattering amplitude with g dependence:

$$\langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle$$

$$\begin{aligned}
&= (-ig) \int d^4x \langle 0|T\{b(\mathbf{k}_3)b(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)\}|0\rangle \\
&+ \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0|T\{b(\mathbf{k}_3)b(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)\}|0\rangle, \\
&+ \mathcal{O}[g^3].
\end{aligned} \tag{4.3.13}$$

Contractions

Consider the $\mathcal{O}[g]$ term. It has an odd number of operators. Which means the time ordered vev. of these operators will be zero. We must calculate the $\mathcal{O}[g^2]$ term. Consider $\langle 0|T\{b^\dagger(\mathbf{k}_3)b^\dagger(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b(\mathbf{k}_1)b(\mathbf{k}_2)\}|0\rangle$. The number of possible complete contractions are enormous, but most will have vanishing contributions. What sorts of contractions have non-vanishing contributions?

Note that any complete contraction with a non-vanishing contribution must include $\overline{\varphi(x)\varphi(y)}$ because none of the other fields (including the initial and final states) contain any part of the φ field (recall that $\overline{\varphi_a\varphi_b} \propto \delta_{ab}$). Moreover, since $\psi \sim b + d^\dagger$ and $\psi^\dagger \sim d + b^\dagger$, any b must contract with $\psi^\dagger$ and $b^\dagger$ with ψ. There are four possible ways of doing this:

1.

$$\overbrace{b(\mathbf{k}_3)b(\mathbf{k}_4)\psi^\dagger(x)\psi(x)\varphi(x)\psi^\dagger(y)\psi(y)\varphi(y)b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)}\,. \tag{4.3.14}$$

2. Same as 1., but with $\overset{\frown}{b(\mathbf{k}_3)\psi^\dagger}(x) \to \overset{\frown}{b(\mathbf{k}_3)\psi^\dagger}(y)$, and $\overset{\frown}{b(\mathbf{k}_4)\psi^\dagger}(y) \to \overset{\frown}{b(\mathbf{k}_4)\psi^\dagger}(x)$.

3. Same as 1., with $x \leftrightarrow y$.

4. Same as 2., with $x \leftrightarrow y$.

Let us take a closer look at individual two field contractions that appear in the above expressions.

- The $\varphi\varphi$ contraction is nothing but the Feynman propagator:

$$\overset{\frown}{\varphi(x)\varphi}(y) = \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = i\Delta_\varphi(x - y) \qquad \text{Feynman Propagator!} \tag{4.3.15}$$

- The ψ contraction with any (incoming) $b^\dagger$ yields:

$$\overset{\frown}{\psi(x)b^\dagger}(\mathbf{k}_j) = e^{-ik_j \cdot x} \qquad \text{incoming} \tag{4.3.16}$$

- The $\psi^\dagger$ contraction with any (outgoing) b yields:

$$\overset{\frown}{b(\mathbf{k}_j)\psi^\dagger}(x) = e^{ik_j \cdot x} \qquad \text{outgoing} \tag{4.3.17}$$

Matrix Element

Using the above calculated building blocks,

$$\langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle = \frac{(-ig)^2}{2!} \int d^4x d^4y\, i\Delta_\varphi(x - y) \left(e^{ik_3 \cdot x}e^{ik_4 \cdot y}e^{-ik_1 \cdot x}e^{-ik_2 \cdot y} + e^{ik_3 \cdot y}e^{ik_4 \cdot x}e^{-ik_1 \cdot x}e^{-ik_2 \cdot y} + x \leftrightarrow y \right)$$

The $x \leftrightarrow y$ simply doubles the contribution, cancelling the 2! (this kind of stuff happens a lot). We end up with

$$\langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y\, i\Delta_\varphi(x - y) \left(e^{-i(k_1 - k_3) \cdot x}e^{-i(k_2 - k_4) \cdot y} + e^{-i(k_1 - k_4) \cdot x}e^{-i(k_2 - k_3) \cdot y} \right)\,. \tag{4.3.18}$$

This is the nontrivial part of the scattering amplitude at order $\mathcal{O}[g^2]$; we will get to higher order contributions later.

I will take this opportunity to introduce one of the more elegant tools of QFT: *Feynman Diagrams*[8] Julian Schwinger, one of the most prominent contributors to the development of QFT said:

"*Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses.*"

Feynman Rules in x-space

There is powerful graphical way of representing the matrix element in eq. (4.3.18). Let me write down *x-space Feynman Rules* for our theory in Fig. 4.4 Using these rules, we now depict the two terms in eq. (4.3.18) as shown below in Fig. 4.5. Note that there is no accumulation of charge at the vertices. Let us try to (heuristically) say in words what is going on in the process, and connect them to various parts of the mathematical expression for the amplitude.

---

[8]For a gentle introduction to Feynman diagrams (suitable even if you have never taken field theory course), can be found at http://www.quantumdiaries.org/2010/02/14/lets-draw-feynman-diagams/. If you are interested in the relevance of Feynman diagrams in post WW2 era physics, see the book *Drawing Theories Apart* by D. Kaiser.
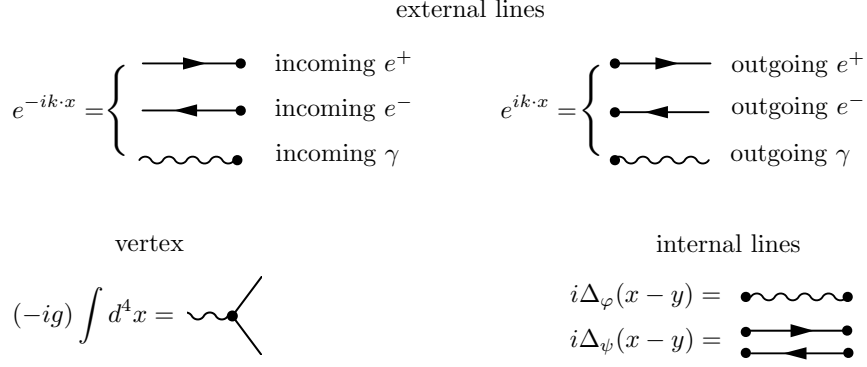
Figure 4.4: x−space Feynman rules for our theory with $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\varphi$.

Figure 4.5: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^- \to e^- + e^-$ scattering amplitude: $\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y\, i\Delta_\varphi(x-y)\left(e^{-i(k_1-k_3)\cdot x}e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x}e^{-i(k_2-k_3)\cdot y}\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in x-space shown in Fig. 4.4 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the amplitude.

- "Electrons" with momenta k_1 and k_2 came in and exchanged a "photon", and came out with momenta k_3 and k_4.

- The integrals $\int d^4x$ and $\int d^4y$ sums over all the locations x and y where the "photon" was exchanged.

- The propagator $\Delta_\varphi(x-y)$ is symmetric in x and y and can be interpreted at the amplitude for the "photon" going from x to y and y to x. This is internal (the "photon" is virtual), and cannot be observed directly.

- The two diagrams are *topologically distinct*. You cannot twist and bend the lines to get from one to another.

Doing calculations in x-space is fine, and sometimes necessary. Nevertheless, it is in Fourier space that the calculations are the simplest.

Feynman Rules in k-space

We can simplify the expression for the matrix element significantly by writing down the Feynman propagator in Fourier space: $\Delta_\varphi(x-y) = \int d^4k\, e^{-ik\cdot(x-y)}\Delta_\varphi(k)$ to get

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y d^4k\, i\Delta_\varphi(k)e^{-ik\cdot(x-y)}\left(e^{-i(k_1-k_3)\cdot x}e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x}e^{-i(k_2-k_3)\cdot y}\right).$$

Now integrate over d^4x and d^4y to get a bunch of delta functions:

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4k\, i\Delta_\varphi(k)\{\delta^{(4)}(k_1-k_3+k)\delta^{(4)}(k_2-k_4-k) + \delta^{(4)}(k_1-k_4+k)\delta^{(4)}(k_2-k_3-k)\}\,,$$

(4.3.19)

Once again, this expression can be depicted graphically using Feynman diagrams. Since the expressions are in Fourier space, let me now provide the Feynman rules for our theory in k-space. Using these rules,
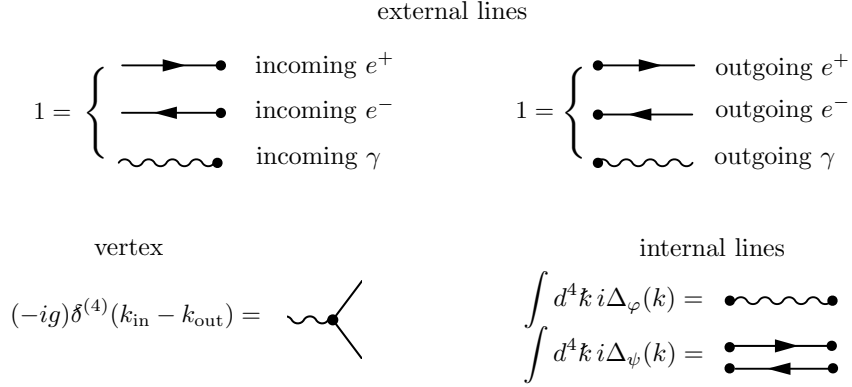

Figure 4.6: $k-$space Feynman Rules for our theory with $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\varphi$.

the matrix element in eq. (4.3.19) can be represented again in terms of Feynman diagrams (see Fig. 4.7). Two diagrams corresponds to the two terms in the expression for the matrix element. You can make the correspondence by carefully looking at the delta functions.

Figure 4.7: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^- \rightarrow e^- + e^-$ matrix element: $\langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y d^4\bar{k}\, i\Delta_\varphi(k)e^{-ip\cdot(x-y)}\left(e^{-i(k_1-k_3)\cdot x}e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x}e^{-i(k_2-k_3)\cdot y}\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in k-space shown in Fig. 4.6 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the matrix element. Note that the choice of direction of the internal momentum k is arbitrary, you just have to be consistent at both vertices once the direction is chosen.

We can even go further here. First note that the products of Dirac-delta functions appearing in eq. (4.3.19) can be combined to yield a momentum conserving delta function that can be moved out of the intergals:

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\delta^4(k_1+k_2-k_3-k_4) \int d^4k\, i\Delta_\varphi(k)\{\delta^{(4)}(k_1-k_3+k) + \delta^{(4)}(k_2-k_3-k)\}\,,$$

(4.3.20)

Now, recall that $-i\delta^{(4)}(k_1+k_2-k_3-k_4)\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle = \langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle$, hence (finally!) the interesting part of our matrix element for this scattering process at order g^2 is

$$-i\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4k\, i\Delta_\varphi(k)\{\delta^{(4)}(k_1-k_3+k) + \delta^{(4)}(k_2-k_3-k)\}\,.$$

(4.3.21)

Furthermore, since we know the form of the momentum-space propagator $i\Delta_\varphi(k) = i/(k^2 - m^2)$. The martrix element then becomes

$$\langle \mathbf{k}_3 \mathbf{k}_4 | \mathcal{M} | \mathbf{k}_1 \mathbf{k}_2 \rangle = g^2 \left[\frac{1}{(k_1 - k_3)^2 - m^2} + \frac{1}{(k_2 - k_3)^2 - m^2} \right]. \tag{4.3.22}$$

We will come back and relate this matrix element to cross section of scattering later. For the moment let us continue calculating matrix elements for different types of scattering.

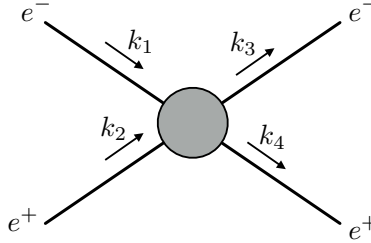### 4.3.3 Scattering: $e^+ + e^- \to e^+ + e^-$

Figure 4.8: "electron"-"positron" scattering.

We will once again carry out the following steps. (1) Write down the initial and final states in terms of creation and annihilation operators of the free fields. (2) Write down the Dyson expansion for the relevant matrix element, and expand to the required non-trivial order in g. (3) Use Wick's theorem; write down the relevant contractions. (4) Write down the expression for the matrix element in position and Fourier space.

Initial and Final States

$$|i\rangle = b^\dagger(\mathbf{k}_1)d^\dagger(\mathbf{k}_2)|0\rangle = | \underbrace{\mathbf{k}_1}_{e^-} \underbrace{\mathbf{k}_2}_{e^+} \rangle \qquad \text{and} \qquad |f\rangle = b^\dagger(\mathbf{k}_3)d^\dagger(\mathbf{k}_4)|0\rangle = | \underbrace{\mathbf{k}_3}_{e^-} \underbrace{\mathbf{k}_4}_{e^+} \rangle, \tag{4.3.23}$$

where for simplicity, we will assume $\mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{k}_3, \mathbf{k}_4$.

Dyson Expansion

The part of the scattering amplitude with g dependence (note that the $\mathcal{O}[g]$ term is zero):

$$\langle \mathbf{k}_3 \mathbf{k}_4 | S - 1 | \mathbf{k}_1 \mathbf{k}_2 \rangle = \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0 | T\{ b(\mathbf{k}_3) d(\mathbf{k}_4) (\psi^\dagger \psi \varphi)_x (\psi^\dagger \psi \varphi)_y b^\dagger(\mathbf{k}_1) d^\dagger(\mathbf{k}_2) \} | 0 \rangle + \mathcal{O}[g^3]. \tag{4.3.24}$$

Contractions

$$\langle 0 | T\{ b(\mathbf{k}_3) d(\mathbf{k}_4) (\psi^\dagger \psi \varphi)_x (\psi^\dagger \psi \varphi)_y b^\dagger(\mathbf{k}_1) d^\dagger(\mathbf{k}_2) \} | 0 \rangle$$

$$= b(\mathbf{k}_3) d(\mathbf{k}_4) (\psi^\dagger \psi \varphi)_x (\psi^\dagger \psi \varphi)_y b^\dagger(\mathbf{k}_1) d^\dagger(\mathbf{k}_2) + x \leftrightarrow y \tag{4.3.25}$$

$$+ b(\mathbf{k}_3) d(\mathbf{k}_4) (\psi^\dagger \psi \varphi)_x (\psi^\dagger \psi \varphi)_y b^\dagger(\mathbf{k}_1) d^\dagger(\mathbf{k}_2) + x \leftrightarrow y$$

Note that $\overset{\frown}{\varphi\varphi}$ contraction is essential since the $|i\rangle$ and $|f\rangle$ do not contain and $a, a^\dagger$. We cannot have any $\psi^\dagger\psi$ or $\psi\psi$ contractions because the operators in $|i\rangle$ and $|f\rangle$ do not yield any non-zero contractions for $|i\rangle \neq |f\rangle$.[9]

Matrix Element and Feynman Diagrams

Evaluating the expressions for the surviving contractions, we have

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y \; i\Delta_\varphi(x-y)\left(e^{-i(k_1+k_2)\cdot y}e^{i(k_3+k_4)\cdot x} + e^{-i(k_1-k_3)\cdot y}e^{-i(k_2-k_4)\cdot x}\right). \tag{4.3.26}$$

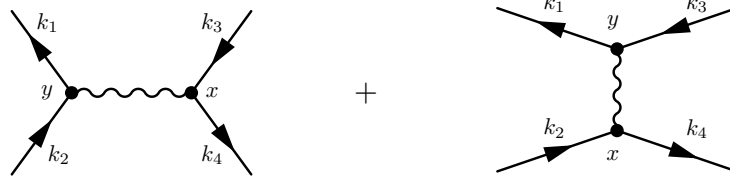Using our Feynman rules, we find that the two terms can be expressed graphically as follows: We can

Figure 4.9: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^+ \to e^- + e^+$ scattering amplitude: $\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y \; i\Delta_\varphi(x-y)\left(e^{-i(k_1+k_2)\cdot y}e^{i(k_3+k_4)\cdot x} + e^{-i(k_1-k_3)\cdot y}e^{-i(k_2-k_4)\cdot x}\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in x-space shown in Fig. 4.4 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the amplitude.

repeat the calculation in momentum space, to get

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle$$
$$= (-ig)^2 \int d^4k \; i\Delta_\varphi(k)\left(\bar\delta^{(4)}(k_1+k_2-k)\bar\delta^{(4)}(k-k_3-k_4) + \bar\delta^{(4)}(k_1+k-k_3)\bar\delta^{(4)}(k_2-k-k_4)\right). \tag{4.3.27}$$

Once again, we may represent these terms in terms of Feynman diagrams in momentum space as shown in Fig. 4.10.

Figure 4.10: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^+ \to e^- + e^+$ matrix element: $\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4\bar{k} \; i\Delta_\varphi(k)\left(\bar\delta^{(4)}(k_1+k_2-k)\bar\delta^{(4)}(k-k_3-k_4) + \bar\delta^{(4)}(k_1-k_3-k)\bar\delta^{(4)}(k_2-k-k_4)\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in k-space shown in Fig. 4.6 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the matrix element. Note that the choice of direction of the internal momentum k is arbitrary.

The expression for the scattering amplitude, can be simplified further. First, note that the products of delta functions yield the usual momentum conserving delta function for the process. Using $S = 1 -$

---

[9]If writing down these contractions gives you a bit of a headaches, be patient for a bit. We will soon get away from its clutches.

$i\delta^{(4)}(p_i - p_f)\mathcal{M}$, we have

$$\langle \mathbf{k}_3 \mathbf{k}_4 | \mathcal{M} | \mathbf{k}_1 \mathbf{k}_2 \rangle = g^2 \int d^4k \ \Delta_\varphi(k) \left(\delta^{(4)}(k - k_1 - k_2) + \delta^{(4)}(k_1 - k_3 - k) \right) ,$$

$$= g^2 \left(\frac{1}{(k_1 + k_2)^2 - m^2} + \frac{1}{(k_1 - k_3)^2 - m^2} \right) . \tag{4.3.28}$$

Exercise 4.3.1 : Consider the following scattering process: $\gamma + e^- \to \gamma + e^-$. Following the same route as in the examples above: (1) Write down the initial and final states in terms of creation and annihilation operators of the free fields. (2) Write down the Dyson expansion for the relevant matrix element, and expand to the required non-trivial order in g. (3) Use Wick's theorem; write down the relevant contractions. (4) Write down the expression for the matrix element in position and Fourier space. Draw and label the relevant Feynman diagrams in position and momentum space.

4.3.4 The Diagrammar's way

So far, we have just noted that the final expressions for the scattering amplitude can be nicely represented in terms of Feynman diagrams, but we have really not taken advantage of the graphical representation. Indeed the power of Feynman diagrams lies in going the other way: draw diagrams, and the diagrams tell you how to organize your calculations and compute amplitudes.[10]

Let me carry out this process for the theory under consideration: $\mathcal{L} = |\partial \psi|^2 - M^2 \psi^\dagger \psi + (1/2)(\partial \varphi)^2 - (1/2)m^2 \varphi^2 - g\varphi\psi^\dagger\psi$. with $\mathcal{L}_{\text{int}} = -g\varphi\psi^\dagger\psi$. It is (typically) easiest to work in momentum space.

1. We will associate squiggly lines with φ and solid lines with $\psi, \psi^\dagger$ (see Fig. 4.6). When they are external, they contribute 1 to the amplitude.

2. Consider the free part of the theory (without $\mathcal{L}_{\text{int}} = 0$). The information about this theory is contained in it's Green's functions, or the transition amplitude. Calculate these, to get:

$$i\Delta_\varphi(k) = i/(k^2 - m^2 + i\epsilon) \qquad \text{and} \qquad i\Delta_\psi(k) = i/(k^2 - M^2 + i\epsilon) \tag{4.3.29}$$

   For internal line of φ contributes $\int d^4k \, i\Delta_\varphi(k)$, whereas of $\psi, \psi^\dagger$ contributes $\int d^4k \, i\Delta_\psi(k)$.

3. The interaction part of the Lagrangian density gives you the strength of the vertex: the three fields represent the coming together of three different lines ($\varphi, \psi, \psi^\dagger$). This is your fundamental vertex. This is the only possible way in which different lines in your diagram can meet. Construct the contribution to the

$$3\text{-point vertex} = i \frac{\partial^3 \mathcal{L}_{\text{int}}}{\partial\psi\partial\psi^\dagger\partial\varphi} \times \delta^{(4)}(\Sigma k) = (-ig)\delta^{(4)}(\Sigma k) \tag{4.3.30}$$

   where the sum is over the momenta meeting at the vertex (by convention, incoming are given a positive sign). Each vertex in the diagram will contribute a factor of g.

4. Now consider the process you are interested in. Say, for example $e^-(k_1) + e^+(k_2) \to e^-(k_3) + e^+(k_4)$ with all momenta being distinct. To calculated the amplitude, draw all possible *topologically distinct* diagrams (up to the order of g you care about, which will fix the number of vertices allowed) with the external momenta fixed (endpoints pinned down). If you can get from one diagram to another

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[10]For a defense of "diagrams first" approach, read the introduction to *Diagrammar* by t'Hooft and Veltman. A recent treatment along these lines is nicely presented in the introduction(s) of *QFT I* notes by Mojzis.

by twisting, but not cutting lines, then the diagrams are the same. (Note: There are annoying combinatorial factors that come with the diagrams in many cases, but we are safe here for the model under consideration). Now, we will draw all diagrams for the process of interest using the above rules. Up to $\mathcal{O}[g^2]$, we can have the following diagrams shown in Fig. 4.11. Once you have drawn the diagrams, write down the integral expressions corresponding to the diagrams, and you are done calculating the amplitude up to that order in g.
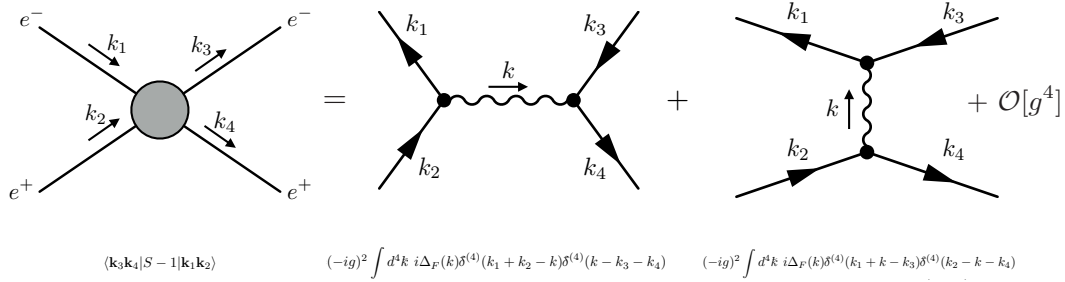

$$\langle \mathbf{k_3 k_4}|S-1|\mathbf{k_1 k_2}\rangle \qquad (-ig)^2 \int d^4k \; i\Delta_F(k)\delta^{(4)}(k_1+k_2-k)\delta^{(4)}(k-k_3-k_4) \qquad (-ig)^2 \int d^4k \; i\Delta_F(k)\delta^{(4)}(k_1+k-k_3)\delta^{(4)}(k_2-k-k_4)$$

Figure 4.11: Feynman diagrams and corresponding expressions for the scattering amplitude for the $e^-(k_1)+e^+(k_2) \to e^-(k_3)+e^+(k_4)$ process. Note that there are no diagrams you can draw with 1 or 3 vertices.

Exercise 4.3.2 : Draw the equivalent figure(s) to the one in Fig. 4.11, along with the corresponding expressions for the following processes at the leading non-trivial order in g: (1) $\gamma(k_1) \to e^+(k_2) + e^-(k_3)$ (2) $e^-(k_1)+e^-(k_2) \to e^-(k_3)+e^-(k_4)$ (3) $e^-(k_1)+\gamma(k_2) \to e^-(k_3)+\gamma(k_4)$ with none of the (individual) initial and final momenta being equal.

Exercise 4.3.3 : Consider the Lagrangian density $\mathcal{L} = (1/2)(\partial\varphi)^2-(1/2)m^2\varphi^2-(\lambda/4!)\varphi^4$ where φ is a real scalar field. Write the down the Feynman rules for this theory in momentum space, which should include (1) rules for incoming and outgoing external lines, (2) propagator for internal lines and (3) the contribution from the vertex. "Follow your nose" in terms of defining the strength of the vertex by referring to the examples we have already considered. (4) Now draw the Feynman diagrams and write the corresponding expression for a $\varphi(k_1)+\varphi(k_2) \to \varphi(k_3)+\varphi(k_4)$ process. Again assume all momenta are distinct, and only consider the leading order diagrams in λ.

4.3.5 Amplitudes to Observables

While we have been pretending otherwise, amplitudes are not "directly" observable. Let us now compute observables like decay rates, scattering cross sections etc. using our scattering amplitudes. The interesting part of the processes like the strength of interaction, momentum dependence etc. are contained in the amplitude, what remains is to be done in putting in kinematics related to the final states (the "phase space" factors). This can be done in a nice and formal way. We will skip the derivation, and just state the answers here for the processes of interest: two-body decay and two-to-two scattering. [11]

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[11] You can find the derivation, for example in Paul Stevenson's notes or in any textbook on QFT or Particle Physics. For a nice introduction, with detailed examples, see chapter 6 in *Introduction to Elementary Particles* by the master of undergraduate textbooks, David J. Griffiths.

Decay rate

For the decay of a particle with mass m_1 (at rest) into two particles with masses m_2 and m_3, the decay rate

$$\Gamma_{1\to2+3} = S \frac{|\mathbf{k}|}{8\pi m_1^2} |\langle f|\mathcal{M}|i\rangle|^2 . \tag{4.3.31}$$

where $\mathbf{k} = (2m_1)^{-1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$ is the momentum of either outgoing particle. The factor S is related to the number of identical particles in the final state. For our $\varphi\psi^\dagger\psi$ theory and the case of $\gamma \to e^+ + e^-$, we have $S = 1$ (if they were identical $S = 1/2!$). The outgoing momenta $|\mathbf{k}| = \sqrt{m^2 - 4M^2}$. Hence

$$\Gamma_{\gamma\to e^+ e^-} = \frac{g^2}{16\pi m}\sqrt{1 - \frac{4M^2}{m^2}} \tag{4.3.32}$$

Note that $m > 2M$ is necessary for decay, as expected (φ clearly is not the electromagnetic field!). In the limit that $M \ll m$, we have $\Gamma \approx g^2/8\pi m$.

Differential cross-section

Now let us consider a $1 + 2 \to 3 + 4$ scattering in the center of momentum frame ($\mathbf{k}_1 = -\mathbf{k}_2, \mathbf{k}_3 = -\mathbf{k}_4$). The differential cross-section (cross-section per unit solid-angle for the outgoing particles),

$$\left(\frac{d\sigma}{d\Omega}\right)_{1+2\to3+4} = \frac{S}{(8\pi)^2} \frac{|\langle f|\mathcal{M}|i\rangle|^2}{(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})^2} \frac{|\mathbf{k}_f|}{|\mathbf{k}_i|} \tag{4.3.33}$$

where $\omega_{\mathbf{k}_1}$ and $\omega_{\mathbf{k}_2}$ are the energies of the incoming particles, $|\mathbf{k}_f|$ is the magnitude of the 3-momentum of either outgoing particle and $|\mathbf{k}_i|$ is the magnitude of either incoming particle. For our $\varphi\psi^\dagger\psi$ theory, and for the particular case of $e^+ + e^- \to e^+ + e^-$ scattering, we have $|\mathbf{k}_f| = |\mathbf{k}_i|$, and the formula simplifies to

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+ e^- \to e^+ e^-} = \frac{1}{64\pi^2} \frac{|\langle f|\mathcal{M}|i\rangle|^2}{E_{\text{cm}}^2} \tag{4.3.34}$$

where we used $(k_1 + k_2)^2 = (\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})^2 = E_{\text{cm}}^2$ which is (the square of) the energy of the system in the center of momentum frame. Recall, that we calculated $\langle f|\mathcal{M}|i\rangle$ for this process in eq. (4.3.28). Thus we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+ e^- \to e^+ e^-} = \frac{g^4}{64\pi^2 E_{\text{cm}}^2} \left(\frac{1}{E_{\text{cm}}^2 - m^2} - \frac{1}{2|\mathbf{k}_f|^2(1 - \cos\theta) + m^2}\right)^2 . \tag{4.3.35}$$

The angle θ is the angle between the initial and final (back to back) particle trajectories in the center of momentum frame. Notice the first term inside the brackets has a denominator which goes to zero at $E_{\text{cm}} \to m$. Thus the cross section diverges (in reality, it will rise and fall fast) near $E_{\text{cm}} = m$ as we scan through E_{cm}. This sharp rise and fall reveals the presence of a particle of mass m as the means by which the interaction between our e^+ and e^- (mass M each) takes place! Note that if $2M > m$, then there is no "resonance" possible.

As we have seen from the calculation of amplitudes, the Lorentz invariant combinations $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$ and $u = (k_1 - k_4)^2$ appear quite naturally. These are called *Mandelstam* variables. They have nice interpretations. For example, $\sqrt{s}$ is the center of momentum energy and t and u are related to momentum transfer. In terms of the Mandelstam variables, that the amplitude for the $e^+ e^- \to e^+ e^-$ scattering (see eq. (4.3.28)) can be written as

$$\langle \mathbf{k}_3 \mathbf{k}_4|\mathcal{M}|\mathbf{k}_1 \mathbf{k}_2\rangle = g^2 \left(\frac{1}{s - m^2} + \frac{1}{t - m^2}\right) \tag{4.3.36}$$

The process related to the diagrams contributing these pieces would be called s-channel (first diagram in Fig. 4.10), and t-channel (second diagram in Fig. 4.10) process respectively.

Amplitudes and cross-sections to forces

Consider the $e^-(k_1) + e^+(k_2) \to e^-(k_3) + e^+(k_4)$ process; the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{64\pi^2 s} \left(\frac{1}{s-m^2} + \frac{1}{t-m^2} \right)^2 . \tag{4.3.37}$$

Let us focus on the non-relativistic limit (that is, $|\mathbf{k}_j| \ll M$) and restrict ourselves to the case where $m \ll M$. In this case $k_j \approx (M, \mathbf{k}_j)$. Hence, $s \approx 4M^2$, $t = -|\mathbf{k}_1 - \mathbf{k}_3|^2 \equiv -|\mathbf{q}|^2$ where $\mathbf{q}$ is the momentum transfer and

$$\frac{d\sigma}{d\Omega} \approx \frac{g^4}{256\pi^2 M^2} \left(\frac{1}{|\mathbf{q}|^2 + m^2} \right)^2 . \tag{4.3.38}$$

Let us now consider the same problem $e^-(k_1) + e^+(k_2) \to e^-(k_3) + e^+(k_4)$ in non-relativistic quantum mechanics. We imagine that e^+ and e^- interact via a potential $V(\mathbf{x})$ where $\mathbf{x}$ is their separation vector. The Born-approximation, then tells us that the scattering amplitude from a momentum eigenstate $|\mathbf{k}_i\rangle$ to $|\mathbf{k}_f\rangle$ is $\langle \mathbf{k}_f | V(\mathbf{x}) | \mathbf{k}_i \rangle \propto \int d^3\mathbf{x} V(\mathbf{x}) e^{-i(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{x}}$, and the differential cross section

$$\frac{d\sigma}{d\Omega} \propto \left| \int d^3\mathbf{x} V(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} \right|^2 = \left| \tilde{V}(\mathbf{q}) \right|^2 , \tag{4.3.39}$$

where $\mathbf{q}$ is $\mathbf{k}_i - \mathbf{k}_f$, ie. it is the momentum transfer and $\tilde{V}(\mathbf{q})$ is the Fourier transform of $V(\mathbf{x})$. Comparing eq. (4.3.39) and eq. (4.3.37), we see that

$$\tilde{V}(\mathbf{q}) \propto \frac{1}{|\mathbf{q}|^2 + m^2} \implies V(\mathbf{x}) \propto \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} . \tag{4.3.40}$$

If this is the interaction potential, the force will be $\mathbf{F} = -\nabla V(\mathbf{x})$. Thus we have discovered that the force between our "electron" and "positron" due to the exchange of a scalar particle of mass m is given by

$$|\mathbf{F}| \propto \begin{cases} 1/|\mathbf{x}|^2 & |\mathbf{x}| \ll m^{-1} \\ e^{-m|\mathbf{x}|}/|\mathbf{x}| & |\mathbf{x}| \gg m^{-1} \end{cases} \tag{4.3.41}$$

That is, we have a "$1/r^2$" force for distances small compared to m^{-1} and a "Yukawa screening" for distances large compared to m^{-1}![12]

Exercise 4.3.4 (i) Verify that in the non-relativistic limit, and with $m \ll M$, eq. (4.3.37) leads to eq. (4.3.38). (ii) Then verify the implication in eq. (4.3.40) (ie. calculate the Fourier transform). (ii) Sketch $V(\mathbf{x})$ as a function of $|\mathbf{x}|$ on a log-log plot. (iv) From the dynamics of the inner planets in our solar system[13], one finds that force law is "$1/r^2$" (or more correctly, consistent with general relativity which has massless gravitons) at order $\mathcal{O}(10^{-8})$. Using our Yukawa type potential to parametrize the effect of a massive graviton, what is the constraint on the mass of the graviton m_g? (give your answer in eV). Order of magnitude is fine, but explain your reasoning. What is the length scale in meters corresponding to this mass?

---

[12]It is worth noting that if we kept track of signs, we would find an attractive force regardless of whether we consider e^+e^- or e^-e^- scattering. Scalar exchange in our theory yields an attractive force. We have to go to Quantum Electrodynamics, the correct theory for electrons, positrons and photons to see that like charges repel, and unlike charges attract. The spin of the force carrier matters! Electrodynamics has a (massless) spin-1 carrier, and gravity, a (massless) spin-2 carrier which determine whether we can have attractive/and or repulsive forces.

[13]Note that $1\text{AU} \approx 1.5 \times 10^{11}$ meters, which you can take as the typical size of orbits of inner planets

4.3.6 Beyond Leading Order

Connected and Amputated Contributions

So far we have always calculated amplitudes at the leading order in the coupling constant. For example, the amplitude for $e^- + e^- \to e^- + e^-$ scattering, we calculated the matrix element $\langle f|S-1|i\rangle$ to order g^2. What happens if we want to calculate amplitudes beyond the leading order. Following the "Diagrammer's way", let us try to draw the diagrams at next order in g, making use of the fundamental vertex. I will be a bit sloppy, and ignore the arrows on the $\psi, \psi^\dagger$ lines, not label external momenta and also not draw diagrams that can be obtained by permutations of external legs, rotation from one to the other etc. We have the following diagrams (shown in Fig. 4.12). Each diagram will have an expression associated with (which can be written down using our Feynman rules), which when added together should yield the total amplitude. Take a close look. Some parts of the diagrams (for example the one inside $(\ldots)$) are disconnected from the external lines. They are "vacuum bubbles". You can convince yourself that you can factor the diagrams into a product of connected diagrams and disconnected vacuum bubbles.[14]
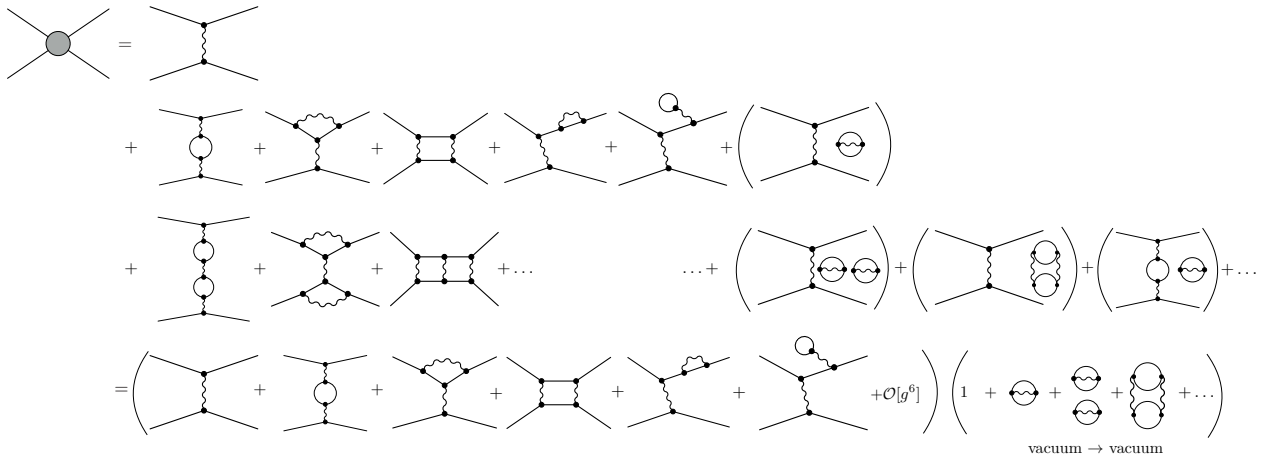
Figure 4.12: The diagrammatic expansion of the amplitude for $e^- e^- \to e^- e^-$. Note the factorization of the vacuum bubbles in the last line. Permutations of external legs has been ignored for simplicity.

The pieces with the vacuum bubbles can be thought of as contributing to the vacuum-to-vacuum transition. Since we normalize the vacuum, this means that at best contribution from the bubbles is a phase $e^{i\theta}$, and does not influence the overall scattering probability density ($|e^{i\theta}|^2 = 1$).[15]

Now look at diagrams with "loops" on external legs. It turns out that we can amputate these legs (for example, cut before the loop begins on the outgoing external legs, or cut after the loop for the incoming external legs). Why is this physically reasonable? This is ok, because as you might recall, we used free-field initial and final states. The external legs with loops is just correcting for the initial and final states in the full theory, and do not have anything to do with the scattering itself. So, with these two rules, we can simplify our scattering calculation as follows:

$$\langle f|S-1|i\rangle = \sum (\text{All connected, amputated Feynman diagrams}) \qquad (4.3.42)$$

---

[14]Our simplifying assumption that the external momenta are not equal precludes disconnected diagrams with external legs.

[15]I am being a bit sloppy here about whether I am talking about the interacting vacuum or the non-interacting vacuum and such. You can look at, for example, Ch. 22 of *QFT for the Gifted Amateur*

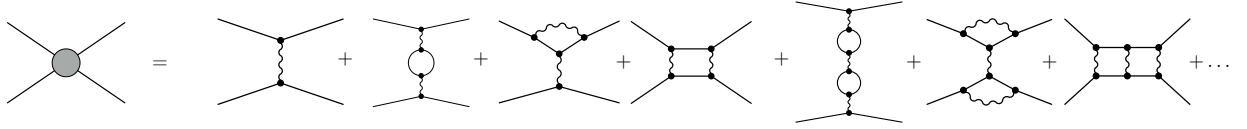Figure 4.13: The diagrammatic expansion, including only the connected amputated diagrams, of the amplitude for $e^- e^- \to e^- e^-$ scattering.

You will find it useful to revisit the Gell-Mann-Low theorem (see eq. (4.2.16)) to see if using connected Feynman diagrams make sense. For example, the denominator is the vacuum to vacuum transition, which precisely cancels the disconnected vaccum bubble contributions from the numerators. The factor of Z in the LSZ reduction formula (see eq. (4.2.15)) can take care of the amputated part.

Divergences at higher orders
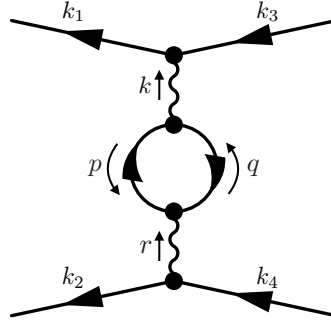
Figure 4.14: The diagrammatic expansion, including only the connected amputated diagrams, of the amplitude for $e^- e^- \to e^- e^-$ scattering.

Let us focus on one of the diagrams at $\mathcal{O}[g^4]$ in the for $e^- e^- \to e^- e^-$ process discussed above (see Fig. 4.14, and try to calculate the integral corresponding to this diagram). Following the Feynman rules in Fig. 4.6, we have

$$
\begin{aligned}
\langle f | S - 1 | i \rangle =& (-ig)^4 \int d\!\!\!/^4 k \, d\!\!\!/^4 p \, d\!\!\!/^4 q \, d\!\!\!/^4 r \, i\Delta_\varphi(k) \, i\Delta_\varphi(r) \, i\Delta_\psi(p) \, i\Delta_\psi(q) \\
& \times \bar\delta^{(4)}(k_1 + k - k_3)\bar\delta^{(4)}(q - p - k)\bar\delta^{(4)}(r - q + p)\bar\delta^{(4)}(k_2 - r - k_4) \, , \\
=& (-ig)^4 \bar\delta^{(4)}(k_1 + k_2 - k_3 - k_4) \, i\Delta_\varphi(k_1 - k_3) \, i\Delta_\varphi(k_2 - k_4) \underbrace{\int d\!\!\!/^4 q \, i\Delta_\psi(q) \, i\Delta_\psi(k_1 - k_3 + q)}_{\text{internal loop}} \, .
\end{aligned}
$$

(4.3.43)

Using our expressions for the propagators in (4.3.29), we can write the contribution from the "internal loop" as

$$
\mathcal{I}(k) = \int d\!\!\!/^4 q \, \frac{i}{q^2 - M^2 + i\epsilon} \frac{i}{(k+q)^2 - M^2 + i\epsilon} \qquad k \equiv k_1 - k_3 \, .
$$

(4.3.44)

Roughly speaking, the integral measure yields q^4, whereas the two propagators also yield q^{-4} as $q \to \infty$. This scaling at large momenta indicates that the integral will diverge logarithmically at large q. This is terrible news! The same is true for (some of) the even higher order diagrams. So the scattering amplitudes have the form $\langle f | S - 1 | i \rangle \sim g^2(\text{finite}) + g^4(\infty?) + \ldots$ This means that the contributions to the amplitude

55

diverges at higher orders in g, and puts our entire scheme of perturbation in small g into question. What did we miss ?

Exercise 4.3.5 : Consider the process $\gamma(k_1) + e^+(k_2) \rightarrow \gamma(k_3) + e^+(k_4)$. (a) Draw all *connected and amputated* Feynman diagrams up to 4-th order in g. You can ignore permutations of external legs (and no need to label the external momenta for this part). (b) Draw a diagram with a disconnected piece at order g^4. (c) Write down the expression corresponding to the "box" diagram $\mathcal{O}[g^4]$ in (Similar, but not identical to the 4-th diagram from the left after the = sign in Fig. 4.13) and simplify as much as you can.

Exercise 4.3.6 : Consider the process $\varphi(k_1) + \varphi(k_2) \rightarrow \varphi(k_3) + \varphi(k_4)$ in the $\mathcal{L} = (1/2)(\partial\varphi)^2 - (1/2)m^2\varphi^2 - (\eta/3!)\varphi^3$ theory. (a) Write down the fundamental vertex in this theory in momentum space. (b) Draw all *connected and amputated* Feynman diagrams up to 4rth order in η. You can ignore permutations of external legs (and no need to label the external momenta for this part). (c) Draw a diagram corresponding to an *s*-channel process at $\mathcal{O}[\eta^2]$ and write down the expression corresponding to this diagram. Simplify as far as possible. (d) Draw a diagram corresponding to an *s*-channel process at $\mathcal{O}[\eta^4]$, with a bubble on the inner propagator and write down the expression corresponding to this diagram. Simplify as far as possible.

4.3.7   A Taste of Renormalization

At this point we need to turn to the important idea of renormalization, which provides an algorithm as well as physical reasoning to remove these infinites. Powerful ideas such as the scale dependence of parameters make an appearance. In this course we do not have time to discuss many aspects of renormalization in detail. Nevertheless, these ideas are too important to skip over entirely, so let us try to do as much as we can in simple examples. In what follows, I rely on A. Zee's *QFT in a Nutshell*.

We encounter infinities in a scattering calculation in the $\varphi\psi^\dagger\psi$ theory when we went beyond leading order in perturbation theory (at least for some, though not necessarily all diagrams). Instead of dealing with this theory (which while simple, still has too many ingredients), I am going to discuss renormalization in an even simpler theory: a real, massive scalar field theory with a φ^4 interaction. We consider the following Lagrangian density for this theory

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 \,. \tag{4.3.45}$$

Note that λ is dimensionless. To make our analysis as simple as possible, we assume that m is smaller than any energy scale of interest where we conduct our scattering experiments. For technical reasons, it is best to keep carrying along a really small, but still non-zero m. The Feynman rules for this theory are straightforward enough, as seen in Fig. 4.15.

In this theory, let us consider the $\varphi(k_1) + \varphi(k_2) \rightarrow \varphi(k_3) + \varphi(k_4)$ scattering. At leading order in λ, we have

$$\begin{aligned}
\mathcal{M}(s,t,u) &= \mathcal{M}^{(1)} + \mathcal{O}[\lambda^2]\,, \\
&= \lambda + \mathcal{O}[\lambda^2]\,,
\end{aligned} \tag{4.3.46}$$

where our $\mathcal{M}$ is really shorthand for $\langle f|\mathcal{M}|i\rangle$ related to the scattering amplitude via $\langle f|S - 1|i\rangle = -i\bar\delta^{(4)}(p_{\text{in}} - p_{\text{out}})\langle f|\mathcal{M}|i\rangle$. Note that the "$(n)$" in $\mathcal{M}^{(n)}$ is the order in λ and $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$ and $u = (k_1 - k_4)^2$ are the Mandelstam variables.

external lines

$$1 = \begin{cases} \rule{2cm}{0.4pt}\bullet \\ \bullet\rule{2cm}{0.4pt} \end{cases}$$

internal lines

$$\int d^4 k \, i\Delta_\varphi(k) = \bullet\rule{2cm}{0.4pt}\bullet$$

vertex

$$(-i\lambda)\,\delta^{(4)}(k_{\text{in}} - k_{\text{out}}) = $$

propagator

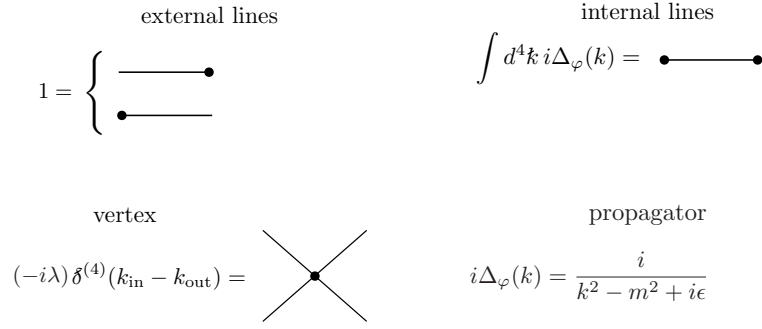$$i\Delta_\varphi(k) = \frac{i}{k^2 - m^2 + i\epsilon}$$
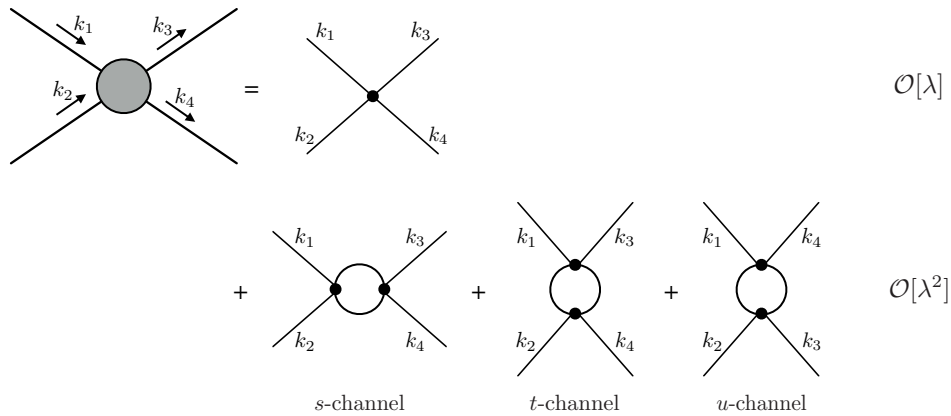
Figure 4.15: k-space Feynman rules for the ϕ^4 theory.

Figure 4.16: The Feynman diagrams contributing to $\varphi(k_1) + \varphi(k_2) \to \varphi(k_3) + \varphi(k_4)$ up to $\mathcal{O}[\lambda^2]$.

We convince the US Department of Energy (DOE) that our φ^4 theory is the long sought-after *Theory of Everything*, we just need to measure the parameter λ. An experiment is built which carries out a $\varphi(k_1^0) + \varphi(k_2^0) \to \varphi(k_3^0) + \varphi(k_4^0)$ scattering, where "0" superscript indicates the actual values used in the experiment, with corresponding s_0, t_0 and u_0. The amplitude of this scattering measured by the experiment has a value λ_P. That is,

$$\mathcal{M}(s_0, t_0, u_0)\bigg|_{\text{exp}} = \lambda_P. \tag{4.3.47}$$

Given this measurement, we can go back to our calculation of the scattering amplitude and identify the parameter $\lambda + \mathcal{O}[\lambda^2] = \lambda_P$. Luckily, it turns out that $\lambda_P \ll 1$. With this triumph, we remember our promise to the DOE that our $\lambda\varphi^4$ is a theory of everything. So we should be able to compute anything that is needed. As a first step, let us at least calculate the scattering cross-section at $\mathcal{O}[\lambda^2]$. Diagrammatically, we write down the contributions up to $\mathcal{O}(\lambda^2)$ in Fig. 4.16. Then,

$$\mathcal{M} = \mathcal{M}^{(1)} + \mathcal{M}_s^{(2)} + \mathcal{M}_t^{(2)} + \mathcal{M}_u^{(2)}. \tag{4.3.48}$$

A quick calculations using the Feynman rules in Fig. 4.15 and the s-channel diagram at $\mathcal{O}[\lambda^2]$ in Fig. 4.16, reveals

$$\mathcal{M}_s^{(2)} = i\lambda^2 \int d^4 q \frac{1}{[q^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]} \qquad \text{where} \qquad k = k_1 + k_2, \tag{4.3.49}$$

Identical expressions hold for $\mathcal{M}_t^{(2)}$ and $\mathcal{M}_u^{(2)}$ with $k = k_1 - k_3$ and $k = k_1 - k_4$ respectively. Note that

integral

$$\mathcal{I}(k) = \int d^4q \frac{1}{[q^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}, \tag{4.3.50}$$

diverges logarithmically at large q! That is disheartening. So much for our theory of everything! The divergence is coming from large q. Let us (at least momentarily) give up the belief that we our $\lambda\varphi^4$ theory is the complete theory, and make the reasonable assumption that we only know our theory to some large energy/momentum scale Λ. We call this large but unknown Λ the cutoff scale of the theory. In that case, we should only compute the integral up to the cutoff, that is,

$$
\begin{aligned}
\mathcal{I}_\Lambda(k) &= \int^\Lambda d^4q \frac{1}{[q^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}, \\
&= \int d^4q \left\{ \frac{1}{[q^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]} - \frac{1}{[q^2 - \Lambda^2 + i\epsilon][(k-q)^2 - \Lambda^2 + i\epsilon]} \right\}.
\end{aligned}
\tag{4.3.51}
$$

The second equality is just one of the many possible ways of "regulating" the integral. This way of making an infinite integral finite is called Pauli-Villars regularization. There are other ways of doing this (for example, Dimensional regularization); regularization schemes should not affect observables in the end. Note that for for $q \gg \Lambda$, the integrand vanishes, whereas for $q \ll \Lambda$ the Λ dependent term will be suppressed compared to the first by Λ^{-4}. This integral $\mathcal{I}_\Lambda(k)$ can now be evaluated (though not trivially by any means). In the limit that $k^2 \gg m^2$:

$$\mathcal{I}_\Lambda(k) = ia \left[\ln \Lambda^2 - \ln s \right] \qquad \text{where} \qquad s = (k_1 + k_2)^2 = k^2 \gg m^2, \tag{4.3.52}$$

and a is a constant. Using this expression for $\mathcal{I}_\Lambda(k)$ and combining all the terms in the amplitude for scattering (up to $\mathcal{O}[\lambda^2]$, see eq. (4.3.48)), we have

$$\mathcal{M}(s,t,u) = \lambda - a\lambda^2 \left[3\ln\Lambda^2 - \ln s - \ln t - \ln u \right] + \mathcal{O}[\lambda^3]. \tag{4.3.53}$$

where we have made the dependence on s, t and u explicit in $\mathcal{M}$. This is not of much help. We still have an unknown Λ in the problem. We know $\lambda + \mathcal{O}[\lambda^2] = \lambda_P$ from eq. (4.3.47). But only to leading order. So to get the amplitude at higher order in λ we have to figure out what λ is, along with the unknown Λ.

Recall that we have experimentally measured the scattering amplitude at $(s,t,u) = (s_0, t_0, u_0)$, and found the experimental value to be λ_P (eq.(4.3.47)). Experimental results do not care about how we do our calculations, or to what order in perturbation theory. A scattering amplitude was measured at some value of incoming and outgoing momenta, and the answer is λ_P. This bit of information is very useful as we now see. Evaluating eq. (4.3.53) at $(s,t,u) = (s_0, t_0, u_0)$ and comparing it with eq. (4.3.47), we have

$$\lambda_P = \lambda - a\lambda^2 \left[3\ln\Lambda^2 - \ln s_0 - \ln t_0 - \ln u_0 \right] + \mathcal{O}[\lambda^3]. \tag{4.3.54}$$

For finite Λ, we can invert this series, and since $\lambda_P \ll 1$, we have

$$\lambda = \lambda_P + a\lambda_P^2 \left[3\ln\Lambda^2 - \ln s_0 - \ln t_0 - \ln u_0 \right] + \mathcal{O}[\lambda_P^3] \tag{4.3.55}$$

Let us substitute this "improved" λ into our general expression eq. (4.3.53). We arrive at

$$\mathcal{M}(s,t,u) = \lambda_P + a\lambda_P^2 \left[\ln(s/s_0) + \ln(t/t_0) + \ln(u/u_0) \right] + \mathcal{O}[\lambda_P^3]. \tag{4.3.56}$$

Magic! This expression does not have the unknown Λ in it, and is written entirely in terms of known quantities based on a previous measurement of $\mathcal{M}(s_0, t_0, u_0)|_{\exp} = \lambda_P$. With this expression we can now compute the scattering amplitude for $\varphi(k_1) + \varphi(k_2) \to \varphi(k_3) + \varphi(k_4)$ for any experiment ie. any s, t and u up to $\mathcal{O}[\lambda_P^2]$.

The lesson we learnt is that perturbation theory in terms of λ is a bit sick. It leads to divergences our calculation of observables, or at least leads to a dependence on large unconstrained cutoff scale physics Λ.

However, if we express our answers in terms of physical coupling constants (ones measured in experiment), then the divergences and cutoff dependences vanish! [16]

In some cases, when we reach some high energy scale Λ in an experiment, we might discover new physics that cures the divergence. Nevertheless, the above procedure shows that we do not need to know the physics at Λ to make predictions at much lower energy scales. That should be a relief.

Flow of Physical Coupling Constants

Recall that $\lambda_P = \mathcal{M}(s_0, t_0, u_0)|_{\exp}$. Let us, for convenience, chose $s_0 = t_0 = u_0 = \mu_0^2$. Note that is impossible to actually do this for any $k_1^0, k_2^0, k_3^0, k_4^0$, but this choice does not affect the subsequent arguments. We interpret λ_P as the physical coupling constant measured at the energy scale μ_0: $\lambda_P(\mu_0)$. Let us re-write $\mathcal{M}(s, t, u)$ expression above with a mildly simplified expression:

$$\mathcal{M}(s, t, u) = \lambda_P(\mu_0) + a\lambda_P^2(\mu_0) \left[\ln(s/\mu_0^2) + \ln(t/\mu_0^2) + \ln(u/\mu_0^2)\right] + \mathcal{O}[\lambda_P^3]. \tag{4.3.58}$$

There is nothing special about μ_0. We could equally have carried out the above procedure at another energy scale μ_1. Then

$$\mathcal{M}(s, t, u) = \lambda_P(\mu_1) + a\lambda_P^2(\mu_1) \left[\ln(s/\mu_1) + \ln(t/\mu_1) + \ln(u/\mu_1)\right] + \mathcal{O}[\lambda_P^3]. \tag{4.3.59}$$

Since the L.H.S $\mathcal{M}(s, t, u)$ in eqns. (4.3.58) and (4.3.59) are the same, we equate the two. Then, as $\Delta\mu = \mu_1 - \mu_0 \to 0$, we get

$$\left.\frac{d\lambda_P(\mu)}{d\ln\mu}\right|_{\mu=\mu_0} = 6a\lambda_P^2(\mu_0) + \mathcal{O}[\lambda_P^3] \tag{4.3.60}$$

Or more generally,

$$\frac{d\lambda_P(\mu)}{d\ln\mu} = 6a\lambda_P^2(\mu) + \mathcal{O}[\lambda_P^3]. \tag{4.3.61}$$

This equation, now tells us how λ_P changes with the energy scale μ. The physical "coupling constant" is not really a constant. If a is positive (which is the case for this theory), λ_P increases with μ, that is the theory is becoming more strongly interacting at higher energies or equivalently, smaller length scales.

In a theory with a number of coupling constants $\{\lambda_0, \lambda_1, \ldots\}$ we will have

$$\frac{d\lambda_i}{d\ln\mu} = \beta_i(\lambda_0, \lambda_1 \ldots), \tag{4.3.62}$$

where we have dropped the subscript "P" for physical coupling constant to avoid clutter. The β_i are called the Beta functions of the theory. The signs of the β_i tell us about how the couplings flow with energy μ. As you might imagine, the fixed points in this space of couplings ($\beta_i(\lambda_1^\star, \lambda_2^\star \ldots) = 0$) are particularly special. [17]

As a couple of physically relevant cases of interest, note that in Quantum Electrodynamics, the "physical charge" of electrons

$$\frac{d\,e_P(\mu)}{d\ln\mu} = \frac{1}{12\pi^2}e_P^3(\mu) + \mathcal{O}[e_P^5]. \tag{4.3.63}$$

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[16] Consider a more general *low energy theory* (compared to Λ) of the form

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n. \tag{4.3.57}$$

Note that since the mass dimensions of $\mathcal{L} = 4$ and $[\phi] = 1$, we have $[\lambda_n] = 4 - n$. Here is some nomenclature. Terms with $[\lambda_n] < 0$ are called *relevant*, terms with $[\lambda_n] = 0$ are called *marginal*, whereas terms with $[\lambda_n] > 0$ are called *irrelevant*. We typically expect only marginal and relevant terms to contribute to our low energy (compared to Λ) processes. Moreover, it turns out that theories with $[\lambda_n] > 0$ are not re-normalizable. However, that does not rule out such terms (just that our theory will have a domain of validity in terms of energies).

[17] Time permitting, we will discuss such a point in the context of current conduction in presence of impurities and *Anderson Localization* in class.

As you probe electrons at higher and and higher energy scales, the charge gets larger and larger. On the other hand, for Quantum chromodynamics, we find that

$$\frac{d\,g_P(\mu)}{d\ln\mu} < 0\,!\qquad\qquad(4.3.64)$$

The coupling gets weaker at higher energies. This is asymptotic freedom; quarks become free at high energies but are strongly bound into mesons and hadrons at low energies.

There is a lot more to learn regarding renormalization. Will we always be able to get rid of the cutoff? What about higher order terms? How do we do all this systematically? We unfortunately have to move on, but hopefully I have given you some flavor of how some of the infinities can be dealt with. For a more in-depth, but still accessible view see A. Zee's textbook. You might also want to consult *QFT for the Gifted Amateur* as well as notes on renormalization by M. Luty. ,

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Exercise 4.3.7 : Learn about "Feynman Parametrization" of integrals and "Dimensional Regularization" on your own. Now evaluate the non-divergent, energy dependent and part of the integral in eq. (4.3.50) at 0th order in the dimensional regularization parameter (ie. the finite part that remains as you take the dimensional regularization parameter to 0). You can leave your answer in terms of the integral over the Feynman parameter, or evaluate it explicitly if you can.

4.4 Summary

Let me graphically summarize all that we have done so far for weakly interacting scalar fields:

Lagrangian

free part   interaction

propagator   mode expansion   vertex

Feynman Rules [or Dyson expansion]

amplitudes (tree level)

(higher order — Renormalization)
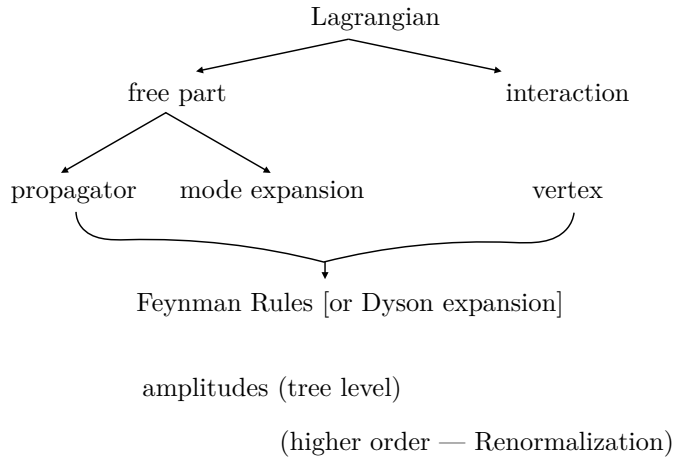
Figure 4.17: Summary of the course so far: QFT for weakly interacting scalar fields.

SYMMETRY AND FIELDS

5.1 Spacetime and Internal Symmetries

Consider the action

$$S = \int d^4x \mathcal{L}(\varphi_a, \partial_\mu \varphi_a),$$

(5.1.1)

where $a = 1, 2 \ldots N$, and φ_a are N scalar fields, or N components of some field. First, let us recall precisely what we mean by symmetries.

A *symmetry transformation* is an operation that does not change the action. Such a transformation must then change the Langrangian density by at most a total derivative:

$$\tilde{\mathcal{L}} = \mathcal{L} + \partial_\mu F^\mu,$$

(5.1.2)

where $F^\mu(x)$ are 4 arbitrary functions of spacetime. Moreover, a *continuous* symmetry transformation means that we are allowed to work with infinitesimal transformations.

Let us make this concrete with an exceptionally simple example. Consider the Lagrangian density of a complex scalar field $\mathcal{L} = |\partial \psi|^2 - M^2 |\psi|^2 - \lambda |\psi|^4$, and the corresponding action $S = \int d^4x \mathcal{L}$. A transformation which rotates the field in the complex plane by the same constant angle α everywhere, ie. $\tilde{\psi}(x) = e^{i\alpha}\psi(x)$ is a symmetry transformation, since it leaves the Lagrangian density (and the action) unchanged. Since the result holds for arbitrarily small α, it is also a continuous symmetry transformation ($\tilde{\psi}(x) = \psi + \delta\psi = \psi + i\alpha\psi$ for small α). For the same Lagrangian, a Lorentz transformation is also a continuous symmetry transformation. It would generate non-zero $\partial_\mu F^\mu$ terms.

5.1.1 Noether's Theorem and its Consequences

Noether's Theorem: Every continuous symmetry of the action yields a current j^μ which is conserved via the equations of motion.
Proof: Consider an infinitestimal transformation which changes the fields as follows: $\tilde{\varphi}_a = \varphi_a + \delta\varphi_a$. Then

under this transformation, the Lagrangian density changes as follows

$$\delta\mathcal{L} = \sum_{a=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial \varphi_a} \delta\varphi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial_\mu(\delta\varphi_a) \right) ,$$

$$= \sum_{a=1}^{N} \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \right]}_{= 0 \text{ if } \varphi_a \text{ satisfies the EOM}} \delta\varphi_a + \sum_{a=1}^{N} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a \right] , \tag{5.1.3}$$

$$= \sum_{a=1}^{N} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a \right] ,$$

Note that we have used that φ_a satisfies the equation of motion. If this transformation is a symmetry transformation, then we must have $\delta\mathcal{L} = \partial_\mu F^\mu$ (see eq. (5.1.2)). Combining with our result above for $\delta\mathcal{L}$, we have

$$\partial_\mu F^\mu = \sum_{a=1}^{N} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a \right] \implies \partial_\mu \left[\sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - F^\mu \right] = 0 . \tag{5.1.4}$$

Thus, we have just constructed a current

$$j^\mu \equiv \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - F^\mu , \qquad \text{such that} \qquad \partial_\mu j^\mu = 0 \qquad \text{on the the eq. of motion} . \tag{5.1.5}$$

The "on the eq. of motion" part refers to the fact that φ_a satisfies the eq. of motion. This completes the proof of Noether's theorem. While not evident right now, F^μ will be obtained by finding $\delta\mathcal{L}$ (which is a scalar) for any particular symmetry transformation without using the expression for $\mathcal{L}$ in terms of the fields.

All this might might seem a bit abstract. We will make Noether's theorem and its implications more concrete by considering examples of continuous symmetries. But before moving on to concrete, physically relevant examples, a few comments are in order:

- **Conserved Charge**: Note that

$$\partial_\mu j^\mu = 0 \implies \partial_0 j^0 = -\nabla \cdot \mathbf{j} ,$$

$$\implies \partial_0 \int d^3x j^0 = -\int d^3x \nabla \cdot \mathbf{j} ,$$

$$\implies \dot{Q} = -\int \mathbf{j} \cdot dS = 0 \qquad \text{where} \qquad Q \equiv \int d^3x j^0 . \tag{5.1.6}$$

  where in the last line we defined a charge Q and assumed that $\mathbf{j}$ vanishes sufficiently rapidly at spatial infinities. Noether's theorem points to the existence of a *conserved charge* for every continuous symmetry transformation.

- It is convenient to define a momentum density π_a^μ, so that

$$j^\mu = \sum_{a=1}^{N} \pi_a^\mu \delta\varphi_a - F^\mu \qquad \text{with} \qquad \pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} . \tag{5.1.7}$$

- **Charges as generators of symmetries**: It is useful to think of Q as a *generator of symmetries*

in the following way. Consider the Poisson bracket of Q with the field φ_a:

$$\{Q, \varphi_a\} = \{\int d^3x \sum_{b=1}^{N} \pi_b^0 \delta\varphi_b - F^0, \varphi_a\},$$

$$= \int d^3x \sum_{b=1}^{N} \{\pi_b^0, \varphi_a\}\delta\varphi_b,$$

$$= -\delta\varphi_a. \tag{5.1.8}$$

where in the the third line we assumed that $\delta\varphi_a$ and F^0 do not depend on π_a^0.[1] Thus we have $\delta\varphi_a = -\{Q, \varphi_a\}$. It is in this sense that Q generates the symmetry transformation.

- While all the above statements have been made for classical fields, the arguments carry over to quantum fields with $f \to \hat{f}$, and with $\{\cdot, \cdot\} \to -i[\cdot, \cdot]$. In particular

$$\delta\hat{\varphi}_a = i[\hat{Q}, \hat{\varphi}_a], \tag{5.1.9}$$

and $\hat{Q}$ becomes a conserved operator, that is

$$\frac{d\hat{Q}}{dt} = i[\hat{H}, \hat{Q}] = 0. \tag{5.1.10}$$

We will again drop the "hats" from our operators.

Let us now turn to some concrete examples. We will initially restrict ourselves to theories with scalar fields, but later generalize the results. Continuous symmetries naturally fall under two classes: (i) Internal Symmetries (ii) Spacetime Symmetries.

5.1.2 Internal Symmetries

Consider the following Lagrangian density:

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \psi^\dagger - M^2 \psi \psi^\dagger. \tag{5.1.11}$$

Consider the $U(1)$ transformation $\tilde{\psi} = e^{i\alpha}\psi$ where α is a real constant. This "rotation" of the field is unrelated to spacetime, and is in an internal "field space". The infinitesimal version of this transformation is

$$\tilde{\psi} = \psi + \underbrace{i\alpha\psi}_{\delta\psi}. \tag{5.1.12}$$

Under this transformation, the value of the Lagrangian density $\tilde{\mathcal{L}}$ for $\tilde{\psi}$ is the same as that of ψ, that is $\delta\mathcal{L} = \tilde{\mathcal{L}} - \mathcal{L} = 0$. For this transformation to to leave the action invariant (ie. to be a symmetry transformation), we must have $\delta\mathcal{L} = \partial_\mu F^\mu$. Hence, $\partial_\mu F^\mu = 0$. That is, $\partial_\mu F^\mu = 0$ without knowing anything about the actual form of the symmetry transformation (apart from the fact that the $\mathcal{L}$ does not change); we will set $F^\mu = 0$. The conserved Noether current (see eq. (5.1.7)) is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\delta\psi^\dagger = i\alpha\left((\partial^\mu \psi^\dagger)\psi - (\partial^\mu \psi)\psi^\dagger\right). \tag{5.1.13}$$

In particular, we have

$$j^0 = i\alpha\left(\dot{\psi}^\dagger \psi - \dot{\psi}\psi^\dagger\right) \quad \text{and} \quad Q = i\alpha \int d^3x \left(\dot{\psi}^\dagger \psi - \dot{\psi}\psi^\dagger\right). \tag{5.1.14}$$

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[1] You might want to remind yourselves of the defining properties of Poisson brackets and functionals discussed in Chapter 2.

Usually, we will divide j^μ and Q by the infinitesimal parameter (in this case α), to define a non-infinitesimal current and charge. Furthermore, we can afford to be a bit sloppy about the ordering of operators. Different orderings will lead to different j^μ, differing by a constant, which is typically irrelevant for conservation of the current. To remove the ambiguity from Q, we can insist on the $\langle 0|Q|0\rangle = 0$; that is, use $Q = \; : Q :$

Q is a generator of the symmetry in the following sense:

$$i[Q, \psi] = -\alpha \int d^3x\, \psi[\dot{\psi}^\dagger, \psi] = i\alpha\psi = \delta\psi \,. \tag{5.1.15}$$

where we used the canonical commutation relations, in particular $[\dot{\psi}^\dagger(x), \psi(y)]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$.

Exercise 5.1.1 : Consider the Lagrangian $\mathcal{L} = (1/2)(\partial\varphi_1)^2 + (1/2)(\partial\varphi_2)^2 - (M^2/2)(\varphi_1^2 + \varphi_2^2)$ where φ_1 and φ_2 are Hermitian fields. This Lagrangian has an internal symmetry. Find the group corresponding to this internal symmetry. Find the (non-infinitesimal) Noether current and conserved charge for this theory. Do not worry about ordering of operators.

Now consider a Lagrangian of the form $\mathcal{L} = (1/2)\partial^\mu\vec{\varphi} \cdot \partial_\mu\vec{\varphi} - V(\vec{\varphi} \cdot \vec{\varphi})$ where $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ and φ_i are Hermitian fields. Repeat the analysis from the earlier part of this exercise. How many Noether charges do you have corresponding to the internal symmetries ?

5.1.3 Spacetime Symmetries

For a theory to be consistent with Special Relativity, we want the action for that theory to be invariant under translation, rotation and boost transformations. The group of such transformations is called the Poincare Group.

$$\text{Poincare Group: Translations} + \underbrace{\text{Lorentz Transformations}}_{\text{Boosts + Rotations}}. \tag{5.1.16}$$

Since translations and Lorentz transformations are continuous symmetry transformations, we will be able to find conserved charges and currents associated with them. Let us first look at translations. For concreteness, we will consider a theory of a Hermitian scalar field $\mathcal{L}(\varphi, \partial_\mu\varphi)$.

Translations

Consider a spacetime translation $\tilde{x}^\mu = x^\mu + \epsilon a^\mu$ where $\epsilon \ll 1$ and a^μ is a constant vector. Let us consider an *active transformation*, ie. translate the field configuration (as opposed to the co-ordinates). Then we can write the field after the translation in terms of the field before the translation as follows

$$\tilde{\varphi}(x^\nu) = \varphi(x^\nu - \epsilon a^\nu) = \varphi(x) - \epsilon a^\alpha \partial_\alpha \varphi(x) \,, \tag{5.1.17}$$

That is, $\delta\varphi(x) = -\epsilon a^\alpha \partial_\alpha \varphi(x)$. Since the Lagrangian is also a scalar (like our field), we also have $\delta\mathcal{L}(x) = -\epsilon a^\alpha \partial_\alpha \mathcal{L}(x)$. Now, if translation is a symmetry transformation, then we must have $\delta\mathcal{L} = \partial_\mu F^\mu$. Hence, we have $F^\mu = -\epsilon a^\mu \mathcal{L}$ (up to an independent divergence free part). Now using the expression for the Noether current (see eq. (5.1.7)), we have

$$j^\mu = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}}_{\pi^\mu = \partial^\mu\varphi} \delta\varphi - F^\mu = -\epsilon a^\nu \left[\pi^\mu \partial_\nu\varphi - \delta^\mu_\nu \mathcal{L}\right] = -\epsilon a^\nu T^\mu{}_\nu \qquad \text{where} \qquad T^\mu{}_\nu \equiv \pi^\mu \partial_\nu\varphi - \delta^\mu_\nu \mathcal{L} \,. \tag{5.1.18}$$

Since $\partial_\mu j^\mu = 0$ and since a^μ can be non-zero for each μ, we have

$$\partial_\mu T^\mu{}_\nu = 0. \tag{5.1.19}$$

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In the above expressions, $T^\mu{}_\nu$ are the components of the *Energy Momentum Tensor*.[2] Since $\nu = 0, 1, 2, 3$, there are 4 different conserved currents (related to 4 independent spacetime translations). It might be helpful to think about the current as $(j^\mu)_\nu = T^\mu{}_\nu$, that is ν simply labels which current we are referring to.

The conserved charges associated with each ν are

$$P_\nu = \int d^3x\, T^0{}_\nu = \text{components of the total 4-momentum of the field system}. \tag{5.1.20}$$

Note in particular, that for $\nu = 0$:

$$P_0 = \int d^3x\, T^0{}_0 = \int d^3x \left(\pi^0 \partial_0 \varphi - \mathcal{L}\right) = \int d^3x\, \mathcal{H} = H = \text{energy or Hamiltonian of the system}. \tag{5.1.21}$$

Let us make a few useful observations about these conserved charges:

- Note that the P_μ are constructed from fields not at a single point, but integrated over space.

- Since P_ν are conserved charges, we must have $\partial_0 P_\nu = 0$ on the equation of motion – energy and momentum of the system are conserved as expected.

- By thinking about the charges (P_ν) as generators of the symmetry (translations), we have

$$i[P_\nu, \varphi] = \partial_\nu \varphi, \qquad \text{which for } \nu = 0 \text{ yields,} \qquad i[H, \varphi] = \partial_0 \varphi, \tag{5.1.22}$$

consistent with Heisenberg's equation of motion for fields.

Boosts and Rotations

Recall that defining property of Lorentz transformations (made up of boosts and rotations) is that for $x^\mu = \Lambda^\mu{}_\nu x^\nu$, the components $\Lambda^\mu{}_\nu$ satisfy

$$\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau g^{\sigma\tau} = g^{\mu\nu}, \tag{5.1.23}$$

where $g_{\mu\nu}$ are the components of the Minkowski metric.[3] An infinitesimal version of the Lorentz transformation can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon a^\mu{}_\nu, \tag{5.1.24}$$

where $a^{\mu\nu} = -a^{\nu\mu}$ (and similarly for $a_{\mu\nu}$, but not $a^\mu{}_\nu$). With both upper and lower indices, "a" is antisymmetric. This follows from the defining property of the Lorentz transformations eq. (5.1.23).

Exercise 5.1.2 : (i) Consider a boost with a velocity $v \ll 1$ in the x direction. Find an explicit form for $\Lambda^\mu{}_\nu$ and $a^\mu{}_\nu$ corresponding to this boost. Is $a^{\mu\nu}$ antisymmetric? (ii) Consider a small rotation by an angle $\theta \ll 1$ around the z axis. Find $\Lambda^\mu{}_\nu$ and $a^\mu{}_\nu$ corresponding to this rotation. Is $a^{\mu\nu}$ antisymmetric ? Note that in our convention, the "a" should not contain the small parameters v or θ. (iii) Show that for a general infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon a^\mu{}_\nu$, we must have $a^{\mu\nu} = -a^{\nu\mu}$ by using eq. (5.1.23).

---

[2]In particular, $T^0{}_0$ is the time component of the momentum flux flowing across a spatial 3-dimensional hypersurface. That is $T^0{}_0$ is the energy density. Similarly $T^i{}_j$ with $i = j$ is the pressure in the ith direction. $T^0{}_i$ is the ith component of the three-momentum density and $T^i{}_j$ with $i \neq j$ are anisotropic stresses.

[3]You might want to go back and consult our Rapid Review Chapter

Under a Lorentz transformation implementing $\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ (with $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon a^\mu{}_\nu$), a scalar field transforms as

$$\tilde{\varphi}(x) = \varphi(\Lambda^{-1}x) = \varphi(\delta^\mu_\nu x^\nu - \epsilon a^\mu{}_\nu x^\nu),$$
$$= \varphi(x) \underbrace{-\epsilon a^\mu{}_\nu x^\nu \partial_\mu \varphi(x)}_{\delta\varphi(x)}, \qquad (5.1.25)$$

where we have assumed $\epsilon \ll 1$. A bit of re-writing of $\delta\varphi(x)$ using the antisymmetry of "a" yields

$$\delta\varphi(x) = -\frac{1}{2}\epsilon a^{\mu\nu} \left(x_\nu \partial_\mu - x_\mu \partial_\nu\right)\varphi(x). \qquad (5.1.26)$$

Since $\mathcal{L}$ is also a scalar, we have

$$\delta\mathcal{L}(x) = -\epsilon a^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}(x) = -\epsilon \partial_\mu \left(a^\mu{}_\nu x^\nu \mathcal{L}\right). \qquad (5.1.27)$$

For the Lorentz transformation to be a symmetry transformation, it must leave the action invariant. That is, we must have $\delta\mathcal{L} = \partial_\mu F^\mu$, which yields $F^\mu = -\epsilon a^\mu{}_\nu x^\nu \mathcal{L}$. Having found F^μ, we can use the expression or the Noether current in eq. (5.1.7) to get

$$\begin{aligned} j^\mu = \pi^\mu \delta\varphi - F^\mu &= -\epsilon \left[\pi^\mu a^\sigma{}_\nu x^\nu \partial_\sigma \varphi - a^\mu{}_\nu x^\nu \mathcal{L}\right], \\ &= -\epsilon a^\sigma{}_\nu \left[x^\nu \pi^\mu \partial_\sigma \varphi - x^\nu \delta^\mu_\sigma \mathcal{L}\right], \\ &= -\epsilon a^\sigma{}_\nu x^\nu T^\mu{}_\sigma, \qquad \text{where} \qquad T^\mu{}_\sigma = \pi^\mu \partial_\sigma \varphi - \delta^\mu_\sigma \mathcal{L}, \\ &= -\frac{1}{2}\epsilon a^{\sigma\nu} \left(x_\nu T^\mu{}_\sigma - x_\sigma T^\mu{}_\nu\right). \end{aligned} \qquad (5.1.28)$$

In the last line we took advantage of the anti-symmetry of "a". This anti-symmetry also tells us that there are 6 independent, non-infinitesimal currents:

$$(j^\mu)_{\nu\sigma} = x_\nu T^\mu{}_\sigma - x_\sigma T^\mu{}_\nu, \qquad \text{which satisfy} \qquad \partial_\mu (j^\mu)_{\nu\sigma} = 0. \qquad (5.1.29)$$

Given the current, the conserved charges $\left(j^0\right)_{\nu\sigma} \equiv J_{\nu\sigma}$ are

$$J_{\nu\sigma} = \int d^3x \left(x_\nu T^0{}_\sigma - x_\sigma T^0{}_\nu\right). \qquad (5.1.30)$$

Note that J_{0i} (three of them for $i = 0, 1, 2, 3$) are conserved charges corresponding to boosts, whereas J_{ij} are the total angular momentum of the field (note that $J_{ij} = -J_{ji}$). The three J_{ij} are the conserved charges corresponding to rotational symmetry around three different axes. Finally, note that these conserved charges generate the symmetry transformation of the field as follows

$$i[J_{\sigma\nu}, \varphi] = (x_\sigma \partial_\nu - x_\nu \partial_\sigma)\varphi. \qquad (5.1.31)$$

Exercise 5.1.3 : Show that *if* the current from translations $(j^\mu)_\nu = T^\mu{}_\nu$ and from Lorentz transformations is $(j^\mu)_{\sigma\nu} = x_\sigma T^\mu{}_\nu - x_\nu T^\mu{}_\sigma$, then $T_{\mu\nu} = T_{\nu\mu}$. [Note: In general, the energy momentum tensor for higher spin fields obtained using our standard procedure with Noether currents is not symmetric. However, we can add divergenceless pieces to the current that do not change the charge, and make the energy momentum tensor symmetric. The Noether currents in this problem are made from such symmetric tensors. Why insist on symmetric energy momentum tensors? We insist on them because the energy momentum tensor in General Relativity is necessarily symmetric.]

Let me summarize what we have learnt regarding spacetime symmetries and Noether's theorem in the

context of a theory with a real scalar field $S = \int d^4x \mathcal{L}(\varphi, \partial_\mu \varphi)$. The symmetry transformations are a combination of translations and Lorentz transformations (the latter contain boosts and rotations).

Translations: $\tilde{x}^\nu = x^\nu + \epsilon a^\nu$. Noether's theorem implies the existence of four conserved 4-currents, and four conserved charges:

$$\text{four currents} : (j^\mu)_\nu = T^\mu{}_\nu \qquad \text{with} \qquad \partial_\mu T^\mu{}_\nu = 0 \qquad \text{where} \qquad T^\mu{}_\nu = \pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}, \qquad \pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}.$$

$$\text{four charges} : P_\nu = \int d^3x T^0{}_\nu, \qquad \text{with} \qquad i[P_\nu, \varphi] = \partial_\nu \varphi.$$

$$(5.1.32)$$

Lorentz Transformations: $\tilde{x}^\nu = x^\nu + \epsilon a^\nu{}_\alpha x^\alpha$ where $\Lambda^\nu{}_\alpha = \delta^\nu_\alpha + \epsilon a^\nu{}_\alpha$. Noether's theorem implies the existence of six conserved currents, and six conserved charges:

$$\text{six currents} : (j^\mu)_{\nu\sigma} = x_\nu T^\mu{}_\sigma - x_\sigma T^\mu{}_\nu, \qquad \text{which satisfy} \qquad \partial_\mu (j^\mu)_{\nu\sigma} = 0.$$

$$\text{six charges} : J_{\nu\sigma} = \int d^3x \left(x_\nu T^0{}_\sigma - x_\sigma T^0{}_\nu \right) \qquad \text{with} \qquad i[J_{\sigma\nu}, \varphi] = (x_\sigma \partial_\nu - x_\nu \partial_\sigma)\varphi.$$

$$(5.1.33)$$

In the above expressions $(j^\mu)_{\sigma\nu} = -(j^\mu)_{\nu\sigma}$ and similarly for the charges. We have a total of ten generators of the Poincare transformations.

$$P_\mu = \text{four generators of translations}$$

$$J_{\sigma\nu} = \text{three (boosts)} + \text{three (rotations)} = \text{six generators of Lorentz transformations}$$

$$(5.1.34)$$

These generators together satisfy the following "algebra":

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \\
&[P_\mu, J_{\rho\sigma}] = i \left(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho \right), \\
&[J_{\mu\nu}, J_{\rho\sigma}] = i \left(g_{\mu\sigma} J_{\nu\rho} + g_{\nu\rho} J_{\mu\sigma} - \rho \leftrightarrow \sigma \right).
\end{aligned} \qquad (5.1.35)$$

The key ingredients needed to verify these identities are (i) Jacobi Identity: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$, (ii) $[P_\mu, \varphi] = -i\partial_\mu \varphi$ and (iii) $[J_{\mu\nu}, \varphi] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi$. We want to first verify that $[P_\mu, P_\nu] = 0$. To do this, let us start with the Jacobi identity

$$[P_\mu, [P_\nu, \varphi]] + [\varphi, [P_\mu, P_\nu]] + [P_\nu, [\varphi, P_\mu]] = 0. \qquad (5.1.36)$$

Upon re-arranging, we have

$$\begin{aligned}
[\varphi, [P_\mu, P_\nu]] &= -[P_\mu, [P_\nu, \varphi]] - [P_\nu, [\varphi, P_\mu]], \\
&= -[P_\mu, -i\partial_\nu \varphi] - [P_\nu, i\partial_\mu \varphi], \\
&= i\partial_\nu [P_\mu, \varphi] - i\partial_\mu [P_\nu, \varphi], \\
&= (\partial_\nu \partial_\mu - \partial_\mu \partial_\nu)\varphi, \\
&= 0.
\end{aligned} \qquad (5.1.37)$$

Now, since $[\varphi, [P_\mu, P_\nu]] = 0$ for any field configuration $\varphi(x)$, we must have $[P_\mu, P_\nu] = 0$, which is what we wanted to verify. The other two identities can also be verified similarly, but with (quite a bit more) algebra.

Exercise 5.1.4: Verify that (i) $[P_\mu, J_{\rho\sigma}] = i\left(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho \right)$, and (ii) $[J_{\mu\nu}, J_{\rho\sigma}] = i\left(g_{\mu\sigma} J_{\nu\rho} + g_{\nu\rho} J_{\mu\sigma} - \rho \leftrightarrow \sigma \right)$.

Exercise 5.1.5 : The 6 generators for Lorentz transformations, $J_{\mu\nu}$, can be decomposed into rotations (J_i) and boosts (K_i) where

$$J_k \equiv \frac{1}{2}\epsilon^{ijk}J_{ij} \qquad \text{and} \qquad K_i \equiv J_{0i}\,. \tag{5.1.38}$$

Verify in general that

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \qquad [J_i, K_i] = i\epsilon_{ijk}K_k\,, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k\,. \tag{5.1.39}$$

Note that the commutator of (i) two rotations is a rotation, (ii) rotation and a boost is a boost, and (iii) two boosts is a rotation. For the case of scalar field, verify explicitly that $K_x K_y - K_y K_x = -iJ_z$.

Finite Transformations

So far, we have restricted ourselves to infinitesimal transformations. For a finite translation $\tilde{x}^\mu = x^\mu + a^\mu$,

$$\varphi(x - a) = e^{-ia^\nu P_\nu}\varphi(x)e^{ia^\nu P_\nu}\,, \tag{5.1.40}$$

where $[P_\nu, \varphi] = -i\partial_\nu\varphi$. We can implement finite transformations using the generators obtained by exploring infinitesimal transformations (close to the identity) is a consequence of the fact that we are dealing with simply connected Lie Groups. Similarly for a Lorentz transformation characterized by $\Lambda^\mu{}_\nu = \delta^\mu_\nu + a^\mu{}_\nu$, we have

$$\varphi(\Lambda^{-1}x) = e^{\frac{i}{2}a^{\mu\nu}J_{\mu\nu}}\varphi(x)e^{-\frac{i}{2}a^{\mu\nu}J_{\mu\nu}}\,. \tag{5.1.41}$$

where $[J_{\mu\nu}, \varphi] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\varphi$.

Let me present another view of the generators. The "algebra" in (5.1.35) satisfied by the generators P_μ and $J_{\mu\nu}$ is also satisfied by the corresponding operators

$$\tilde{P}_\mu = -i\partial_\mu \qquad \text{and} \qquad \tilde{J}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\,. \tag{5.1.42}$$

In this case, we have

$$\varphi(x - a) = e^{-ia^\mu \tilde{P}_\mu}\varphi(x) \qquad \text{and} \qquad \varphi(\Lambda^{-1}x) = e^{-\frac{i}{2}a^{\mu\nu}\tilde{J}_{\mu\nu}}\varphi(x)\,. \tag{5.1.43}$$

Exercise 5.1.6 : (i) Consider the transformations (5.1.40) and (5.1.41), but replace $a^\nu \to \epsilon a^\nu$ and $a^{\mu\nu} \to \epsilon a^{\mu\nu}$ with $\epsilon \ll 1$. Show that these finite transformation laws reduce to the infinitesimal ones in (5.1.17) and (5.1.26) respectively at leading order in ϵ. (ii) Same exercise, but now start with (5.1.43).

5.2 Lorentz Transformations of Fields

We now understand how scalar fields transform under Lorentz transformations. But we know that nature has other types of fields such as vector fields, tensor fields, spinor fields etc. Where do these fit in?

Consider an N component field $\phi_\alpha(x)$. Each component is a scalar field, however, the different components can mix with each other under a Lorentz transformation. In general, this field transforms as

$$\phi_\alpha(x) \xrightarrow{\Lambda} [\mathsf{M}(\Lambda)]_\alpha{}^\beta\phi_\beta(\Lambda^{-1}x) \tag{5.2.1}$$

where $\mathsf{M}(\Lambda)$ is a matrix which depends on the Lorentz transformation under consideration, and satisfy $\mathsf{M}(\Lambda_1)\mathsf{M}(\Lambda_2) = \mathsf{M}(\Lambda_1\Lambda_2)$, and $\mathsf{M}(\Lambda^{-1}) = [\mathsf{M}(\Lambda)]^{-1}$. $\mathsf{M}(\Lambda)$ is called a representation of the Lorentz group. Different types of fields transform under different representations. For example,

1. If $\phi_\alpha(x)$ are members of a collection of scalar fields, then $\mathsf{M}(\Lambda) = \mathbb{1}$. That is

$$\phi_\alpha(x) \xrightarrow{\Lambda} \delta_\alpha^\beta \phi_\beta(\Lambda^{-1}x) = \phi_\alpha(\Lambda^{-1}x) \tag{5.2.2}$$

2. If $\phi_\alpha(x)$ are components of a Lorentz vector field, then $\mathsf{M}(\Lambda) = \Lambda$. That is

$$\phi_\alpha(x) \xrightarrow{\Lambda} \Lambda_\alpha{}^\beta \phi_\beta(\Lambda^{-1}x) \tag{5.2.3}$$

Previously, we dealt with infinitesimal Lorentz transformations. $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon a^\mu{}_\nu$. This allows us to write $\phi_\alpha(\Lambda^{-1}x) = \phi_\alpha(x) - \epsilon a^\mu{}_\nu x^\nu \partial_\mu \phi_\alpha(x)$. What about $\mathsf{M}(\Lambda)$? Under an infinitesimal Lorentz transformation

$$[\mathsf{M}(\Lambda)]_\alpha{}^\beta = \delta_\alpha^\beta - \frac{i}{2}\epsilon\, a^{\mu\nu}[\Sigma_{\mu\nu}]_\alpha{}^\beta \tag{5.2.4}$$

Note that $\epsilon a^{\mu\nu}$ is determined by the Lorentz transformation $(\Lambda^\mu{}_\nu)$, and $\Sigma_{\mu\nu}$ is a matrix which depends on the type of field under consideration. Note that $\Sigma_{\mu\nu}$ is a matrix (there are 6 of them) whose entries are labelled by α and β. For $\mathsf{M}(\Lambda)$ to be a representation of the Lorentz group, $\Sigma_{\mu\nu}$ must satisfy [check!]

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = i\left(g_{\mu\sigma}\Sigma_{\nu\rho} + g_{\nu\rho}\Sigma_{\mu\sigma} - \rho \leftrightarrow \sigma\right). \tag{5.2.5}$$

We are now in a position to find the Noether currents and charges associated with the Lorentz transformation for our fields (without the restriction to scalar fields). Under a Lorentz transformation, the fields transform as

$$\begin{aligned}
\phi_\alpha(x) \quad &\xrightarrow{\Lambda} \quad [\mathsf{M}(\Lambda)]_\alpha{}^\beta \phi_\beta(\Lambda^{-1}x)\,, \\
&= \left[\delta_\alpha^\beta - \frac{i}{2}\epsilon\, a^{\mu\nu}[\Sigma_{\mu\nu}]_\alpha{}^\beta\right]\left[\phi_\beta(x) - \frac{\epsilon}{2}a^{\mu\nu}\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right)\phi_\beta(x)\right]\,, \\
&= \phi_\alpha(x) + \underbrace{\frac{\epsilon}{2}a^{\mu\nu}\left[\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\delta_\alpha^\beta - \frac{i}{2}[\Sigma_{\mu\nu}]_\alpha{}^\beta\right]\phi_\beta(x)}_{\delta\phi_\alpha(x)}\,,
\end{aligned} \tag{5.2.6}$$

where in the second line, we used eq. (5.2.4) for the first square bracket and eq. (5.1.26) for the second. Note that the same $\epsilon a^{\mu\nu}$ appears in both, since we are talking about the same underlying Lorentz transformation. In the third line, we have simply expanded the expression to leading order in ϵ. The Noether current is then given by

$$j^\mu = (\Pi^\mu)^\alpha \delta\phi_\alpha - F^\mu\,. \tag{5.2.7}$$

The non-infinitesimal version, after "peeling" off the $\epsilon a^{\mu\nu}$ are given by

$$(j^\mu)_{\nu\sigma} = x_\nu T^\mu{}_\sigma - x_\sigma T^\mu{}_\nu - i(\Pi^\mu)^\alpha\left[\Sigma_{\nu\sigma}\right]_\alpha{}^\beta \phi_\beta(x)\,. \tag{5.2.8}$$

The Noether charges are given by

$$J_{\nu\sigma} = \int d^3x\left[\left(x_\nu T^0{}_\sigma - x_\sigma T^0{}_\nu\right) - i(\Pi^0)^\alpha\left[\Sigma_{\nu\sigma}\right]_\alpha{}^\beta \phi_\beta(x)\right]\,. \tag{5.2.9}$$

The intergral of the first term can be interpreted as the orbital angular momentum (which we will now refer to by $L_{\mu\nu}$ whereas the second is associated with "spin" angular momentum. As you can check, $J_{\mu\nu}$ satisfies: $[J_{\mu\nu}, J_{\rho\sigma}] = i\left(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - \rho \leftrightarrow \sigma\right)$.

Spin: The term "spin" was introduced rather abruptly here. We will have more to say about this after some examples. For the moment, let us just get a quick sense of spin and establish nomenclature. First, for simplicity, assume that the fields are independent of spacetime.

- If a field transforms as a scalar under the 3-dimensional rotation group, we have a spin-0 field.

- If a field transforms as a 3-vector under the 3-dimensional rotation group, we have a spin-1 field.

We will discuss other spins later. There is another term that often appears in this context: *helicity*. While spin is the intrinsic angular momentum of a particle, helicity is the projection of the spin along the momentum of the particle. Note that the projection of orbital angular momentum along the linear momentum is zero.

5.3 Spin 1

Consider a spin-1 field (vector field) $A_\alpha(x)$. It transforms as

$$
\begin{aligned}
A_\alpha(x) \quad &\xrightarrow{\Lambda} \quad [\mathsf{M}(\Lambda)]_\alpha{}^\beta A_\beta(\Lambda^{-1}x) , \\
&= \left[\delta_\alpha^\beta - \frac{i}{2} \epsilon \, a^{\mu\nu} [\Sigma_{\mu\nu}]_\alpha{}^\beta \right] A_\beta(\Lambda^{-1}x) ,
\end{aligned}
\tag{5.3.1}
$$

with

$$
[\Sigma_{\mu\nu}]_\alpha{}^\beta = i \left(g_{\mu\alpha}\delta_\nu^\beta - g_{\nu\alpha}\delta_\mu^\beta \right) .
\tag{5.3.2}
$$

As you will verify in the exercise below, $A_\alpha(x) \xrightarrow{\Lambda} \Lambda_\alpha{}^\beta A_\beta(\Lambda^{-1}x)$.

Exercise 5.3.1 : (i) Consider a boost with a small velocity $v \ll 1$ in the x^1 direction. Explicitly write down the change in the components of the field (at leading order in v). That is

$$
\{A_0(x^\mu), A_1(x^\mu), A_2(x^\mu), A_3(x^\mu)\} \xrightarrow{\Lambda} ?
$$

For this part, use the transformation law in eq. (5.3.1) along with eq. (5.3.2). (ii) Using eq. (5.3.2), show that $A_\alpha(x) \xrightarrow{\Lambda} \Lambda_\alpha{}^\beta A_\beta(\Lambda^{-1}x)$ in general. Verify explicitly that the specific transformation derived in part (i) matches with the result obtained using the transformation law part (ii).

5.3.1 Massive Vector Fields

Let us consider the following Lagrangian density

$$
\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\nu A^\nu ,
\tag{5.3.3}
$$

where the *field-strength tensor*

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu} .
\tag{5.3.4}
$$

As you can check, this is a Lorentz scalar, hence the action will be invariant under a Lorentz transformation (which is something we want for all our theories). As a general rule, contract all your Lorentz indices when writing down a Lagrangian density with Lorentz vector fields and you will be fine. This theory is like free Electromagnetism, but in Electromagnetism $m = 0$. We chose $m \neq 0$ here to avoid complications from "gauge invariance" (a redundancy in the description of our theory) which we will discuss later.

Exercise 5.3.2 : Using the transformation properties of vector fields (∂_μ is also a vector field), verify that eq. (5.3.3) is a Lorentz scalar. That is $\mathcal{L}(x) \xrightarrow{\Lambda} \mathcal{L}(\Lambda^{-1}x)$.

Equations of Motion

The equations of motion for the vector field are given by the *Proca* equation (derivation is left as an exercise below):

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \,. \tag{5.3.5}$$

Note that since $F^{\mu\nu} = -F^{\nu\mu}$, we have $\partial_\mu \partial_\nu F^{\mu\nu} = 0$. Using this result in eq. (5.3.5) yields

$$\partial_\nu A^\nu = 0 \,. \tag{5.3.6}$$

Taking the 4-divergence of eq. (5.3.5) and then using eq. (5.3.6), we arrive at

$$(\partial_\mu \partial^\mu + m^2)A^\alpha = 0 \qquad \text{with} \qquad \partial_\alpha A^\alpha = 0 \,. \tag{5.3.7}$$

That is, each component $A^\alpha(x)$ satisfies the Klein-Gordon equation. While this makes it appear that each component behaves independently as a free-massive scalar field, this is not quite true. There is an additional important constraint coming from $\partial_\mu A^\mu = 0$ which links the different components together.

Exercise 5.3.3 : (i) Use the Lagrangian in eq. (5.3.3) to derive the Proca equation (5.3.5) using the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \,. \tag{5.3.8}$$

In the process, you will need to first show that

$$\Pi^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \,. \tag{5.3.9}$$

(ii) Starting with the Proca equation, arrive at eqns. (5.3.7). (iii) Verify that

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \,. \tag{5.3.10}$$

This is the *Bianchi Identity*.

Plane-Wave Solutions & Mode Expansion

Since the A_μ satisfy the Klein-Gordon equation, the general solution has the form (see Exercise 3.2.7):

$$
\begin{aligned}
A_\mu(x) &= \int d^4k\, \bar\delta(k^2 - m^2)\xi_\mu(k^\alpha)e^{-ik\cdot x} \,, \\
&= \int (dk)\left[\xi_\mu(\mathbf{k})e^{-ik\cdot x} + \xi_\mu^*(\mathbf{k})e^{ik\cdot x} \right] \,,
\end{aligned}
\tag{5.3.11}
$$

where $k_0 = \omega_\mathbf{k} \equiv \sqrt{m^2 + |\mathbf{k}|^2}$ in the second line.[4] The solution must also satisfy the constraint $\partial_\mu A^\mu = 0$. If we have

$$k^\mu \xi_\mu(\mathbf{k}) = 0 \,, \tag{5.3.12}$$

then this constraint is satisfied. That is, the $\xi_\mu(\mathbf{k})$ are restricted to a three dimensional space perpendicular to k^μ. An orthonormal basis set for this three dimensional vector space $\{\xi_\mu^{(\lambda)}(\mathbf{k})\}$ with (λ labelling 3 basis vectors) must satisfy:

$$
\begin{aligned}
\xi^{\mu(\lambda')*}(\mathbf{k})\xi_\mu^{(\lambda)}(\mathbf{k}) &= -\delta^{\lambda\lambda'} && \text{orthonomality} \,, \\
\sum_\lambda \xi_\mu^{(\lambda)*}(\mathbf{k})\xi_\nu^{(\lambda)}(\mathbf{k}) &= -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) \equiv \mathcal{P}_{\mu\nu}(k) && \text{completeness} \,.
\end{aligned}
\tag{5.3.13}
$$

---

[4]We have abused notation a bit to use the same symbol ξ in both lines.

In the second line, we have defined the *projection operator*, $\mathcal{P}_{\mu\nu}$, which we will use shortly. The index λ goes over three values.[5]

We can now expand our vector $\xi_\mu(\mathbf{k})$ as $\xi_\mu(\mathbf{k}) = \sum_\lambda \xi_\mu^{(\lambda)}(\mathbf{k}) a_\lambda(\mathbf{k})$ where for the moment $a_\lambda(\mathbf{k})$ are just numbers. Hence, a general solution of the equations of motion can be written as

$$A_\mu(x) = \int (dk) \sum_\lambda \left[\xi_\mu^{(\lambda)}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-ik\cdot x} + \xi_\mu^{(\lambda)*}(\mathbf{k}) a_\lambda^*(\mathbf{k}) e^{ik\cdot x} \right] . \tag{5.3.14}$$

Next let us turn to quantization of the field.

Canonical Quantization

In the Hamiltonian formalism, the conjugate momentum density to A_α is given by

$$\Pi^{0\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\alpha)} = -F^{0\alpha} . \tag{5.3.15}$$

Hence, $\Pi^{00} = 0$. That is, A_0 is not a dynamical degree of freedom. Hence, following our nose, the canonical commutation relations are given by

$$\left[A_i(t, \mathbf{x}), F^{0j}(t, \mathbf{y}) \right] = -i\delta^3(\mathbf{x} - \mathbf{y}) \delta_i^j , \tag{5.3.16}$$

where note that the minus sign comes from $\Pi^{0i} = -F^{0i}$. For this canonical quantization condition to be satisfied, along with the conditions on the basis vectors in (5.3.13), the $a_\lambda(\mathbf{k})$ appearing in the general solution (5.3.14) must be elevated to operators satisfying:

$$\left[a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}') \right] = 2\omega_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'} . \tag{5.3.17}$$

Propagator

Recall that in the case of scalar fields, the Feynman Propagator was defined as the vacuum expectation value of the time-ordered product of fields (see Sec. (4.2.3)). In the same way

$$
\begin{aligned}
i\Delta_{\mu\nu}(x - y) &\equiv \langle 0|T\{A_\mu(x) A_\nu(y)\}|0\rangle , \\
&\equiv \Theta(x^0 - y^0)\langle 0|A_\mu(x) A_\nu(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|A_\nu(y) A_\mu(x)|0\rangle , \\
&= \Theta(x^0 - y^0) \int (dk) e^{-ik\cdot(x-y)} \mathcal{P}_{\mu\nu}(k) + \Theta(y^0 - x^0) \int (dk) \left[e^{-ik\cdot(y-x)} \mathcal{P}_{\mu\nu}(k) \right] , \\
&= \int d^4k\, e^{-ik\cdot(x-y)} \frac{i\mathcal{P}_{\mu\nu}(k)}{k^2 - m^2 + i\epsilon} , \\
&= \int d^4k\, e^{-ik\cdot(x-y)} i\Delta_{\mu\nu}(k) .
\end{aligned}
\tag{5.3.18}
$$

where $\epsilon = 0^+$. In the above derivation, we used completeness relation in equations (5.3.13) and (5.3.17). For future use, note that the propagator in Fourier space is

$$i\Delta_{\mu\nu}(k) = \frac{i\mathcal{P}_{\mu\nu}(k)}{k^2 - m^2 + i\epsilon} . \tag{5.3.19}$$

Exercise 5.3.4 : We will fill in the steps used in the derivation of the propagator. (i) First, show that

$$\langle 0|A_\mu(x) A_\nu(y)|0\rangle = \int (dk) e^{-ik\cdot(x-y)} \mathcal{P}_{\mu\nu}(k) . \tag{5.3.20}$$

---

[5]We have a choice in selecting the basis. Usually, they are chosen to be eigenvectors of the helicity operator.

This will require using (5.3.13) and (5.3.17). (ii) Now, show that

$$\int d^4 k\, e^{-ik\cdot(x-y)} \frac{i\mathcal{P}_{\mu\nu}(k)}{k^2 - m^2 + i\epsilon} = \Theta(x^0 - y^0)\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|A_\nu(y)A_\mu(x)|0\rangle. \quad (5.3.21)$$

This will require remembering how to do contour integrals with residues.

5.3.2 Massless Vector Fields

Let us now focus on the Lagrangian density in eq. (5.3.3), but with $m = 0$.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad (5.3.22)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Euler Lagrange equations yield

$$\partial_\mu F^{\mu\nu} = \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = 0, \quad (5.3.23)$$

and we still have the Bianchi identity $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$.

Gauge Freedom

The massless theory has an important redundancy, a *gauge freedom*, which was not present in the massive theory. To see this note that $F_{\mu\nu}$ is unchanged if we change A_μ to $A_\mu + \partial_\mu f$. That is we can add an arbitrary 4 gradient of a scalar function, $\partial_\mu f$, to A_μ without affecting $F_{\mu\nu}$ or the equations of motion $\partial_\mu F^{\mu\nu} = 0$. For example, we can use this freedom to set $A^0 = 0$ and $\partial_i A^i = 0$[6], or $\partial_\mu A^\mu = 0$. These choices correspond to a choice of gauge. The former is called the Coulumb gauge, whereas the latter is called the Lorenz gauge. In these gauges, the equations of motion become

$$\begin{aligned}
\partial_\mu\partial^\mu A^i(x) &= 0 \qquad \text{Coulumb Gauge, } A^0 = 0 \text{ and } \partial_i A^i = 0, \\
\partial_\mu\partial^\mu A^\nu(x) &= 0 \qquad \text{Lorenz Gauge, } \partial_\mu A^\mu = 0.
\end{aligned} \quad (5.3.24)$$

Connecting to the "usual" Maxwell's Equations

We can make contact with the "usual" Maxwell's equations by identifying

$$\begin{aligned}
E_i &\equiv F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A^0 && \text{– these are components of the electric field}, \\
B_k &\equiv -\frac{1}{2}\epsilon_{ijk}F^{ij} = -\epsilon_{ijk}\partial^i A^j = \epsilon_{ijk}\partial_i A^j && \text{– these are components of the magnetic field},
\end{aligned} \quad (5.3.25)$$

where recall that A^μ and ∂^μ are 4-vectors, with $A_\mu = g_{\mu\nu}A^\nu$ and $\partial_\mu = g_{\mu\nu}\partial^\nu$. That is, $A^\mu = (A^0, A^i) = (A_0, -A_i)$ and $\partial^\mu = (\partial^0, \partial^i) = (\partial_0, -\partial_i)$. However, E_i and B_i are not 4-vectors. The ϵ_{ijk} is the Levi-cevita symbol: $\epsilon_{ijk} = 1$ for even permutations of $ijk = 123$, and $\epsilon_{ijk} = -1$ for odd permutations of $ijk = 123$, and zero otherwise. The equations of motion[7]

$$\begin{aligned}
\partial^\mu F_{\mu\nu} &= 0, \\
\Longrightarrow \partial_i E_i &= 0 && \nu = 0, \\
\partial_0 E_k &= \epsilon_{ijk}\partial_i B_j && \nu = k.
\end{aligned} \quad (5.3.26)$$

Moreover the Bianchi Identity (see eq. (5.3.10)) yields:

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \Longrightarrow \partial_i B_i = 0 \quad \text{and} \quad \partial_0 B_k = -\epsilon_{ijk}\partial_i E_j. \quad (5.3.27)$$

Together we have our complete set of sourceless Maxwell's equations.

---

[6]More carefully, we can set $\partial_i A^i = 0$, and then $A^0 = 0$ is allowed when we have no sources.

[7]Note that in the above equations, summation is implied for repeated indices even when the indices are not upper and lower.

Solutions in Coulumb Gauge

Let us work with the Coulumb gauge. Since $A_0 = 0$ in this gauge, we only have to think about $A_i(x)$. As with the massive vector field, $A_i(x)$ satisfy the Klein-Gordon equation, and the general solution is given by

$$A_j(x) = \int d^4k \, \delta(k^2) \xi_j(k^\alpha) e^{-ik \cdot x} \,,$$
$$= \int (dk) \left[\xi_j(\mathbf{k}) e^{-ik \cdot x} + \xi_j^*(\mathbf{k}) e^{ik \cdot x} \right] \,,$$

(5.3.28)

where $k_0 = \omega_{\mathrm{k}} \equiv |\mathbf{k}|$ in the second line, and we do not write the 0 component since it is zero in the Coulumb gauge. Also, the solution must satisfy $\partial^i A_i = 0$ in the Coulumb gauge. This condition is satisfied if we pick $\xi_i(\mathbf{k})$ such that

$$\delta^{ij} k_j \xi_i(\mathbf{k}) = 0 \,.$$

(5.3.29)

That is, the three dimensional vector $\vec{\xi}(\mathbf{k})$ is restricted to live in the plane perpendicular to $\mathbf{k}$. We pick a complete, orthonormal basis in this plane $\{\vec{\xi}^{(\lambda)}(\mathbf{k})\}$ (with λ ranging over 2 values):

$$\delta^{ij} \xi_i^{(\lambda)*}(\mathbf{k}) \xi_j^{(\lambda')}(\mathbf{k}) = \delta^{\lambda\lambda'} \,, \qquad \sum_\lambda \xi_j^{(\lambda)}(\mathbf{k}) \xi_j^{(\lambda)*}(\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) \,.$$

(5.3.30)

We can then write $\xi_j(\mathbf{k}) = \sum_\lambda \xi_j^{(\lambda)}(\mathbf{k}) a_\lambda(\mathbf{k})$. The general solution for the equation of motion becomes

$$A_j(x) = \int (dk) \sum_\lambda \left[\xi_j^{(\lambda)}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-ik \cdot x} + \xi_j^{(\lambda)*}(\mathbf{k}) a_\lambda^*(\mathbf{k}) e^{ik \cdot x} \right] \,.$$

(5.3.31)

Canonical Quantization in Coulumb Gauge

Let us turn to quantization. Since $A_0 = 0$, we have to only worry about A_j and its conjugate momentum $\Pi^{0i} = -F^{0i} = E_i$. Let us try to impose canonical commutation relationship on the field and its conjugate momentum. Naively, we would write

$$\left[A_i(x), \Pi^{0j}(y) \right]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_i^j \implies [A_i(x), E_j(y)]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{ij} \,.$$

(5.3.32)

However, this is inconsistent with Maxwell's equations. To see this, note that

$$\partial^i \partial^j [A_i(x), E_j(y)]_{x^0=y^0} = i\nabla^2 \delta^{(3)}(\mathbf{x} - \mathbf{y}) \,.$$

(5.3.33)

The right hand side is non-zero for $\mathbf{x} = \mathbf{y}$, however, the left hand side is zero because $\partial_j E_j = 0$ and $\partial^i A_i = 0$. Hence, we need a canonical commutation relation that is consistent with our equations of motion. As you can check, such a relation is given by (using Poisson brackets first?)

$$[A_i(x), E_j(y)]_{x^0=y^0} = i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \,.$$

(5.3.34)

Note that the inverse Laplacian is appropriately interpreted in Fourier space.

To be consistent with these commutation relations for the fields and conjugate momenta, we need the $a_\lambda(\mathbf{k})$ appearing in the general solution to satisfy:

$$\left[a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}') \right] = 2\omega_{\mathrm{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'} \,.$$

(5.3.35)

Finally, we can calculate the propagator in this gauge for the massless vector field:

$$i\Delta_{ij}(x - y) = \langle 0|T\{A_i(x)A_j(y)\}|0\rangle \,,$$
$$i\Delta_{ij}(k) = \frac{i}{k^2 + i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) \,.$$

(5.3.36)

This entire exercise can be repeated in the Lorenz gauge; details can be found in David Tong's lecture notes.

5.4 Spin $1/2$

Consider a 4-component spin-1/2 field: $\psi_\alpha(x)$. This field transforms as

$$\psi_\alpha(x) \quad \xrightarrow{\Lambda} \quad [\mathsf{M}(\Lambda)]_\alpha{}^\beta \psi_\beta(\Lambda^{-1}x),$$
$$= \left[\delta_\alpha^\beta - \frac{i}{2}\epsilon\, a^{\mu\nu}[\Sigma_{\mu\nu}]_\alpha{}^\beta\right]\psi_\beta(x), \tag{5.4.1}$$

where the six $\Sigma_{\mu\nu}$ matrices are constructed out of the commutator of "gamma" matrices.

$$\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. \tag{5.4.2}$$

Each γ_μ (for $\mu = 0, 1, 2, 3$) is a 4×4 matrix, with their defining properties being

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\mathbb{1}g_{\mu\nu}. \tag{5.4.3}$$

We also have $\gamma^\mu \equiv g^{\mu\nu}\gamma_\nu$. As can be checked, $\Sigma_{\mu\nu}$ satisfy the Lorentz algebra

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = i\left(g_{\mu\sigma}\Sigma_{\nu\rho} + g_{\nu\rho}\Sigma_{\mu\sigma} - \rho \leftrightarrow \sigma\right). \tag{5.4.4}$$

Exercise 5.4.1 : Check that the $\Sigma_{\mu\nu}$ defined in terms of gamma matrices satisfy the Lorenz algebra. Hint: First calculate $[\Sigma_{\mu\nu}, \gamma_\rho]$.

5.4.1 Dirac Spinors

Instead of writing things down in terms of the components, it is sometimes cleaner to write down the full object

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \tag{5.4.5}$$

which transforms as

$$\psi(x) \quad \xrightarrow{\Lambda} \quad \mathsf{M}(\Lambda)\psi(\Lambda^{-1}x) \qquad \text{where} \qquad \mathsf{M}(\Lambda) = \mathbb{1} - \frac{i}{2}\epsilon a^{\mu\nu}\Sigma_{\mu\nu}. \tag{5.4.6}$$

The explicit form of γ_μ matrices is not unique. For the case of massive, charged, spin 1/2 particles (Dirac Spinors – electrons/positrons), we can chose them to be made of Pauli matrices as follows:

$$\gamma_i = \left(\begin{array}{c|c} 0 & -\sigma_i \\ \hline \sigma_i & 0 \end{array}\right) \qquad \gamma_0 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \tag{5.4.7}$$

where σ_i are the Pauli Matrices. Note that $\Sigma_{ij} = (1/2)\epsilon_{ijk}\sigma_k$ where ϵ_{ijk} is the Levi-Civita symbol, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.4.8}$$

The Pauli matrices satisfy $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$.

Exercise 5.4.2 : We all know that rotating a vector by an angle of 2π around any axis gives us back the same vector. Is the same true for our four component object ψ? To check this, we need to first write down the non-infinitesimal version of the transformation law for the spinor (assume ψ does not depend on space-time co-ordinates):

$$\psi \quad \xrightarrow{\Lambda} \quad \mathsf{M}(\Lambda)\psi \qquad \text{where} \qquad \mathsf{M}(\Lambda) = \exp\left(-\frac{i}{2}a^{\mu\nu}\Sigma_{\mu\nu}\right). \tag{5.4.9}$$

(i) Find $a^{\mu\nu}$ so that it represents a rotation around the z axis by an angle θ.
(ii) Then show that $\psi \to -\psi$ when $\theta \to 2\pi$.

Let us collect a few results, which can be checked using the explicit forms of the matrices above.

1. In general $\gamma_\mu^\dagger \neq \gamma_\mu$, and $\Sigma_{\mu\nu}^\dagger \neq \Sigma_{\mu\nu}$. This leads to $\mathsf{M}^\dagger(\Lambda)\mathsf{M}(\Lambda) \neq \mathbb{1}$.

2. Note that $\gamma_0^\dagger = \gamma_0$ and $\gamma_0^2 = \mathbb{1}$. This motivates us to define and $\bar\gamma_\mu \equiv \gamma_0(\gamma_\mu)^\dagger\gamma_0 = \gamma_\mu$, $\bar\Sigma_{\mu\nu} \equiv \gamma_0(\Sigma_{\mu\nu})^\dagger\gamma_0 = \Sigma_{\mu\nu}$, so that $\gamma_0\mathsf{M}^\dagger(\Lambda)\gamma_0\mathsf{M}(\Lambda) = \mathbb{1}$. That is, $\gamma_0\mathsf{M}^\dagger(\Lambda)\gamma_0 = \mathsf{M}^{-1}(\Lambda)$.

3. It is convenient to define $\bar\Sigma_{\mu\nu} \equiv \gamma_0(\Sigma_{\mu\nu})^\dagger\gamma_0 = \Sigma_{\mu\nu}$.

4. It is also useful to define
$$\bar\psi(x) \equiv \psi^\dagger(x)\gamma_0, \tag{5.4.10}$$

because although $\psi^\dagger(x)\psi(x)$ does not transform as a Lorentz scalar, $\bar\psi(x)\psi(x)$ does transform as a Lorentz scalar. Similarly, $\bar\psi(x)\gamma^\mu\psi(x)$ transforms as a Lorentz vector.

Exercise 5.4.3 : Using the explicit forms for the γ matrices, show that

(i) $(\Sigma_{\mu\nu})^\dagger \neq \Sigma_{\mu\nu}$ and $\gamma_0(\Sigma_{\mu\nu})^\dagger\gamma_0 = \Sigma_{\mu\nu}$
(ii) $(\mathsf{M}[\Lambda])^\dagger\mathsf{M}[\Lambda] \neq \mathbb{1}$ and $\gamma^0(\mathsf{M}[\Lambda])^\dagger\gamma^0 = (\mathsf{M}[\Lambda])^{-1}$ up to order ϵ.
(iii) $\bar\psi(x)\gamma^\mu\psi(x) \longrightarrow \Lambda^\mu{}_\nu\bar\psi(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x)$

The Dirac Equation and its Solutions

Using our understanding of how $\psi(x)$ and $\bar\psi(x)\gamma^\mu\psi(x)$ transform under Lorentz transformations, we can check that the following Lagrangian density[8]

$$\mathcal{L} = \bar\psi(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x), \tag{5.4.11}$$

is real valued, and transforms as a Lorentz scalar. Note that really there should be $m\mathbb{1}$ in the above equation, but we will suppress the identity matrix. Using the fact that $S = \int d^4x\mathcal{L}$ is dimensionless, it is easy to check that the mass dimension of ψ is $3/2$.

Equation of Motion

To derive the equations of motion, as always, we turn to the Euler-Lagrange equations (treating ψ and $\bar\psi$ as independent, same as we did for charged scalar fields). One of these equations is much easier to derive

---

[8]There is an interesting historical account of how Dirac arrived at this Lagrangian, but we will not pursue it here.

than the other (though both are consistent with each other)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \,,$$

(5.4.12)

$$\implies (i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0 \,.$$

This is the Dirac equation. Note that the first term in the EL equations was zero. Let us operate on both sides of the Dirac equation with $i\gamma^\mu \partial_\mu + m$. This yields

$$(\partial_\mu \partial^\mu + m^2)\psi(x) = 0 \,.$$

(5.4.13)

Mode Expansion

The general solution, $\psi(x)$, which satisfied the Klein-Gordon equation will have the form.

$$\psi(x) = \int d^4 k \bar{\delta}(k^2 - m^2) W(k^\alpha) e^{-ik\cdot x} \,,$$

$$= \int (dk) \left(u(\mathbf{k}) e^{-ik\cdot x} + v(\mathbf{k}) e^{ik\cdot x} \right) \,,$$

(5.4.14)

where $u(\mathbf{k}) = W(\omega_k, \mathbf{k})$, and $v(\mathbf{k}) = W(-\omega_k, -\mathbf{k})$ and in the exponentials $k_0 = \omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$. Since the components of $\psi(x)$ are not real valued, $v(\mathbf{k}) \neq u^*(\mathbf{k})$. If

$$(\gamma^\mu k_\mu - m)u(\mathbf{k}) = 0 \qquad \text{and} \qquad (\gamma^\mu k_\mu + m)v(\mathbf{k}) = 0 \,,$$

(5.4.15)

then $(i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0$. We can expand $u(\mathbf{k})$ and $v(\mathbf{k})$ in an appropriate basis sets $\{u^{(\lambda)}(\mathbf{k})\}$ and $\{v^{(\lambda)}(\mathbf{k})\}$. That is

$$u(\mathbf{k}) = \sum_\lambda u^{(\lambda)}(\mathbf{k}) b_\lambda(\mathbf{k}) \qquad \text{and} \qquad v(\mathbf{k}) = \sum_\lambda v^{(\lambda)}(\mathbf{k}) d_\lambda^*(\mathbf{k}) = 0 \,,$$

(5.4.16)

where for the moment $b_\lambda(\mathbf{k})$ and $d_\lambda^*(\mathbf{k})$ are numbers. The λ ranges over two values. Note that the Dirac equation has 4-independent solutions since it is first order in time[9]. A careful choice of basis, allows us to chose λ such that we the two basis vectors correspond to two different spins. The explicit form of the basis vectors can be found in lectures 14 and 15 of Prof. P. Stevenson's notes. With appropriate normalization, the basis vectors satisfy

$$\bar{u}^{(\lambda)}(\mathbf{k}) u^{(\lambda')}(\mathbf{k}) = 2m\delta^{\lambda\lambda'} \qquad \text{and} \qquad \sum_\lambda u^{(\lambda)}(\mathbf{k}) \bar{u}^{(\lambda)}(\mathbf{k}) = \gamma_\mu k^\mu + m \,,$$

$$\bar{v}^{(\lambda)}(\mathbf{k}) v^{(\lambda')}(\mathbf{k}) = -2m\delta^{\lambda\lambda'} \qquad \text{and} \qquad \sum_\lambda v^{(\lambda)}(\mathbf{k}) \bar{v}^{(\lambda)}(\mathbf{k}) = \gamma_\mu k^\mu - m \,.$$

(5.4.17)

With these basis vectors, our mode expansion is

$$\psi(x) = \int (dk) \sum_\lambda \left[u^{(\lambda)}(\mathbf{k}) b_\lambda(\mathbf{k}) e^{-ik\cdot x} + v^{(\lambda)}(\mathbf{k}) d_\lambda^*(\mathbf{k}) e^{ik\cdot x} \right] \,,$$

(5.4.18)

where $k_0 = \omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$.

---

[9]It is useful to go to the rest frame $k_\mu = (m, 0)$ to see the structure of these equations

Quantization

Let us follow our nose again and try to quantize our theory. First, we need the conjugate momentum (in component form)

$$\pi_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = i\psi_\alpha^\dagger(x) \,. \tag{5.4.19}$$

The usual commutation relation would have been

$$\left[\psi_\alpha(x), i\psi_\beta^\dagger(y)\right]_{x^0 = y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} \,. \tag{5.4.20}$$

If we plug in our mode expansion (along with elevating b_λ and d_λ to operators), we will find that this commutation relation is *not* consistent with

$$\left[b_\lambda(\mathbf{k}), b_{\lambda'}^\dagger(\mathbf{q})\right] = 2\omega_k\delta^{(3)}(\mathbf{k} - \mathbf{q})\delta_{\lambda\lambda'} \qquad \text{and} \qquad \left[d_\lambda(\mathbf{k}), d_{\lambda'}^\dagger(\mathbf{q})\right] = 2\omega_k\delta^{(3)}(\mathbf{k} - \mathbf{q})\delta_{\lambda\lambda'} \,. \tag{5.4.21}$$

You can see this in lecture 16 of P. Stevenson's notes. However, what does work is imposing "anti-commutation" relations. That is

$$\left\{\psi_\alpha(x), i\psi_\beta^\dagger(y)\right\}_{x^0 = y^0} \equiv \psi_\alpha(x)i\psi_\beta^\dagger(y) + i\psi_\beta^\dagger(y)\psi_\alpha(x)\bigg|_{x^0 = y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} \,, \tag{5.4.22}$$

and correspondingly

$$\left\{b_\lambda(\mathbf{k}), b_{\lambda'}^\dagger(\mathbf{q})\right\} = 2\omega_k\delta^{(3)}(\mathbf{k} - \mathbf{q})\delta_{\lambda\lambda'} \qquad \text{and} \qquad \left\{d_\lambda(\mathbf{k}), d_{\lambda'}^\dagger(\mathbf{q})\right\} = 2\omega_k\delta^{(3)}(\mathbf{k} - \mathbf{q})\delta_{\lambda\lambda'} \,. \tag{5.4.23}$$

with all other anti-commutators being zero. Note that there is no classical Poisson Bracket version of this. The mode expansion now becomes

$$\psi(x) = \int (dk) \sum_\lambda \left[u^{(\lambda)}(\mathbf{k})b_\lambda(\mathbf{k})e^{-ik\cdot x} + v^{(\lambda)}(\mathbf{k})d_\lambda^\dagger(\mathbf{k})e^{ik\cdot x}\right] \,. \tag{5.4.24}$$

An immediate consequence of the anti-commuting operators is that we cannot have a momentum state two particles with the same λ. That is

$$b_\lambda^\dagger(\mathbf{k})b_\lambda^\dagger(\mathbf{k})|0\rangle = -b_\lambda^\dagger(\mathbf{k})b_\lambda^\dagger(\mathbf{k})|0\rangle = 0 \,. \tag{5.4.25}$$

That is the occupation number for a given λ and $\mathbf{k}$ has to be 0 or 1. This is how fermi-Dirac statistics arise! Moreover there is a famous theorem called the "Spin-Statistics Theorem" which states that integer spin particles give rise to Bose-Einstein statistics, whereas half-integer spin particles yield Fermi-Dirac statistics.

Soon, we will want to carry out scattering calculations. Such scattering calculations yield are simplified by defining and working with normal ordered products of creation and annihilation operators. Recall that for integer spin particles (bosons), we defined normal ordering as $: a^\dagger a := a^\dagger a$ and $: aa^\dagger := a^\dagger a$. For fermions, normal ordering is defined as

$$: b_\lambda^\dagger b_{\lambda'}' := b_\lambda^\dagger b_{\lambda'} \qquad \text{and} \qquad : b_\lambda b_{\lambda'}^\dagger : -b_{\lambda'}^\dagger b_\lambda \,. \tag{5.4.26}$$

Moreover note that the time-ordering

$$T\{\psi(x)\bar{\psi}(y)\} = \Theta(x^0 - y^0)\psi(x)\bar{\psi}(y) - \Theta(y^0 - x^0)\bar{\psi}(y)\psi(x) \,. \tag{5.4.27}$$

This is required by Lorentz Invariance (see D. Tong's notes). The fermion propagator

$$
\begin{aligned}
i\Delta_\psi(x-y) &= \langle 0|T\{\psi(x)\bar{\psi}_\beta(y)\}|0\rangle\,, \\
&= \Theta(x^0-y^0)\int (dk)(k^\mu\gamma_\mu+m)e^{-ik\cdot(x-y)} - \Theta(y^0-x^0)\int (dk)(k^\mu\gamma_\mu-m)e^{ik\cdot(x-y)}\,, \\
&= \int d^4k\, \frac{i(k^\mu\gamma_\mu+m)}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)}\,, \\
\implies i\Delta_\psi(k) &= \frac{i(k^\mu\gamma_\mu+m)}{k^2-m^2+i\epsilon}\,.
\end{aligned}
\tag{5.4.28}
$$

Exercise 5.4.4 : Using the anti-commutation relations in (5.4.23), and the mode expansion in (5.4.24), show that
(i)

$$
\{\psi_\alpha(x),\pi_\beta(y)\}_{x^0=y^0} = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}
$$

where $\pi_\alpha = i\psi^\dagger$.
(ii) The Hamiltonian density for our spin 1/2 fields is obtained in the usual way via a Legendre transform of the Lagrangian denisty:

$$
\mathcal{H} = \sum_\alpha \pi_\alpha\dot{\psi}_\alpha - \mathcal{L}\,,
\tag{5.4.29}
$$

Show that the normal ordered Hamiltonian is (when evaluated on the solution):

$$
:H: = :i\int d^3x\,\psi^\dagger\partial_0\psi := \int (dp)\sum_{\lambda=1}^2 \omega_{\mathbf{p}}\left(b_\lambda^\dagger(\mathbf{p})b_\lambda(\mathbf{p}) + d_\lambda^\dagger(\mathbf{p})d_\lambda(\mathbf{p})\right)\,.
\tag{5.4.30}
$$

5.5 Quantum Electrodynamics

Having dealt with free spin 1 (massless) and massive spin 1/2 fields, let us now consider interactions between them. Consider the Lagrangian density

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\nu\partial_\nu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\,.
\tag{5.5.1}
$$

While not obvious, this Lagrangian leads to an action which is invariant under $A_\mu \to A_\mu + \partial_\mu\alpha$ (we will come back to this soon). We have already derived the mode expansions and propagators from the first and second terms. To carry out perturbative calculations, we need the vertex function

$$
i\frac{\partial\mathcal{L}_{\text{int}}}{\partial A_\mu\partial\bar{\psi}\partial\psi} = (-ie)\gamma^\mu\,.
\tag{5.5.2}
$$

Let us now write down the Feynman rules associated with our theory (see Fig. 5.1), where

$$
\begin{aligned}
i\Delta_{\mu\nu}(k) &= -\frac{ig_{\mu\nu}}{k^2+i\epsilon}\,, \\
i\Delta_\psi(k) &= \frac{i(k^\mu\gamma_\mu+m)}{k^2-m^2+i\epsilon}\,.
\end{aligned}
\tag{5.5.3}
$$

The first line is an extension of the propagator derived in the Coulomb gauge (see (5.3.36)). The reason it looks different here is two fold. First, we included the 0 index here. This is necessary because while we

can impose $\partial_i A^i = 0$, it does not lead to $A_0 = 0$ (which we used earlier) in presence of the interaction term. Second, you might have expected a term like $k_\mu k_\nu / k^2$ to also appear with the $g_{\mu\nu}$. However, it turns out that terms like $k_\mu \bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p})$ which appear in scattering calculations vanish after taking into account the addition of momenta at the interaction vertex, and the constraints on u and $\bar{u}$ coming from the Dirac equation. After some work, one can show that the above form of the propagator is the correct one to use for our theory. See D. Tong's notes for details.
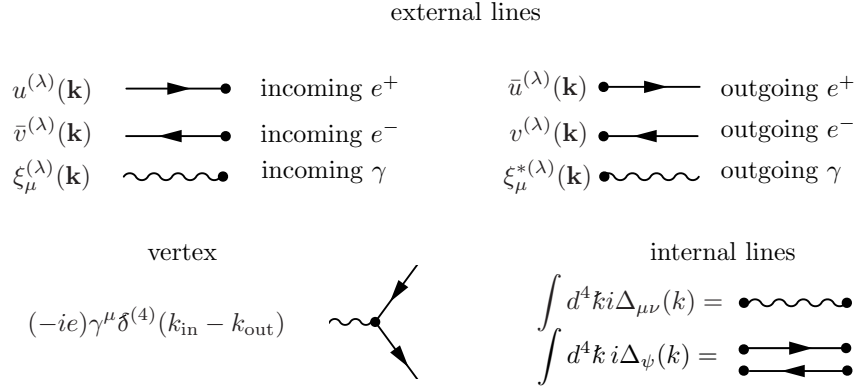
external lines

Figure 5.1: $k-$space Feynman Rules for Quantum Electrodynamics

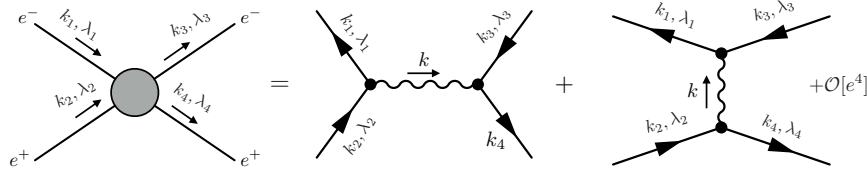Let us carry out a sample calculation for the $e^+ e^- \to e^+ e^-$ process using our Feynman rules.

Figure 5.2: Leading order $e^- e^+ \to e^- e^+$ scattering.

Let us focus on the first of the two diagrams (the s-channel one) that contribute at order e^2. To write down the integral expression, it helps to follow the fermion lines in the direction opposite to the charge arrows. For example, we start with the k_1, λ_1 part, pass through the vertex and end at the k_2, λ_2 end. This would yield a factor $\bar{v}^{(\lambda_1)}(\mathbf{k}_1)(-ie)\delta^{(4)}(k_1 + k_2 - k)\gamma^\mu u^{(\lambda_2)}(\mathbf{k}_2)$. Similarly, the we follow k_4, λ_4 through the vertex and out to k_3, λ_3 to get $\bar{u}^{(\lambda_4)}(\mathbf{k}_4)(-ie)\delta^{(4)}(k - k_3 - k_4)\gamma^\nu v^{(\lambda_3)}(\mathbf{k}_3)$. Now sandwich the propagator in the middle to get (with the integral outside)

$$\text{s-channel} = -e^2 \int d^4k \delta^{(4)}(k_1 + k_2 - k)\delta^{(4)}(k - k_3 - k_4)\bar{v}^{(\lambda_1)}(\mathbf{k}_1)\gamma^\mu u^{(\lambda_2)}(\mathbf{k}_2)i\Delta_{\mu\nu}(k)\bar{u}^{(\lambda_4)}(\mathbf{k}_4)\gamma^\nu v^{(\lambda_3)}(\mathbf{k}_3),$$

$$= -ie^2\delta^{(4)}(k_1 + k_2 - k_3 - k_4)\frac{\left[\bar{v}^{(\lambda_1)}(\mathbf{k}_1)\gamma^\mu u^{(\lambda_2)}(\mathbf{k}_2)\right]\left[\bar{u}^{(\lambda_4)}(\mathbf{k}_4)\gamma_\mu v^{(\lambda_3)}(\mathbf{k}_3)\right]}{(k_1 + k_2)^2}.$$

(5.5.4)

In an exercise below you will calculate the t-channel contribution in a similar way.

Exercise 5.5.1 : Write down contribution of the t-channel process to the matrix element in $e^- e^+ \to e^- e^+$ scattering. You can leave the expression in terms of u, v etc, and do not worry about the overall sign.

There can be pesky minus signs in front of different diagrams which require us to go back to the Dyson

expansion (at the end of the day, they come from the anti-commuting nature of our fields). I am not going to go through that here. You can find a discussion elsewhere, for example, in D. Tong's notes, or the book by Peskin and Schroeder. These signs are important, since they can introduce relative signs between diagrams (so be careful about using the above expressions!). There are also rules related to signs when loops are present.

Typically one holds of on simplifying the matrix elements further, since the evaluation is easier when considering a sum over spins as is often the case in evaluating cross-sections. See more in P. Stevenson's notes (QFTlec17.pdf) if you are interested.

5.5.1 Local Gauge Invariance

Global Symmetries

The interaction term in the QED Lagrangian that we wrote down earlier is not arbitrary. First note that the Lagrangian is invariant under a continuous "global" internal symmetry, $U(1)$ transformations,

$$\psi(x) \to e^{i\alpha}\psi(x) \qquad \text{where } \alpha \text{ is an arbitrary constant.} \tag{5.5.5}$$

Noether's theorem then guarantees the existence of a conserved current (which you should derive):

$$j^\mu = e\bar{\psi}\gamma^\mu\psi\,, \qquad \partial_\mu j^\mu = 0\,. \tag{5.5.6}$$

Hence the structure of the interaction terms is of the form $A_\mu j^\mu$ (with $\partial_\mu j^\mu = 0$ on the solutions). That is, we are coupling A_μ to a conserved current.

Local Symmetries

What if we made α appearing in (5.5.5) a function of spacetime, $\alpha(x)$, and demand that the Lagrangian remain invariant under a "local" $U(1)$ transformation:

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \qquad \text{where } \alpha(x) \text{ is an arbitrary real-valued function of spacetime} \tag{5.5.7}$$

Just using this transformation, you would find that (5.5.1) transforms as

$$\mathcal{L} \to \mathcal{L} - \bar{\psi}\gamma^\mu\partial_\mu\alpha(x)\psi\,. \tag{5.5.8}$$

which is obviously not invariant. However, the Lagrangian will be invariant under this local $U(1)$ transformation for any $\alpha(x)$, if we specify a transformation rule

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)\,, \tag{5.5.9}$$

for our vector fields as well. With this in mind, it is convenient and conceptually beneficial to introduce a "covariant" derivative:

$$\mathcal{D}_\mu \equiv \partial_\mu + ieA_\mu\,. \tag{5.5.10}$$

In terms of this derivative, the QED Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\nu\mathcal{D}_\nu - m)\psi\,. \tag{5.5.11}$$

Note that the interaction term in hidden from view, but it arises from $\mathcal{D}_\mu$. It is also worth noting that in terms of the covariant derivative

$$F_{\mu\nu} = -\frac{i}{e}\left(\mathcal{D}_\nu\mathcal{D}_\mu - \mathcal{D}_\nu\mathcal{D}_\mu\right) = -\frac{i}{e}[\mathcal{D}_\mu, \mathcal{D}_\nu]\,. \tag{5.5.12}$$

It is typically best to feed a trial function to both sides when verifying such relations involving derivatives.

Let us change our perspective, and consider the following steps:

1. Start with the Dirac Lagrangian, $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$, which has a global symmetry $\psi \to e^{i\alpha}\psi$ ($\alpha=$ const.).

2. Insist on making this a local symmetry, i.e. insist that $\mathcal{L}$ must remain unchanged under $\psi(x) \to e^{i\alpha(x)}\psi(x)$.

3. This can be achieved by replacing ∂_μ with $\mathcal{D}_\mu \equiv \partial_\mu + ieA_\mu$ where $A_\mu(x) \to A_\mu(x) - (1/e)\partial_\mu\alpha(x)$ under the local $U(1)$ transformation.

4. We should have a kinetic term for the field A_μ that we introduced (which must respect the local symmetry). This leads us to adding $F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian. So we end up with QED: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\nu\mathcal{D}_\nu - m)\psi$.

5. In making our Lagrangian invariant under the local symmetry, we introduced a gauge field A_μ and an appropriate interaction $eA_\nu\bar{\psi}\gamma^\nu\psi$.

5.5.2 Yang-Mills Theories

In the previous section we started with the Dirac Lagrangian which has a global $U(1)$ symmetry, and by making this symmetry local, we ended up with Quantum Electrodynamics. Let us generalize this by considering a Lagrangian which is invariant under $SU(N)$.

Before we do this, let us recall some properties of $SU(N)$. Recall that an element of $U(1) = SU(1)$ looks like $\mathsf{U} = e^{i\alpha}$ which is obviously a one dimensional unitary "matrix" with unit determinant. Elements of $SU(N)$ (in the fundamental representation) are $N \times N$ unitary matrices (i.e. $\mathsf{U}^\dagger\mathsf{U} = \mathbb{1}$) with det $\mathsf{U} = 1$. An important difference between the $SU(1)$ case and $SU(N)$ with $N \neq 1$ case is that the elements of $SU(N)$ do not necessarily commute with each other – ie. the group is non-Abelian, whereas in the $SU(1)$ case the group is Abelian. $SU(N)$ is a Lie group, and the elements can be written as

$$\mathsf{U} = e^{i\alpha_A\mathsf{T}_A} \qquad \text{with} \qquad [\mathsf{T}_A, \mathsf{T}_B] = if_{ABC}\mathsf{T}_C\,, \tag{5.5.13}$$

where A runs from 1 to $N^2 - 1$ and summation over repeated indices is assumed. Here, T_B are generators of the group; they are Hermitian and traceless matrices (follows from the U being unitary and det $\mathsf{U} = 1$). We will denote an entry of the U or T_A matrix as U_{ab} or $[T_A]_{ab}$.

Consider the following Lagrangian:

$$\mathcal{L} = \bar{\psi}_a(i\gamma^\mu\partial_\mu - m)\psi_a\,, \tag{5.5.14}$$

where for each a, ψ_a is a 4-component Dirac Spinors, and a sum from $a = 1$ to N is implied. Under the a $SU(N)$ transformation, each ψ_a transforms as

$$\psi_a \to U_{ab}\psi_b = e^{i\alpha_A[T_A]_{ab}}\psi_b\,, \tag{5.5.15}$$

where a summation over $b = 1 \ldots N$ is implied. Moreover A is summer over 1 to $N^2 - 1$. The Lagrangian is trivially invariant under a global $SU(N)$ transformation, where α_A are all constants.

Now let us make consider a local $SU(N)$ transformation, that is, we consider $\alpha_A(x)$. Under this tranformation, the Lagrangian is no longer invariant. However, it can be made so by changing the regular derivative to a covariant derivative

$$\mathcal{D}_\mu \equiv \partial_\mu + ie\mathsf{T}_AW_\mu^A(x)\,, \tag{5.5.16}$$

where we introduced $N^2 - 1$ gauge fields with the transformation property

$$W_\mu^A(x) \to W_\mu^A(x) - \frac{1}{e}\partial_\mu\alpha_A(x) - f_{ABC}\mathsf{T}_BW_\mu^C(x)\,. \tag{5.5.17}$$

We now need to add kinetic terms for the gauge fields $W_\mu^A(x)$ which does not destroy the local symmetry of the action. We do so with the help of

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = ie\mathsf{T}_C \left(\partial_\mu W_\nu^C - \partial_\nu W_\mu^C - ef_{ABC}W_\mu^A W_\nu^B \right) . \tag{5.5.18}$$

The term $(\ldots) \equiv F_{\mu\nu}^C$. Hence our final Lagrangian has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^C F_C^{\mu\nu} + \bar{\psi}_a(i\gamma^\mu \mathcal{D}_\mu - m)\psi_a , \tag{5.5.19}$$

where all repeated indices are summed over. This Lagrangian is invariant under local $SU(N)$ transformations, and has $N^2 - 1$ gauge fields and N spinor fields.

This Lagrangian has a number of interaction terms. Schematically, there is a two fermions and gauge field vertex coming from the $\bar{\psi}W\psi$ term. A similar vertex was present in QED. There is a three gauge field vertex, and a four gauge field vertex coming from the FF term, this is new to our Yang-Mills theory.

Everything below TO BE COMPLETED

5.6   Broken Symmetries

5.6.1   Preliminaries – Nambu-Goldstone Theorem

5.6.2   Anderson-Higgs Mechanism and Applications

5.6.3   Defects

Topological Solitons

5.7   QFT in Nontrivial Classical Backgrounds

MATHEMATICAL PRELIMINARIES

A.1 Fourier Transforms

Consider a function $f(\mathbf{x})$ which satisfied periodic boundary conditions in a box of size L (volume $V = L^3$). Then

$$f(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}$$

$$f_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int_V d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \tag{A.1.1}$$

$$\mathbf{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

If $L \to \infty$,

$$f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})$$

$$f(\mathbf{k}) = \int_V d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \tag{A.1.2}$$

The finite and infinite box cases are related by

$$\sum_{\mathbf{k}} \to V \int \frac{d^3k}{(2\pi)^3}$$

$$f_{\mathbf{k}} \to \frac{1}{\sqrt{V}} f(\mathbf{k}) \tag{A.1.3}$$

In $3 + 1$ spacetime dimensions $x = x^\mu = (x^0, \mathbf{x})$ and $k = k^\mu = (k^0, \mathbf{k})$, with $k \cdot x = k^0 x^0 - \mathbf{k} \cdot \mathbf{x}$ (also see the review of Special Relativity).

$$f(x) = \int d^4k \, e^{-ik\cdot x} f(k)$$

$$f(k) = \int d^4x \, e^{ik\cdot x} f(x) \tag{A.1.4}$$

A.2 Delta Functions

<u>Kroneckar Delta</u>: The Kroneckar Delta function is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \,, \\ 0 & i \neq j \,. \end{cases} \tag{A.2.1}$$

For Fourier Transforms in a finite box,

$$\delta_{\mathbf{q},\mathbf{k}} = \begin{cases} 1 & \mathbf{q} = \mathbf{k} \,, \\ 0 & \mathbf{q} \neq \mathbf{k} \,. \end{cases} \tag{A.2.2}$$

A useful representation of the Kroneckar Delta function is

$$\delta_{\mathbf{q},\mathbf{k}} = \frac{1}{V} \int_V d^3 x e^{-i(\mathbf{q}-\mathbf{k})\cdot\mathbf{x}} \,. \tag{A.2.3}$$

<u>Dirac Delta</u>: The Dirac Delta function is defined by

$$\int d^3 x \delta(\mathbf{x} - \mathbf{y}) f(\mathbf{x}) = f(\mathbf{y}) \,, \tag{A.2.4}$$

for sufficiently well behaved $f(\mathbf{x})$. A useful representation of the Dirac Delta function is

$$\delta(\mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \,. \tag{A.2.5}$$

The Dirac Delta "function" is an example of a distribution. Two distributions D_1 and D_2 are equal if

$$\int dx D_1(x) f(x) = \int dx D_2(x) f(x) \,. \tag{A.2.6}$$

Some useful identities related to the Dirac-Delta function (which can be appropriately generalized to arbitrary number of dimensions), are listed below:

$$\delta(ax) = \frac{1}{|a|} \delta(x) \,,$$
$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i) \qquad \text{where } f(x_i) = 0 \,, \tag{A.2.7}$$
$$\frac{d}{dx} \delta(x) = -\delta(x) \frac{d}{dx} \,.$$

The last property can be checked using integration by parts: $\int dx f(x) \delta'(x) = -\int dx f'(x) \delta(x)$.

We will see the combination $(2\pi)^3 \delta(\mathbf{k})$ as well as $d^3 k/(2\pi)^3$ often, hence it is useful to define:

$$d^3 k \equiv \frac{d^3 k}{(2\pi)^3} \,,$$
$$\bar{\delta}(\mathbf{k}) \equiv (2\pi)^3 \delta(\mathbf{k}) \,. \tag{A.2.8}$$

These generalize to an arbitrary number of dimensions as well.

A.3 Functionals

We can think of a function $f(x)$ as taking an argument x and returning a number $f(x)$. A functional $F[f]$, takes an entire function f and returns a number $F[f]$. For example, let $f(x) = x^2$ and

$$F[f] = \int_{-1}^{1} dx f(x) = \int_{-1}^{1} dx x^2 = \frac{2}{3}. \tag{A.3.1}$$

You have been using functionals all along, without calling them by that name. The function $f(x)$ can also be though of as a specific functional:

$$F[f] = \int dy \delta(x - y) f(y) = f(x). \tag{A.3.2}$$

though we will call such "local" functionals, just functions. Thinking in terms of functionals is quite useful for variational problems (extremizing the action), as well as in, unsurprisingly, the functional formulation of QFT. Let us learn some rules for dealing with functionals. The derivative of a functional at a location $x = y$ is

$$\frac{\delta F}{\delta f(y)} \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ F[f(x) + \epsilon \delta(x - y)] - F[f(x)] \right\}. \tag{A.3.3}$$

This definition captures the following. What is the rate of change of the functional $F[f]$ as we perturb $f(x)$ at the location y. The usual rules of derivative such as the chain rule apply to functional derivatives as well.

As a very relevant application, consider the *action functional*:

$$S = \int_{t_i}^{t_f} dt L(q, \dot{q}, t), \tag{A.3.4}$$

Extremizing the action corresponds to finding the path $q(t)$, such that for $q(t) \to q(t) + \delta q(t)$, $\delta S / \delta q = 0$ for all $\delta q(t)$ such that $\delta q(t_i) = \delta q(t_f) = 0$. Let us calculate $\delta S / \delta q(t)$.

$$\begin{aligned}
\frac{\delta S}{\delta q(t)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ S[q(\tau) + \epsilon \delta(\tau - t)] - S[q(\tau)] \right\}, \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_i}^{t_f} d\tau \left\{ L(q(\tau) + \epsilon \delta(\tau - t), \dot{q}(\tau) + \epsilon \dot{\delta}(\tau - t)) - L(q(\tau), \dot{q}(\tau)) \right\}, \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_i}^{t_f} d\tau \left\{ \epsilon \frac{\partial L(q, \dot{q})}{\partial q} \delta(\tau - t) + \epsilon \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \dot{\delta}(\tau - t) \right\}, \\
&= \frac{\partial L(q, \dot{q})}{\partial q} - \frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right).
\end{aligned} \tag{A.3.5}$$

The first line is just the definition of a functional derivative. In the third line, we treat L as a usual function (takes $\{q(t), \dot{q}(t)\}$ and spits out a number) and use the chain rule. In going from the penultimate line to the last line, used integration by parts. Extremizing the action

$$\frac{\delta S}{\delta q} = 0 \implies \frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) = \frac{\partial L(q, \dot{q})}{\partial q}. \tag{A.3.6}$$

which are our usual Euler-Lagrange equations.

Let us now consider the action functional for a relativistic scalar field $\varphi(x) = \varphi(t, \mathbf{x})$:

$$S = \int d^4 y \mathcal{L}(\varphi(y), \partial_\mu \varphi(y)). \tag{A.3.7}$$

where $\mathcal{L}$ is the Lagrangian density. Repeating the steps we took in the previous case, (and assuming the boundary at infinity),

$$
\begin{aligned}
\frac{\delta S}{\delta \varphi(x)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ S[\varphi(y) + \epsilon \delta(y - x)] - S[\varphi(y)] \right\} , \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4 y \left\{ \mathcal{L}(\varphi(y) + \epsilon \delta(y - x), \partial_\mu \varphi(y) + \epsilon \partial_\mu \delta(y - x)) - \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) \right\} , \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4 y \left\{ \epsilon \frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial \varphi} \delta(y - x) + \epsilon \frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial \varphi} \partial_\mu \delta(y - x) \right\} , \\
&= \frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial_\mu \varphi} \right) .
\end{aligned}
\tag{A.3.8}
$$

where in going from the penultimate to the last line, we have used the 3+1 dimensional version of the divergence theorem along with integration by parts. Extremizing the action

$$
\frac{\delta S}{\delta \varphi} = 0 \implies \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial_\mu \varphi} \right) = \frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial \varphi} .
\tag{A.3.9}
$$

Further Reading: For a quick and gentle introduction to functionals, see for example, section 1.3 in *QFT for the Gifted Amateur* by Lancaster and Blundell.

A.4 Green's Functions

Consider a linear differential equation

$$
\mathcal{D}\varphi(x) = f(x) ,
\tag{A.4.1}
$$

where $\mathcal{D}$ is some linear differential operator, say for example $\mathcal{D} = \partial^\mu \partial_\mu$. The general solution for the above equation can be written as

$$
\varphi(x) = \varphi_h(x) + \int d^4 y \, G(x, y) f(y) ,
\tag{A.4.2}
$$

where

$$
\begin{aligned}
\mathcal{D} G(x, y) &= \delta(x - y) , \\
\mathcal{D} \varphi_h(x) &= 0 .
\end{aligned}
\tag{A.4.3}
$$

where $\varphi_h(x)$ is the *homogeneous* solution. The function $G(x, y)$ is the *Green's function* for the system. One needs to specify initial/boundary conditions to determine it. Once $G(x, y)$ is determined, our problem is solved for arbitrary $f(x)$! This is why Green's functions are useful.

A.5 Contour Integration

Let $f(z)$ be a function in the complex plane, and C a closed curve in the complex plane. Then

$$
\oint_C dx f(z) = 2\pi i \sum_j \left(\text{Residue of } f \text{ at } z_j \text{ inside } C \right) ,
\tag{A.5.1}
$$

where the integration is done in the anti-clockwise direction, and

$$
\text{Residue of } f \text{ at } z_j = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_j)^n f(z) \right] ,
\tag{A.5.2}
$$

The largest n for which Residue of f at $z_j \neq 0$, determined the nature of the "Pole" in the function f at z_j. For example, if $n = 1$, we have a "Simple Pole" at $z = z_j$, whereas if $n = 2$, we have a "Double Pole". Said another way, recall that the series expansion of a function $f(z)$ in the complex plane (Laurent Series)

$$f(z) = b_0 + \frac{b_1}{(z - z_j)} + \frac{b_2}{(z - z_j)^2} + \ldots + a_1(z - z_j) + a_2(z - z_j)^2 + \ldots \tag{A.5.3}$$

Then if $b_n = 0$ for all $n \geq 2$, $f(z)$ is said to have a simple pole at $z = z_j$.

We will often use this "Residue Theorem" to evaluate integrals along the real line. As an example, consider the following integral (which is related to an integral we will encounter in the discussion of "Propagators")

$$I = \int_{-\infty}^{\infty} dx \frac{e^{-ixy}}{x^2 - a^2 + i\epsilon} \, , \tag{A.5.4}$$

where $a > 0$ and $y < 0$ and $\epsilon \to 0^+$. The above integral is along the real line. To evaluate this integral, consider the following integral which is to be done over a closed curve in the complex plane.

$$\mathcal{I} = \oint dz \frac{e^{-izy}}{z^2 - a^2 + i\epsilon} = 2\pi i \sum_j (\text{Residue of } f \text{ at } z_j \text{ inside } C) \, , \tag{A.5.5}$$

where the curve will be chosen in a way so that upon evaluation, this integral is equal to I. The chosen curve runs along the real axis and closes with a large semi-circle in the upper half of the complex plane. Why upper half? To see this, let us move to polar co-ordinates $z = re^{i\theta}$. Then the integral becomes

$$\mathcal{I} = \lim_{r \to \infty} \int_{-r}^{r} dx \frac{e^{-ixy}}{x^2 - a^2 + i\epsilon} + \lim_{r \to \infty} \int_0^{\pi} d\theta r \frac{e^{ry\sin\theta} e^{-iry\cos\theta}}{r^2 e^{i2\theta} - a^2 + i\epsilon} \, . \tag{A.5.6}$$

Note that since $y < 0$, the exponent drives the second term to 0 as $r \to \infty$. Note that if $y > 0$, we would have chosen to close the integral in the lower half plane (along with an associated minus sign since the curve would now be clockwise). Thus we have shown that $\mathcal{I} = I$ with this chosen contour C.

Now let us use the residue theorem. Thanks to $\epsilon > 0$, there is only one simple pole in the upper-half plane at $z = -a + i\epsilon/(2a)$. To see this, note that $z^2 - a^2 + i\epsilon = (z - (a - i\epsilon/2a))(z - (-a + i\epsilon/2a))$. Thus

$$(\text{Residue of } f \text{ at } z_* = -a + i\epsilon/(2a)) = (z - z_*)f(z_*) \to \frac{e^{iay}}{-2a} \, . \tag{A.5.7}$$

Putting this all together, we have

$$I = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dx \frac{e^{-ixy}}{x^2 - a^2 + i\epsilon} = -i\pi \frac{e^{iay}}{a} \, . \tag{A.5.8}$$

Further Reading: For a quick review of useful complex analysis, see the Appendix B in *QFT for the Gifted Amateur* by Lancaster and Blundell.

A.6   Groups and Representation Theory

It is not essential for you to know about Lie Groups and Representation Theory inside out, but a bit of familiarity will go a long way in helping you make sense of the second part of the course. You can read chapter 2 of Maggiore's *A Modern Introduction to QFT*. You may also want to read chapter 9 of *QFT for the Gifted Amateur*.