An Introduction to Relativistic Quantum Field Theory

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Abstract

These are QFT Lecture Notes for Phys 622 at Rice University. At the moment, they constitute a work in progress, with half of the notes typeset, the other half (chapter 5 onwards) hand-written. Please report any typos/errors to mustafa.a.amin@rice.edu.

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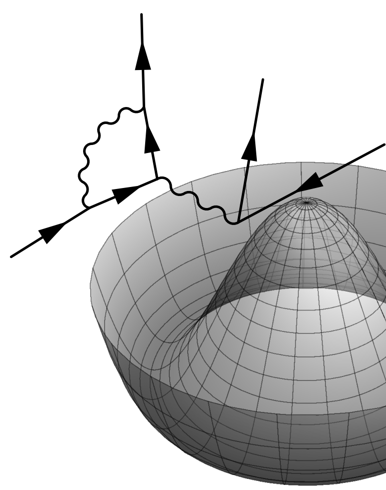

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Preface

These are notes for the Phys 622 course, *An Introduction to QFT*, at Rice University. This is *not* a comprehensive textbook on QFT. Typically, students from high energy physics and condensed matter theory take this course at Rice. The condensed matter students go on to take many-body physics as a second course, whereas the high energy physicists take a more Standard Model focused course next. Along with high energy physics and condensed matter theory, at times some undergrads, cosmologists/astrophysicists, biophysicists and theoretical chemists also take this course. This diversity of students, and their academic paths have guided the preparation of these notes – though it is still focussed on relativistic field theory. My goal is to set up the foundations of QFT, give a flavor for calculations (with sufficient, simplified examples) but leave the inevitable complications of the real world to later, more specialized courses.

The course is roughly divided into two parts. In the first part, I only deal with scalar field QFT. It culminates in the derivation of Feynman rules, and a number of calculations of scattering at tree level. Spin is only introduced in the second half of the semester, where we focus on how symmetries guide and constrain the nature of different spin fields. In a one semester course, it was possible to introduce, but not do any calculations in non-Abelian Gauge theories. I include spontaneous symmetry breaking in the context of Anderson-Higgs mechanism for particle physics and also its relevance, for example, in superconductivity here as well. Dictated by student interest, I have also included some brief notes on solitons and topological considerations in field theories.

At the time of posting (Dec 2017), only the first half of the course is typeset in LaTeX, the rest of the notes are handwritten. Next year, I will hopefully typeset the rest as well.

Acknowledgements

These notes do not contain much in terms of original material. QFT is a mature subject, with many comprehensive textbooks and pedagogical notes that are easily available. In preparing these notes, I benefited from Prof. Paul Stevenson's excellent notes for this class at Rice, which he taught for many years. His experience showed especially in a wonderfully economical path through the material. Prof. David Tong's (Cambridge) notes were the other set of notes I turned to repeatedly in preparing these notes.

As far as textbooks go, I learnt QFT from Peskin and Schroeder, as well as by dipping in and out of A. Zee's *QFT in a Nutshell* for inspiration and anecdotes.[2] In preparing these notes, I referred to them repeatedly. In addition, I thoroughly enjoyed *QFT for the Gifted Amateur* by Tom Lancaster and Stephen Blundell. The short chapters in both Zee and Lancaster & Blundell, the appropriately simplified sample calculations, the pedagogical (and at times whimsical) prose and diagrams made them a pleasure to read. Both books were also helpful in helping me learn (at a cursory level) condensed matter applications of QFT from their self-contained chapters.

[2] In addition to the teachers, I learnt most from my fellow students when taking QFT in graduate school. A special thanks to John Conley, MyPhuong Le, Navin Sivanandam, David Starr, Tommer Wizansky and Jeremy Verkaik who patiently helped me through the first quarter, for which I was ill-prepared at the time.

CONTENTS

CHAPTER 1

AN INVITATION

In "Classical" physics, particles are objects localized in space. Their position varies with time according to ordinary differential equations. In single particle quantum mechanics, position became an operator and the notion of a particle became "fuzzier" (think about localized probability density in terms of wavefunctions, uncertainty principle etc.)

Classical fields on the other hand are extended objects, with values at each point in space. Typically, partial differential govern the time evolution of fields (think of the electromagnetic fields: $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ governed by Maxwell's equations). How can we think about quantum mechanics of fields ? What is the a connection between fields and particles?

In this course, we will take the view point of the field and particles being part of the same physical entity. For example, photons are quantized excitations of the electromagnetic field. Equivalently we can think of the electromagnetic field as a collection of quantized excitations: photons. Similar statements hold for the electron and the electron field, quarks and a quark field ... you get the idea.

What is this course about?

This course is about learning the rules that govern the behavior of fields and their excitations (particles), insisting on consistency with Quantum Mechanics and Special Relativity. Symmetries will play an important role in determining these rules.[1]

1.1 Some Highlights from QFT

- QFT is one of the most successful theoretical frameworks we have. In Quantum Electrodynamics (QED), agreement between theory and experiment for the *anamolous muon magnetic moment* $(g_\mu - 2)/2$ is within one part in 10^{10}.

$$\begin{aligned}\left(\frac{g_\mu - 2}{2}\right)_{\text{exp}} &= 0.001\,159\,652\,180\,73(28)\\ \left(\frac{g_\mu - 2}{2}\right)_{\text{th}} &= 0.001\,159\,652\,181\,78(77)\end{aligned} \tag{1.1.1}$$

- All particles of a field are indistinguishable (just excitations of the same field, you cannot label them).

[1] Occasionally we will work in the non-relativistic limit, and sometimes only talk about classical fields also. The tools developed here will remain useful.

- "particles" = excitations, can be created and destroyed.
- Bose & Fermi Statistics emerge naturally.
- QFT plays a central role in condensed matter as well as atomic and molecular physics. Important especially for collective dynamics, phase transitions etc.
- Cosmology: QFT important in understanding the origin of density perturbations in our universe, as well as the hot big bang.

Before diving into QFT, we will review (1) Special Relativity (2) Lagrangian and Hamiltonian Mechanics (3) Quantum Mechanics and finally (4) Classical Fields. Putting these together, we will get to QFT. Let us begin with units.

1.2 Units

- Note the dimensions of $[\hbar] = ML^2T^{-1}$ and $[c] = LT^{-1}$. In "everyday" units, $\hbar \approx 1.05\times 10^{-34}\,\mathrm{m}^2\,\mathrm{kg}\,\mathrm{s}^{-1}$, whereas speed of light $c \approx 3\times 10^8 \mathrm{ms}^{-1}$. We work in units where $\hbar = c = 1$.
- In this case, mass, energy and momentum can be measured with the same unit. We will typically use a GeV as a unit for these quantities. Note that $1\,\mathrm{GeV} \approx 1.60\times 10^{-10}\,\mathrm{kg}\,\mathrm{m}\,\mathrm{s}^{-2}$.
- length and time intervals have the same units. We will typically use GeV^{-1} as a unit for them.
- examples: mass of proton $= 0.938\,\mathrm{GeV} = 0.938\,\mathrm{GeV/c}^2 \approx 10^{-27}\,\mathrm{kg}$), size of a proton $\sim 5\,\mathrm{GeV}^{-1} = 5\mathrm{GeV}^{-1}\,\hbar c \approx 10^{-15}\,\mathrm{m} = 1\,\mathrm{fermi}$).

CHAPTER 2

RAPID REVIEW

In this chapter I review relevant aspects of Einstein's Special Theory of Relativity, Lagrangian and Hamiltonian Mechanics as well as Quantum Mechanics for a countable (and finite) number of degrees of freedom. Though you are likely familiar with at least some of this material, some formal aspects such as Poisson Brackets and time-evolution operators introduced here might not have been covered in earlier courses.

For most of this course we will work in natural units with $\hbar = c = 1$ where $\hbar$ and c are the Planck's constant and speed of light respectively. In this review chapter, I set $c = 1$ so length and time have the same units, but do not set $\hbar = 1$ since it helps in seeing quantum aspects more clearly.

2.1 Special Relativity

In two sentences, here is Einstein's Special Theory of Relativity:

- The laws of physics are the same in all intertial frames.
- The speed of light (in vacuum) c is independent of the reference frame.

The inifinitesimal, frame-invariant spacetime interval:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad \mu, \nu = 0, 1, 2, 3\,. \tag{2.1.1}$$

where "0" labels the time co-ordinate. We have adopted the Einstein summation convention, where repeated upstairs and downstairs indices are summed over. $g_{\mu\nu}$ are components of the metric tensor which (among other responsibilities), determines the interval between events. Think of $g_{\mu\nu}$ as entires of a matrix g, that is $\mathsf{g}(\mu, \nu) \equiv g_{\mu\nu}$, where the first index labels the row, and the second the column with $\mu, \nu = 0, 1, 2, 3$. In cartesian co-ordinates (and with slight abuse of notation):

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.1.2}$$

Explicity $ds^2 = (dx^0)^2 - \delta_{ij} dx^i dx^j = dt^2 - d\mathbf{x} \cdot d\mathbf{x}$. If the interval corresponding to nearby events $ds^2 < 0 (> 0)$, then the events are said to be connected by a space-like (time-like) interval. $ds^2 = 0$ corresponds to a light-like interval. The same definitions carry over for finite intervals (obtained by joining together infinitesimal intervals $\sim \int ds$). One event can influence another only if the interval between

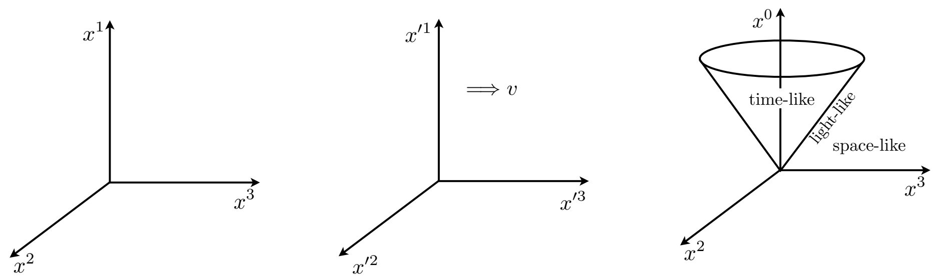

Figure 2.1

them is not spacelike. All the events along the trajectory of a massive (massless) particle are connected by a time-like (light-like) interval. This can be visualized as a "light-cone" (see Fig. 2.1). A Lorentz transformation allows us to change reference frames. A familiar example is the transformation of coordinates from one inertial frame to another moving at a velocity v along along one of the cartesian axes (see Fig. 2.1).

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}\,, \qquad \text{where} \qquad \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{2.1.3}$$

with $\gamma = (1-v^2)^{-1/2}$ and recall that $c = 1$ in our units. More generally, the defining property of Lorentz transformations is

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} g_{\mu\nu} = g_{\rho\sigma}\,. \tag{2.1.4}$$

There is a possibility of confusion here when one tries to write this in matrix form. So let me say a few more words. We have a matrix $\mathsf{\Lambda}$ whose entry in the μ row and ν column is given by $\mathsf{\Lambda}(\mu,\nu) \equiv \Lambda^{\mu}{}_{\nu}$. The defining property of the Lorentz transformation in matrix form is $\mathsf{\Lambda}^T \mathsf{g} \mathsf{\Lambda} = \mathsf{g}$. To see this, note that in terms of matrix entries, the left hand side is $\sum_{\nu,\mu=0}^{3} \mathsf{\Lambda}^T(\rho,\mu)\mathsf{g}(\mu,\nu)\mathsf{\Lambda}(\nu,\sigma) = \sum_{\nu,\mu=0}^{3} \mathsf{\Lambda}(\mu,\rho)\mathsf{g}(\mu,\nu)\mathsf{\Lambda}(\nu,\sigma)$. Since we have defined $\mathsf{\Lambda}(\mu,\nu) \equiv \Lambda^{\mu}{}_{\nu}$ and $\mathsf{g}(\mu,\nu) \equiv g_{\mu\nu}$, we have $\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} g_{\mu\nu} = g_{\rho\sigma}$ where now we revert back to Einstein summation. It is worth noting that for the inverse transformation, $\mathsf{\Lambda}^{-1}(\mu,\nu) = \Lambda_{\nu}{}^{\mu}$, whereas the inverse metric $\mathsf{g}^{-1}(\mu,\nu) = g^{\mu\nu}$.

An immediate consequence of the defining property of the Lorentz transformation is that $|\det\Lambda| = 1$. Hence, 4-volume elements d^4x are Lorentz invariant: $d^4x' = d^4x|\det\Lambda| = d^4x$.

Exercise 2.1.1 : Verify that the Lorentz transformation in eq. (2.1.3) satisfies its defining property (2.1.4).

In component form, a Lorentz four-vector is a 4-component object which transforms as $A'^{\mu} = \Lambda^{\mu}{}_{\nu}A^{\nu}$. Explicitly:

$$A^{\mu} = (A^0, \mathbf{A}) \qquad \text{and} \qquad A_{\mu} = g_{\mu\nu}A^{\nu} = (A^0, -\mathbf{A})\,, \tag{2.1.5}$$

where with $\mathbf{A} = \{A^1, A^2, A^3\}$. In the second equality, note that the metric allows us to raise and lower indices. We will often use the following shorthand(s)

$$\begin{aligned} x = x^{\mu} &= (x^0, \mathbf{x}) = (t, \mathbf{x})\,, \\ k = k^{\mu} &= (k^0, \mathbf{k}) = (E, \mathbf{k})\,, \end{aligned} \tag{2.1.6}$$

where in the last line we are thinking of k as the four-momentum of a particle, with E being the energy. Their dot product is Lorentz invariant (ie. its value does not change under Lorentz transformations)

$$\begin{aligned} x \cdot k &= g_{\mu\nu} x^\mu k^\nu = x_\mu k^\mu = x^\mu k_\mu = Et - \mathbf{x} \cdot \mathbf{k} \,, \\ k^\mu k_\mu &= E^2 - |\mathbf{k}|^2 = m^2 \,. \end{aligned} \tag{2.1.7}$$

where m is the rest-mass of the particle.

Some useful differential operators are listed below:

$$\begin{aligned} \partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \nabla) \qquad &\text{and} \qquad \partial^\mu = (\partial_0, -\nabla) \,, \\ \Box = \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2 \qquad &\text{and} \qquad \partial \cdot A = \partial_\mu A^\mu = \partial_0 A^0 + \nabla \cdot \mathbf{A} \,. \end{aligned} \tag{2.1.8}$$

Exercise 2.1.2 : Show that $f(x) = e^{ik\cdot x}$ satisfies $(\Box + m^2) f(x) = 0$ only if $k^\mu k_\mu = m^2$.

Exercise 2.1.3 : Consider two reference frames related by the Lorentz transformation in eq. (2.1.3). Let $x^\mu(P) = \left(x_P^0, 0, 0, x_P^3\right)$ and $x^\mu(Q) = \left(x_Q^0, 0, 0, x_Q^3\right)$ be the co-ordinates of two events P and Q in the "unprimed" frame. The co-ordinates of these events in the "primed" frame are given by ${x'}^\mu(P) = \left({x'}_P^0, 0, 0, {x'}_P^3\right)$ and ${x'}^\mu(Q) = \left({x'}^0_Q, 0, 0, {x'}^3_Q\right)$. Show that if the events are not simultaneous ($x_P^0 \neq x_Q^0$), and are space-like separated in the unprimed frame (i.e. $\Delta x^\mu \Delta x_\mu < 0$ where $\Delta x^\mu = x^\mu(P) - x^\mu(Q)$), then there exists a velocity v such that in the primed frame, the events are simultaneous ${x'}_P^0 = {x'}_Q^0$. Find this velocity v.

While simultaneity is frame-dependent, intervals are not. That is, $\Delta {x'}^\mu \Delta x'_\mu = \Delta x^\mu \Delta x_\mu$. Hence the space-like, time-like and light-like nature of intervals is invariant under Lorentz transformations.

2.2 Classical Mechanics

I am going to go through a quick, formal review of classical mechanics with an eye towards Quantum Mechanics.

2.2.1 Lagrangian Mechanics

Start with a (given) Lagrangian $L(q_\alpha, \dot{q}_\alpha, t)$ where q_α are the generalized co-ordinate of the system ($\alpha = 1, 2 \ldots N$). The equations of motion for q_α are obtained by extremizing the *Action*:

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt L(q_\alpha, \dot{q}_\alpha, t) \,, \tag{2.2.1}$$

That is $q_\alpha(t)$ are such that for $q_\alpha(t) \to q_\alpha(t) + \delta q_\alpha(t)$ (where $\delta q_\alpha(t)$ are arbitrary apart from $\delta q(t_\mathrm{i}) = \delta q(t_\mathrm{f}) = 0$) we have $\delta S = 0$. For $\delta S = 0$, q_α must satisfy (see Appendix A.3):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha} \qquad \text{Euler-Lagrange equations} \tag{2.2.2}$$

For example, for a collection of coupled harmonic oscillators with unit mass and *time-independent* couplings[1] $M_{\alpha\rho}$:

$$
\begin{aligned}
L(q_\alpha, \dot{q}_\alpha) &= \sum_{\alpha=1}^{N} \left(\frac{1}{2}\dot{q}_\alpha^2 - \sum_{\rho=1}^{N} \frac{1}{2} M_{\alpha\rho}\, q_\alpha q_\rho \right) , \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) &= \frac{\partial L}{\partial q_\alpha} \\
\Longrightarrow \ddot{q}_\alpha + \sum_{\rho=1}^{N} M_{\alpha\rho} q_\rho &= 0 \, .
\end{aligned}
\tag{2.2.3}
$$

2.2.2 Hamiltonian Mechanics

The Hamiltonian is a Legendre transform of the Lagrangian and contains the same information as the Lagrangian. In Quantum Mechanics and QFT, the Hamiltonian is often more convenient to work with, so lets do a quick review:

$$
\begin{aligned}
p_\alpha &\equiv \frac{\partial L}{\partial \dot{q}_\alpha} && \text{conjugate momentum} \\
H &\equiv \sum_{\alpha=1}^{N} (p_\alpha \dot{q}_\alpha) - L && \text{Hamiltonian} \\
\dot{q}_\alpha &= \frac{\partial H}{\partial p_\alpha} \, , \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} && \text{Hamilton's equations}
\end{aligned}
\tag{2.2.4}
$$

For the coupled harmonic oscillators example,

$$
\begin{aligned}
p_\alpha &= \dot{q}_\alpha \, , \\
H &= \sum_{\alpha=1}^{N} \left(\frac{1}{2} p_\alpha^2 + \sum_{\rho=1}^{N} \frac{1}{2} M_{\alpha\rho}\, q_\alpha q_\rho \right) , \\
\dot{p}_\alpha &= -\sum_{\rho=1}^{N} M_{\alpha\rho}\, q_\rho \, .
\end{aligned}
\tag{2.2.5}
$$

2.2.3 Poisson Brackets

The Poisson Bracket is defined as:

$$
\{f(q_\alpha, p_\alpha), g(q_\alpha, p_\alpha)\} \equiv \sum_{\alpha=1}^{N} \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} \right) , \tag{2.2.6}
$$

where f and g are arbitrary functions on the space of generalized co-ordinates and their conjugate momenta. The time evolution of $f(q_\alpha, p_\alpha, t)$ is given by

$$
\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \, . \tag{2.2.7}
$$

which you can see immediately by noting that $df/dt = \sum_\alpha [(\partial f/\partial q_\alpha)\dot{q}_\alpha + (\partial f/\partial p_\alpha)\dot{p}_\alpha] + \partial f/\partial t$ and using Hamilton's equations of motion for $\dot{p}_\alpha$ and $\dot{q}_\alpha$. If the function f does not explicitly depend on time, then

$$
\frac{df}{dt} = \{f, H\} \, . \tag{2.2.8}
$$

[1] Typically, we consider $M_{\alpha\rho}$ that only couples nearest neighbors.

The time evolution of f is generated by H. In particular, for $f = q_\alpha$ and $g = p_\rho$, we have

$$\{q_\alpha, p_\rho\} = \delta_{\alpha\rho} \,. \tag{2.2.9}$$

Similarly, $\{q_\alpha, q_\rho\} = \{p_\alpha, p_\rho\} = 0$. The time evolution equations for q_α and p_ρ become

$$\frac{dq_\alpha}{dt} = \{q_\alpha, H\}\,, \qquad \text{and} \qquad \frac{dp_\alpha}{dt} = \{p_\alpha, H\}\,. \tag{2.2.10}$$

The last two equations are equivalent to Hamilton's equations of motion, and also to the Euler-Lagrange equations.

Exercise 2.2.1 : For the coupled harmonic oscillator example in eq. (3.2.8), evaluate the Poisson Brackets $\{p_\alpha, H\}$, to recover $\dot{p}_\alpha = -\sum_\rho M_{\alpha\rho} q_\rho$.

2.3 Quantum Mechanics

I am going to review relevant aspects of Quantum Mechanics; the results here are the most relevant part of this review chapter.

2.3.1 Canonical Quantization

One route to getting from classical to quantum mechanics is as follows (thanks to Dirac[2]):

- replace the co-ordinates and momenta by operators (think of them as matrices)

$$q_\alpha, p_\alpha \longrightarrow \hat{q}_\alpha, \hat{p}_\alpha \,. \tag{2.3.1}$$

  The functions f and g inherit the operator structure from p and q: $f, g \to \hat{f}, \hat{g}$.

- Replace the Poisson Bracket by the "Commutator"

$$\{f, g\} \to -\frac{i}{\hbar}\left[\hat{f}, \hat{g}\right] \,. \tag{2.3.2}$$

  Note the appearance of i and Planck's constant $\hbar$.The commutator is defined as

$$\left[\hat{f}, \hat{g}\right] \equiv \hat{f}\hat{g} - \hat{g}\hat{f} \,. \tag{2.3.3}$$

  Operators $\hat{f}$ and $\hat{g}$ do not necessarily commute. For $\hat{f} = \hat{q}_\alpha$ and $\hat{q} = \hat{p}_\rho$, we get

$$[\hat{q}_\alpha, \hat{p}_\rho] = i\hbar\delta_{\alpha\rho} \,. \tag{2.3.4}$$

  This should look familiar! Also note that $[\hat{q}_\alpha, \hat{q}_\rho] = [\hat{p}_\alpha, \hat{q}_\rho] = 0$. One could directly start from these commutation relations as well, without going through Poisson Brackets.

- The time evolution of $\hat{f}(\hat{q}_\alpha, \hat{p}_\alpha)$ (no explicit time dependence) is given by

$$\frac{d\hat{f}}{dt} = -\frac{i}{\hbar}\left[\hat{f}, \hat{H}\right] \,. \tag{2.3.5}$$

  For co-ordinates and momenta,

$$\frac{d\hat{q}_\alpha}{dt} = -\frac{i}{\hbar}\left[\hat{q}_\alpha, \hat{H}\right]\,, \qquad \text{and} \qquad \frac{d\hat{p}_\alpha}{dt} = -\frac{i}{\hbar}\left[\hat{p}_\alpha, \hat{H}\right] \,. \tag{2.3.6}$$

[2] Dirac, *Principles of Quantum Mechanics*, Oxford University Press (1982)

Comments:

- This above quantization procedure procedure is not a guarantee of finding the correct quantum theory. Higher order terms in $\hbar$ might be relevant. The above procedure is motivated by recovering classical physics in the limit "$\hbar \to 0$". There also exists an ambiguity in the above procedure regarding the orderings of operators; which one is correct? $q^3p^2 \to \hat{q}^3\hat{p}^2$ or $q^3p^2 \to \hat{p}^2\hat{q}^3$? Ultimately, you have to check with nature whether you have the correct quantum Hamiltonian.

- **Heisenberg Picture** : Note that we are working in the "Heisenberg Picture" where the operators $\hat{f}(\hat{q}_\alpha, \hat{p}_\alpha)$ evolve with time according to[3]

$$\frac{d\hat{f}}{dt} = -\frac{i}{\hbar}\left[\hat{f}, \hat{H}\right] , \qquad \text{or equivalently} \qquad \hat{f}(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{f}(t_0)e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} . \tag{2.3.7}$$

  The states $|\psi\rangle$ of the system is time-independent. To find the expectation value of an observable corresponding to the operator $\hat{f}(t)$ in a given state $|\psi\rangle$, we have to calculate $\langle\psi|\hat{f}(t)|\psi\rangle$. Notice that it is a combination of operators sandwiched between states that appears in the expectation values.

- **Schrödinger Picture**: A mathematically equivalent way of thinking about time evolution of quantum systems is to think of states $|\psi(t)\rangle_\mathrm{s}$ evolving with time, and the operators $\hat{f}_\mathrm{s}$ being time-independent. States evolve according to the Schrödinger equation[4]:

$$\frac{d}{dt}|\psi(t)\rangle_\mathrm{s} = -\frac{i}{\hbar}\hat{H}|\psi(t)\rangle_\mathrm{s} , \qquad \text{or equivalently} \qquad |\psi(t)\rangle_\mathrm{s} = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\psi(t_0)\rangle_\mathrm{s} . \tag{2.3.8}$$

  As you can check, by setting $|\psi(t_0)\rangle_\mathrm{s} = |\psi\rangle$ and $\hat{f}(t_0) = \hat{f}_\mathrm{s}$, the expectation values constructed in either picture will yield the same answer ${}_\mathrm{s}\langle\psi(t)|\hat{f}_\mathrm{s}|\psi(t)\rangle_\mathrm{s} = \langle\psi|\hat{f}(t)|\psi\rangle$. The same argument works for arbitrary matrix elements: $f_{ab} = {}_\mathrm{s}\langle a(t)|\hat{f}_\mathrm{s}|b(t)\rangle_\mathrm{s} = \langle a|\hat{f}(t)|b\rangle$ (think about transition probabilities), thus observables will be equal when calculated in either picture.

Exercise 2.3.1 : Verify that $\hat{f}(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{f}(t_0)e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ is a solution of $d\hat{f}/dt = -(i/\hbar)[\hat{f}, \hat{H}]$. Be careful about the fact that $\hat{H}$ is an operator, not a number. You should interpret $e^{\hat{A}} = \sum_{n=0}^{\infty}(1/n!)\hat{A}^n$.

2.3.2 Worked Example: Harmonic Oscillators

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction – Sidney Coleman

Let us revert once more to coupled harmonic oscillators with unit mass; see equations (2.2.3) and (3.2.8). For time-independent couplings $M_{\alpha\rho}$ there exists $C_{\alpha\rho}$ such that if $\tilde{q}_\alpha = \sum_\rho C_{\alpha\rho}q_\rho$, then

$$H = \sum_{\alpha=1}^{N}\left(\frac{1}{2}\tilde{p}_\alpha^2 + \frac{1}{2}\omega_\alpha^2\,\tilde{q}_\alpha^2\right) . \tag{2.3.9}$$

The "tilde" co-ordinates are the normal-modes of the system. For example, for a collection of 2-masses and three springs (see Fig. 2.2), the two normal modes would be the modes where the masses oscillate together (in phase) or with an opposite phase. Normal modes are exceptionally convenient, because they

[3] We assume that $\hat{H}$ has no explicit time-dependence for simplicity.

[4] The familiar wave-function in position space corresponding to the state $|a(t)\rangle_\mathrm{s}$ is obtained via $\Psi_a(t, \mathbf{x}) = {}_\mathrm{s}\langle\mathbf{x}|a(t)\rangle_\mathrm{s}$.

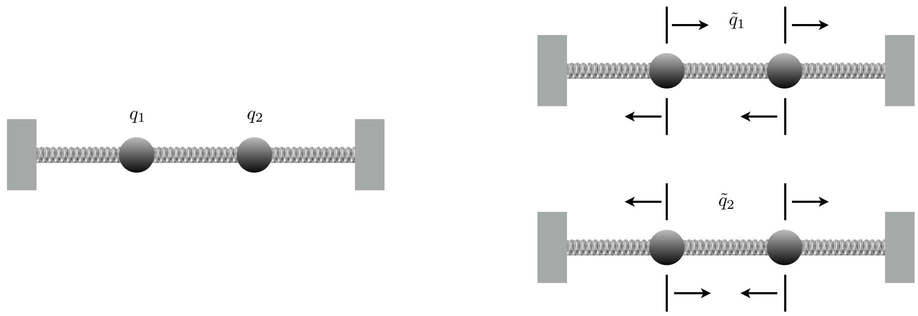

Figure 2.2: Normal modes for two masses connected by springs.

evolve independently from each other! For our system (dropping the "tilde" now), we have Hamilton's equations

$$\dot{q}_\alpha = p_\alpha \,, \qquad \text{and} \qquad \dot{p}_\alpha = -\omega_\alpha^2 q_\alpha \,. \tag{2.3.10}$$

Equivalently, the Euler-Lagrange equations are:

$$\ddot{q}_\alpha + \omega_\alpha^2 q_\alpha = 0 \,, \tag{2.3.11}$$

with the solutions $q_\alpha \propto e^{\pm i\omega_\alpha t}$.

Canonical Quantization

Let us turn the crank of quantizing our theory. For the system at hand $q_\alpha, p_\alpha \longrightarrow \hat{q}_\alpha, \hat{p}_\alpha$ with $[\hat{q}_\alpha, \hat{p}_\rho] = i\hbar\delta_{\alpha\rho}$. The Hamiltonian and equations of motion are

$$\hat{H} = \sum_{\alpha=1}^{N} \left(\frac{1}{2}\hat{p}_\alpha^2 + \frac{1}{2}\omega_\alpha^2\,\hat{q}_\alpha^2 \right) \,, \qquad \frac{d\hat{q}_\alpha}{dt} = \hat{p}_\alpha \,, \qquad \text{and} \qquad \frac{d\hat{p}_\alpha}{dt} = -\omega_\alpha^2 \hat{q}_\alpha \,. \tag{2.3.12}$$

Exercise 2.3.2 : For the above Hamiltonian, evaluate the commutator $[\hat{p}_\alpha, \hat{H}]$ using the commutators for $\hat{q}_\alpha$ and $\hat{p}_\rho$. Then use the time evolution equation $(d/dt)\hat{p}_\alpha = -(i/\hbar)[\hat{p}_\alpha, \hat{H}]$ to recover $(d/dt)\hat{p}_\alpha = -\omega_\alpha^2\hat{q}_\alpha$. It is often useful to remember the following identity for commutators: $[\hat{a}\hat{b}, \hat{c}] = \hat{a}[\hat{b}, \hat{c}] + [\hat{a}, \hat{c}]\hat{b}$.

Next, we introduce the formalism of creation and annihilation operators, which will turn out to be quite useful when we deal with fields in the next chapter.

Creation and Annihilation Operators

It is convenient to define the "creation" and "annihilation" operators

$$\hat{a}_\alpha(t) = \sqrt{\frac{\omega_\alpha}{2\hbar}} \left(\hat{q}_\alpha(t) + i\frac{\hat{p}_\alpha(t)}{\omega_\alpha} \right) \qquad \text{and} \qquad \hat{a}_\alpha^\dagger(t) = \sqrt{\frac{\omega_\alpha}{2\hbar}} \left(\hat{q}_\alpha(t) - i\frac{\hat{p}_\alpha(t)}{\omega_\alpha} \right) \,. \tag{2.3.13}$$

Recall that $\hat{q}_\alpha$ and $\hat{p}_\alpha$ are Hermitian because they correspond to observables (i.e. they must have real eigenvalues). These can be inverted to yield

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_\alpha}} \left(\hat{a}_\alpha(t) + \hat{a}_\alpha^\dagger(t) \right) \qquad \text{and} \qquad \hat{p}_\alpha(t) = -i\sqrt{\frac{\hbar\omega_\alpha}{2}} \left(\hat{a}_\alpha(t) - \hat{a}_\alpha^\dagger(t) \right) \,. \tag{2.3.14}$$

The time dependence of $\hat{a}_\alpha(t)$ and $\hat{a}^\dagger_\alpha(t)$ can be obtained by using our knowledge of $d\hat{q}_\alpha/dt$ and $d\hat{p}_\alpha/dt$, which yields $(d/dt)\hat{a}_\alpha(t) = -i\omega_\alpha \hat{a}_\alpha(t)$ and $(d/dt)\hat{a}^\dagger_\alpha(t) = i\omega_\alpha \hat{a}^\dagger_\alpha(t)$. The solutions are

$$\hat{a}_\alpha(t) = \hat{a}_\alpha(0) e^{-i\omega_\alpha t} \qquad \text{and} \qquad \hat{a}^\dagger_\alpha(t) = \hat{a}^\dagger_\alpha(0) e^{+i\omega_\alpha t} \, . \tag{2.3.15}$$

Exercise 2.3.3 : Derive $(d/dt)\hat{a}_\alpha(t) = -i\omega_\alpha \hat{a}_\alpha(t)$ and $(d/dt)\hat{a}^\dagger_\alpha(t) = i\omega_\alpha \hat{a}^\dagger_\alpha(t)$.

Using eq. (2.3.14), we have the "mode-expansion" of $\hat{q}_\alpha$:

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_\alpha}} \left(\hat{a}_\alpha(0) e^{-i\omega_\alpha t} + \hat{a}^\dagger_\alpha(0) e^{i\omega_\alpha t} \right) \, . \tag{2.3.16}$$

To reduce clutter, We drop the (0) part in the time-independent creation and annihilation operators, and simply write

$$\hat{q}_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_\alpha}} \left(\hat{a}_\alpha e^{-i\omega_\alpha t} + \hat{a}^\dagger_\alpha e^{i\omega_\alpha t} \right) \, . \tag{2.3.17}$$

From now on, when we refer to $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$, we will always mean the time-independent ones. The corresponding expression for p_α is

$$\hat{p}_\alpha(t) = -i\sqrt{\frac{\hbar\omega_\alpha}{2}} \left(\hat{a}_\alpha e^{-i\omega_\alpha t} - \hat{a}^\dagger_\alpha e^{i\omega_\alpha t} \right) \, . \tag{2.3.18}$$

Why are $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$ useful? It is worth reminding ourselves of the important properties of $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$:

$$\begin{aligned} \left[\hat{a}_\alpha, \hat{a}^\dagger_\rho\right] &= \delta_{\alpha\rho} \, , \\ \left[\hat{a}_\alpha, \hat{a}_\rho\right] &= \left[\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\rho\right] = 0 \, , \\ \hat{H} &= \sum_\alpha \left(\hat{a}^\dagger_\alpha \hat{a}_\alpha + \frac{1}{2} \right) \hbar\omega_\alpha \, . \end{aligned} \tag{2.3.19}$$

Exercise 2.3.4 : Derive the above expressions using the commutators for the co-ordinates and momenta, the Hamiltonian in eq. (2.3.12) as well as the mode expansions above.

Why the name "creation" and "annihilation" operators? Note another property (that you should verify):

$$\left[\hat{H}, \hat{a}_\alpha\right] = -\hbar\omega_\alpha \hat{a}_\alpha \, , \qquad \text{and} \qquad \left[\hat{H}, \hat{a}^\dagger_\alpha\right] = \hbar\omega_\alpha \hat{a}^\dagger_\alpha \, . \tag{2.3.20}$$

The last line tells us about creation and annihilation of quanta. To see this, note that if $|\psi\rangle$ is an eigenstate of the $\hat{H}$ with energy E, then $\hat{a}^\dagger_\alpha|\psi\rangle$ has an energy $E + \hbar\omega_\alpha$:

$$\hat{H}\hat{a}^\dagger_\alpha|\psi\rangle = (E + \hbar\omega_\alpha)\hat{a}^\dagger_\alpha|\psi\rangle \, . \tag{2.3.21}$$

That is, $\hat{a}^\dagger_\alpha$ raises the energy by $\hbar\omega_\alpha$; it "creates" a quantum $\hbar\omega_\alpha$. Similarly,

$$\hat{H}\hat{a}_\alpha|\psi\rangle = (E - \hbar\omega_\alpha)\hat{a}_\alpha|\psi\rangle \, . \tag{2.3.22}$$

That is, $\hat{a}_\alpha$ lowers the energy by $\hbar\omega_\alpha$; it "annihilates" a quantum $\hbar\omega_\alpha$. The *vacuum state* is defined as the state that is annihilated by any $\hat{a}_\alpha$:

$$|0\rangle \equiv |0, 0 \ldots 0\rangle, \qquad \text{where} \qquad \hat{a}_\alpha|0\rangle = 0 \, . \tag{2.3.23}$$

Each "slot" in $|0, 0 \dots 0\rangle$ corresponds to different $\alpha = 1, \dots N$. We can build normalized eigenstates of energy by repeatedly acting on the vacuum state using our creation operators

$$|n_1, n_2 \dots n_N\rangle = \prod_{\alpha=1}^{N} \frac{\left(\hat{a}_\alpha^\dagger\right)^{n_\alpha}}{\sqrt{n_\alpha!}} |0\rangle \,, \tag{2.3.24}$$

where n_α are the number of quanta (occupation number) with energy $\hbar\omega_\alpha$ is this state. Note that $\hat{a}_\alpha^\dagger \hat{a}_\alpha |n_1, n_2 \dots n_N\rangle = n_\alpha |n_1, n_2 \dots n_N\rangle$. The total energy of this state

$$E_{\text{tot}} = \sum_{\alpha=1}^{N} E_\alpha = \sum_{\alpha=1}^{N} \left(n_\alpha + \frac{1}{2}\right) \hbar\omega_\alpha \,. \tag{2.3.25}$$

Even if there are no quanta $n_\alpha = 0$ for all α, we still have vacuum energy $E_{\text{vac}} = \sum_{\alpha=1}^{N} (1/2)\hbar\omega_\alpha$. This is important when absolute values of energy matter (e.g. when gravity is involved ("Cosmological Constant" problem)) or when dealing with non-trivial boundary conditions ("Casimir effect"). In this course we can ignore this vacuum contribution; for us only differences in energy matter.

Exercise 2.3.5 : Evaluate $\langle 0|(\hat{a}_\alpha + \hat{a}_\alpha^\dagger)^4|0\rangle$ using the commutation relations for $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ as well as their action on the vacuum.

2.4 Special Relativity and Single-Particle Quantum Mechanics

For simplicity of expressions, let $\hbar = c = 1$. Consider a relativistic particle with a Hamiltonian such that $\hat{H}|\mathbf{k}\rangle = \sqrt{\mathbf{k}^2 + m^2}|\mathbf{k}\rangle$ for a particle with mass m in its momentum eigenstate. Let us calculate the "amplitude" $\mathcal{A}$ (where the probability $\propto |\mathcal{A}|^2$) of a localized particle moving from $(t_0 = 0, \mathbf{x}_0)$ to $(t, \mathbf{x})$. That is, we take a localized state $|\mathbf{x}_0\rangle$, time evolve it: $e^{-i\hat{H}(t-t_0)}|\mathbf{x}_0\rangle$, and ask whether this state has an overlap with $\langle\mathbf{x}|$. To be consistent with special relativity, we must get $\mathcal{A} = 0$ if $(t_0, \mathbf{x}_0)$ and $(t, \mathbf{x})$ are space-like separated. Upon an explicit calculation[5]

$$\mathcal{A} = \langle\mathbf{x}|e^{-i\hat{H}(t-t_0)}|\mathbf{x}_0\rangle \sim e^{-m|\mathbf{x}-\mathbf{x}_0|} \neq 0 \tag{2.4.1}$$

where we have assumed that $|\mathbf{x} - \mathbf{x}_0|^2 \gg (t - t_0)^2$. This is small, but non-zero. Thus single-particle Quantum Mechanics is inconsistent with Special Relativity. As we will soon see next, a radical new approach is needed where particle number is not conserved. In anticipation of what is to follow in the next chapter, and to avoid clutter in this subsection, I have set $\hbar = 1$ and will continue to do so from now onwards.

Comment: QM can be consistent with SR. We have only shown that single-particle QM is inconsistent with SR. Note that in QFT, the Schrödinger equation: $i(d/dt)|\psi\rangle_{\text{s}} = \hat{H}|\psi\rangle_{\text{s}}$ is still valid. You have to make sure you use the correct Hamiltonian made out of operator valued fields, whose evolution is consistent with SR, and also use appropriate states. Using the Schrödinger equation proves to be rather unwieldy when dealing with fields, so the formalism (while mathematically equivalent), will look quite different from the Schrödinger equation.

[5] see for example, section 8.3 in Lancaster and Blundell, or section 2.1 in Peskin and Schroeder.

CHAPTER 3

QUANTIZING SCALAR FIELDS

We are now ready to talk about fields. In this chapter we will limit ourselves to *scalar* fields. We start with classical, relativistic scalar fields and then quantize them. Most of this chapter is devoted to *free* scalar fields. Interactions will be taken up in the next chapter. We get a taste of what we mean by particles as well as how causality is built into our field theory.

3.1 Classical Scalar Fields

We now turn to fields, specifically real, relativistic scalar fields $\varphi(t, \mathbf{x}) = \varphi(x)$.[1] The Lagrangian density

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\,, \tag{3.1.1}$$

where $V(\varphi) = \frac{1}{2}m^2\varphi^2 + (1/3!)\lambda_3\varphi^3 + (1/4!) \pm \lambda\varphi^4 \ldots$ is the potential, which contains the "mass" m, and "self-interaction" terms with strengths λ_3, λ etc. Note that $(1/2)\partial^\mu\varphi\partial_\mu\varphi = (1/2)\dot{\varphi}^2 - (1/2)(\nabla\varphi)^2$ which includes the "kinetic" and "gradient" terms. Why does the Lagrangian density have this particular form? Wait till the second half of this semester to get answers to this very relevant question. Hint: Symmetries, and Lorentz invariance will play a role. Of course, as always, these only provide guidance, with experiments having the final say. For the moment take this Lagrangian density as it is.

The Lagrangian and Lagrangian density are related by

$$L = \int d^3x\mathcal{L}\,. \tag{3.1.2}$$

We will get sloppy and call $\mathcal{L}$ the Lagrangian as well. By extremizing the action $S = \int d^4x\mathcal{L}$, we arrive at the Euler-Lagrange equations[2], that is, the equations of motion for the field φ:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi} \implies \partial_\mu\partial^\mu\varphi + \partial_\varphi V(\varphi) = 0\,. \tag{3.1.3}$$

It is initially useful to think about $\mathbf{x}$ as a label. That is $\varphi(t, \mathbf{x}) = \varphi_{\mathbf{x}}(t)$, we can now think of $q_\alpha(t) \leftrightarrow \varphi_{\mathbf{x}}(t)$. We can define a conjugate momentum *density*, and a Hamiltonian density

$$\begin{aligned} \pi &\equiv \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} && \text{conjugate momentum density} \\ \mathcal{H} &= \pi\dot{\varphi} - \mathcal{L} && \text{Hamiltonian density} \end{aligned} \tag{3.1.4}$$

[1] An example of such a field would be the (as yet to be detected) axion field, which might constitute dark matter.
[2] See Appendix A.3

More explicitly

$$\pi = \dot{\varphi}\,, \qquad \text{and} \qquad \mathcal{H} = \frac{\pi^2}{2} + \frac{(\nabla\varphi)^2}{2} + V(\varphi)\,. \tag{3.1.5}$$

The Hamiltonian density is the energy density, and each term has a meaning:

$$\begin{aligned} \frac{\pi^2}{2} &= \text{kinetic energy density; time variation costs energy}\,, \\ \frac{(\nabla\varphi)^2}{2} &= \text{gradient energy density; spatial variation costs energy}\,, \\ V(\varphi) &= \text{potential energy density}\,. \end{aligned} \tag{3.1.6}$$

The Hamiltonian H and the Hamiltonian density $\mathcal{H}$ are related by $H = \int d^3x \mathcal{H}$ which is also consistent with $H = \int d^3x \pi\dot{\varphi} - L$ as it should be. Think about $\int d^3x$ as the sum over all degrees of freedom. Again, it gets cumbersome to say momentum density and Hamiltonian density, so we will get sloppy and call them the conjugate momentum and density instead. Hamilton's equations:

$$\dot{\varphi} = \frac{\delta H}{\delta \pi} = \pi \qquad \text{and} \qquad \dot{\pi} = -\frac{\delta H}{\delta \varphi} = \nabla^2\varphi - V'(\varphi)\,. \tag{3.1.7}$$

where $\delta H/\delta\pi$ and $\delta H/\delta\varphi$ are *functional* derivatives[3]. Again, see Appendix A.3.

Exercise 3.1.1 : Show that $\delta H/\delta\pi = \pi$ and $-\delta H/\delta\varphi = \nabla^2\varphi - V'(\varphi)$. You might want to consult Appendix A.3 for the definition of functional derivatives. Think of H as a functional of π and φ. This exercise should also convince you that the definition of conjugate momentum density $\pi = \partial\mathcal{L}/\partial\dot{\varphi} = \delta L/\delta\dot{\varphi}$ where you can treat L as a functional of $\dot{\varphi}$ and φ.

3.1.1 Poisson Brackets

The Poisson Bracket of two functionals (with all quantities evaluated at the same time): $F = \int d^3x \mathcal{F}(\varphi, \pi)\,, G = \int d^3x \mathcal{G}(\varphi,\pi)\,,$ is defined as

$$\{F, G\} \equiv \int d^3x \left(\frac{\delta F}{\delta\varphi}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\pi}\frac{\delta G}{\delta\varphi} \right)\,, \tag{3.1.8}$$

whereas the time evolution (using Hamilton's equations of motion) is given by

$$\frac{dF}{dt} = \{F, H\}\,. \tag{3.1.9}$$

Consider the special case where $F = \varphi(t, \mathbf{y})$, and $G = \pi(t, \mathbf{z})$. In this case, the Poisson bracket and time evolution are given by

$$\begin{aligned} &\{\varphi(t,\mathbf{y}), \pi(t,\mathbf{z})\} = \delta^{(3)}(\mathbf{y} - \mathbf{z})\,, \\ &\dot{\varphi} = \{\varphi, H\} \qquad \text{and} \qquad \dot{\pi} = \{\pi, H\}\,. \end{aligned} \tag{3.1.10}$$

Exercise 3.1.2 : Show that $dF/dt = \{F, H\}$ using Hamilton's equations. Then show that $\delta\varphi(t,\mathbf{y})/\delta\varphi(t,\mathbf{x}) = \delta^{(3)}(\mathbf{y} - \mathbf{x})$ (similarly for π), which together yields the desired equations (3.1.10).

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[3] A functional can be a function of an entire function, not just its value at a given point. A function is special case of functionals.

3.2 Canonical Quantization of Scalar Fields

We follow our nose and put hats on our field and its conjugate momenta, and specify the fundamental commutation relation between them

$$\begin{aligned} &\varphi(t,\mathbf{x}),\pi(t,\mathbf{x}) \longrightarrow \hat{\varphi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x})\,, \\ &[\hat{\varphi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \end{aligned} \tag{3.2.1}$$

We have set $\hbar$ to unity, and have followed $\{\cdot,\cdot\} \to -i[\cdot,\cdot]$. Note that this is an *equal time* commutation relation. The commutator at different times can be obtained from the equations of motion. The time evolution of these operators follows from Hamilton's equations

$$\frac{d\hat{\varphi}}{dt} = -i\left[\hat{\varphi},\hat{H}\right] = \hat{\pi}\,, \qquad \text{and} \qquad \frac{d\hat{\pi}_\alpha}{dt} = -i\left[\hat{\pi},\hat{H}\right] = \nabla^2\hat{\varphi} - V'(\hat{\varphi})\,. \tag{3.2.2}$$

Combining the above two equations, we have

$$\frac{d^2\hat{\varphi}}{dt^2} - \nabla^2\hat{\varphi} + V'(\hat{\varphi}) = 0\,. \tag{3.2.3}$$

This is the equation of motion for our operator valued scalar field.

3.2.1 Free Quantum Scalar Field

Life is easy when we are free –

Let us consider the special case where $V(\varphi) = (1/2)m^2\varphi^2$. The equation of motion

$$\frac{d^2\hat{\varphi}}{dt^2} - \nabla^2\hat{\varphi} + m^2\hat{\varphi} = 0\,. \tag{3.2.4}$$

For this special case, the equation of motion is *linear* in the field. This invites us to move to Fourier Space. Let us imagine that our field is restricted to a box of volume $V = L^3$ and satisifies periodic boundary conditions. Then the Fourier transform of the field (and its inverse) are given by

$$\hat{\varphi}(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\varphi}_{\mathbf{k}}\,, \qquad \hat{\varphi}_{\mathbf{k}} = \frac{1}{\sqrt{V}}\int_V d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{\varphi}(\mathbf{x})\,, \qquad \text{with} \qquad \mathbf{k} = \frac{2\pi}{L}(n_1,n_2,n_3)\,, \tag{3.2.5}$$

where n_i are integers, and similar expressions hold for $\hat{\pi}$ and $\hat{\pi}_{\mathbf{k}}$. Plugging the Fourier transform into the equation of motion we get

$$\frac{d^2\hat{\varphi}_{\mathbf{k}}}{dt^2} + \omega_{\mathrm{k}}^2\hat{\varphi}_{\mathbf{k}} = 0\,, \qquad \text{with} \qquad \omega_{\mathrm{k}}^2 = |\mathbf{k}|^2 + m^2\,. \tag{3.2.6}$$

where we note that ω_{k} is a function of k = the magnitude of $\mathbf{k}$. The Fourier modes $\hat{\varphi}_{\mathbf{k}}$ are decoupled from each other and each one satisfies the same equation as a harmonic oscillator! This would not have been possible if the $V(\varphi)$ had terms beyond the mass term. What about the commutation relations? We inherit it from the commutation relation in position space

$$[\hat{\varphi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \Longrightarrow [\hat{\varphi}_{\mathbf{k}}(t),\hat{\pi}_{\mathbf{q}}(t)] = i\delta_{\mathbf{k},-\mathbf{q}}\,. \tag{3.2.7}$$

Note that the delta function in Fourier space is a Kronekor delta, a consequence putting the field in a box with periodic boundary conditions, which resulted in discrete $\mathbf{k}$. Also notice the minus sign in $\delta_{\mathbf{k},-\mathbf{q}}$.

The Hamiltonian can be written as

$$\hat{H} = \int_V d^3x\left(\frac{\hat{\pi}^2}{2} + \frac{(\nabla\hat{\varphi})^2}{2} + \frac{m^2}{2}\hat{\varphi}^2\right) = \sum_{\mathbf{k}}\left(\frac{1}{2}\hat{\pi}_{\mathbf{k}}\hat{\pi}_{-\mathbf{k}} + \frac{\omega_{\mathrm{k}}^2}{2}\hat{\varphi}_{\mathbf{k}}\hat{\varphi}_{-\mathbf{k}}\right)\,, \tag{3.2.8}$$

where we have used the fact that $\hat{\varphi}(x)$ is Hermitian, which implies $\hat{\varphi}^\dagger_{\mathbf{k}} = \hat{\varphi}_{-\mathbf{k}}$. The time evolution of $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$

$$\frac{d\hat{\varphi}_{\mathbf{k}}}{dt} = \hat{\pi}_{\mathbf{k}}\,, \qquad \frac{d\hat{\pi}_{\mathbf{k}}}{dt} = -\omega_{\mathrm{k}}^2 \hat{\varphi}_{\mathbf{k}}\,. \tag{3.2.9}$$

Exercise 3.2.1 : Derive the form of the Hamiltonian in Fourier space (second equality in eq. (3.2.8)), as well as the time evolution equations (3.2.9).

You should compare the Hamiltonian and the time evolution equations to our corresponding equations for harmonic oscillators (normal modes). In Fourier space, the free scalar field is just a collection of harmonic oscillators.

In complete analogy with the harmonic oscillators, it is convenient to introduce creation and annihilation operators, and write the mode expansion of the field in terms of the (time-independent) creation and annihilation operators

$$\hat{\varphi}_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_{\mathrm{k}}}}\left(\hat{a}_{\mathbf{k}} e^{-i\omega_{\mathrm{k}} t} + \hat{a}^\dagger_{-\mathbf{k}} e^{i\omega_{\mathrm{k}} t}\right)\,. \tag{3.2.10}$$

Notice the minus sign in $\hat{a}^\dagger_{-\mathbf{k}}$.

Exercise 3.2.2 : Following the derivation in the harmonic oscillator example from earlier in the notes, define the time-dependent creation and annihilation operators in terms of $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$. Be careful about $\hat{\varphi}^\dagger_{\mathbf{k}} = \hat{\varphi}_{-\mathbf{k}}$. Invert the relations to get $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ in terms of $\hat{a}_{\mathbf{k}}(t)$ and $\hat{a}^\dagger_{-\mathbf{k}}(t)$. Find the time evolution equations for $\hat{a}_{\mathbf{k}}(t)$ and $\hat{a}^\dagger_{-\mathbf{k}}(t)$. Use these to finally arrive at our mode expansion above.

Let us move back to position space:

$$\hat{\varphi}(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}}\left(\hat{a}_{\mathbf{k}} e^{-i\omega_{\mathrm{k}} t + i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} e^{i\omega_{\mathrm{k}} t - i\mathbf{k}\cdot\mathbf{x}}\right)\,. \tag{3.2.11}$$

Notice that I have flipped the sign of the subscript of $\hat{a}^\dagger_{\mathbf{k}}$ term, as well as that of $i\mathbf{k}\cdot\mathbf{x}$ in the exponent multiplying $\hat{a}^\dagger_{\mathbf{k}}$. We will do this repeatedly.

Exercise 3.2.3 : Quickly verify that $\sum_{\mathbf{k}} f(\mathrm{k}) g(\mathbf{k}) = \sum_{\mathbf{k}} f(\mathrm{k}) g(-\mathbf{k})$. This justifies our sign flip in the term containing $\hat{a}^\dagger$ above.

We now take advantage of our nice relativistic notation $ik\cdot x = i\omega_{\mathrm{k}} t - i\mathbf{k}\cdot\mathbf{x}$ and $-ik\cdot x = -i\omega_{\mathrm{k}} t + i\mathbf{k}\cdot\mathbf{x}$ to write the most useful form of our fields and conjugate momenta

$$\begin{aligned}\hat{\varphi}(x) &= \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}}\left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}^\dagger_{\mathbf{k}} e^{ik\cdot x}\right)\,, \\ \hat{\pi}(x) &= -\frac{i}{\sqrt{V}}\sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathrm{k}}}{2}}\left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} - \hat{a}^\dagger_{\mathbf{k}} e^{ik\cdot x}\right)\,.\end{aligned} \tag{3.2.12}$$

Let us recall the important properties of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$. You should verify as many of these as possible.

$$\begin{aligned}&[\hat{\varphi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{q}}(t)] = i\delta_{\mathbf{k},-\mathbf{q}} \\ \Longrightarrow\ &\left[\hat{a}_{\mathbf{k}}(t), \hat{a}^\dagger_{\mathbf{q}}(t)\right] = \delta_{\mathbf{k},\mathbf{q}}\,.\end{aligned} \tag{3.2.13}$$

Note that there is no pesky minus sign or i in the $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}$ commutation relation. The Hamiltonian can be written as

$$\hat{H} = \sum_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{2} \right) \omega_{\mathrm{k}} \,. \tag{3.2.14}$$

Note that if $|\psi\rangle$ is an eigenstate of the $\hat{H}$ with energy E, then $\hat{a}^\dagger_{\mathbf{k}}|\psi\rangle$ has an energy $E + \omega_{\mathrm{k}}$:

$$\hat{H}\hat{a}^\dagger_{\mathbf{k}}|\psi\rangle = (E + \omega_{\mathrm{k}})\hat{a}^\dagger_{\mathbf{k}}|\psi\rangle \,. \tag{3.2.15}$$

That is, $\hat{a}^\dagger_{\mathbf{k}}$ raises the energy by ω_α; it creates a "particle" of energy ω_{k}. Similarly, $\hat{a}_{\mathbf{k}}$ lowers the energy by ω_{k}; it annihilates a particle of energy ω_{k}. These "particles" are completely delocalized (have infinite spatial extent), since they have a fixed momentum. A localized particle can be created by creating a wave-packet from the superposition of such definite momentum "particles".

A general, normalized eigenstate of the Hamiltonian is

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2} \ldots\rangle = \prod_{\mathrm{s}} \frac{\left(\hat{a}^\dagger_{\mathbf{k}_{\mathrm{s}}}\right)^{n_{\mathbf{k}_{\mathrm{s}}}}}{\sqrt{n_{\mathbf{k}_{\mathrm{s}}}!}} |0\rangle \,, \tag{3.2.16}$$

with energy

$$E_{\mathrm{tot}} = \sum_{\mathrm{s}} E_{n_{\mathbf{k}_{\mathrm{s}}}} = \sum_{\mathbf{k}} \left(n_{\mathbf{k}} + \frac{1}{2} \right) \omega_{\mathrm{k}} \,. \tag{3.2.17}$$

Note that $n_{\mathbf{k}}$ are the number of particles with momentum $\mathbf{k}$. Notice that even when $n_{\mathbf{k}} = 0$ for all $\mathbf{k}$, we have an infinite vacuum energy: $E_{\mathrm{vac}} = \sum_{\mathbf{k}} (1/2)\omega_{\mathrm{k}}$.

A single particle state with momentum $\mathbf{k}_1$ is described by $n_{\mathbf{k}_1} = 1$, $n_{\mathbf{k}_{\mathrm{s}}} = 0$ for $s \neq 1$:

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2} \ldots\rangle = |1, 0 \ldots\rangle \equiv |\mathbf{k}_1\rangle = \hat{a}^\dagger_{\mathbf{k}_1}|0\rangle \,, \tag{3.2.18}$$

An example of a two-particle state is described by $n_{\mathbf{k}_1} = 1$, $n_{\mathbf{k}_2} = 1$ and $n_{\mathbf{k}_{\mathrm{s}}} = 0$ for $s \neq 1, 2$:

$$\begin{aligned} |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3} \ldots\rangle = |1, 1, 0 \ldots\rangle &\equiv |\mathbf{k}_1, \mathbf{k}_2\rangle \\ &= \hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}|0\rangle \,, \\ &= \hat{a}^\dagger_{\mathbf{k}_2}\hat{a}^\dagger_{\mathbf{k}_1}|0\rangle \,, \\ &= |\mathbf{k}_2, \mathbf{k}_1\rangle \,. \end{aligned} \tag{3.2.19}$$

Bose Statistics emerge naturally! Above, we used the fact that $\hat{a}^\dagger_{\mathbf{k}_1}$ and $\hat{a}^\dagger_{\mathbf{k}_2}$ commute with each other.

Exercise 3.2.4 : Write down a normalized eigenstate of the Hamiltonian with two $\mathbf{k}_1$ momentum particles, one $\mathbf{k}_2$ momentum particles and five $\mathbf{k}_3$ momentum particles (in terms of the creation operators acting on the vacuum).

So far, we have restricted the field to a box of volume V. Let us now allow $V \to \infty$. The field in position and momentum space are related by

$$\begin{aligned} \hat{\varphi}(t, \mathbf{x}) &= \int d^3\!k\, \hat{\varphi}(t, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \,, \\ \hat{\varphi}(t, \mathbf{k}) &= \int d^3x\, \hat{\varphi}(t, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \,. \end{aligned} \tag{3.2.20}$$

where $d^3\bar{k} = d^3k/(2\pi)^3$. Notice, that I am writing $\hat{\varphi}(t, \mathbf{k})$ instead of $\hat{\varphi}_{\mathbf{k}}(t)$ just for the sake of distinguishing it from the finite box case. Similar expressions hold for the conjugate momentum as well. We can get from the finite to the infinite box expressions through $\sum_{\mathbf{k}} \to V \int d^3\bar{k}$ and $\hat{\varphi}_{\mathbf{k}}(t) \to V^{-1/2}\hat{\varphi}(t, \mathbf{k})$.

The fundamental commutation relations become

$$[\hat{\varphi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \Longrightarrow \quad [\hat{\varphi}(t, \mathbf{k}), \hat{\pi}(t, \mathbf{q})] = i\bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}) \,. \tag{3.2.21}$$

where $\bar{\delta}(\mathbf{k}) = (2\pi)^3\delta(\mathbf{k})$ is the (scaled) Dirac delta function. There are two ways of deriving the Fourier space commutation relation. First, one can simply use our finite V to infinite V conversion, and recognize

$$\lim_{V\to\infty} V\delta_{\mathbf{k},-\mathbf{q}} = \bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}) \,. \tag{3.2.22}$$

Another way is to substitute the inverse Fourier transform:

$$\begin{aligned} [\hat{\varphi}(t, \mathbf{k}), \hat{\pi}(t, \mathbf{q})] &= \int d^3x d^3y \, e^{i(\mathbf{k}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})} \left[\hat{\varphi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\right] , \\ &= i \int d^3x d^3y \, e^{i(\mathbf{k}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \,, \\ &= i \int d^3x \, e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}} \,, \\ &= i\bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}) \,. \end{aligned} \tag{3.2.23}$$

where in the last line we used following representation of the Dirac delta function: $\bar{\delta}^{(3)}(\mathbf{k} + \mathbf{q}) = \int d^3x e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}$.

We can now continue with our now familiar program of defining creation annihilation operators and such. However, before we do so, let us digress a bit to note that $d^3\bar{k}$ is not a Lorentz invariant measure. A Lorentz invariant measure would be

$$(dk) \equiv \frac{d^3\bar{k}}{(2\omega_{\mathrm{k}})} \,. \tag{3.2.24}$$

Exercise 3.2.5 : Here, I guide you through the proof that shows that (dk) is indeed Lorentz invariant. First show that $\int (dk) F(k^\mu) = \int d^4\bar{k}\bar{\delta}(k^2 - m^2)\Theta(k^0)F(k^\mu)$ for arbitrary F. Here, $k^2 = k^\mu k_\mu$, $\Theta(x) = 0$ for $x < 0$ and 1 for $x > 0$. Now realize that $\int d^4\bar{k}\bar{\delta}(k^2 - m^2)\Theta(k^0)$ is Lorentz invariant because $d^4\bar{k}\bar{\delta}(k^2 - m^2)$ is manifestly Lorentz invariant, and so is $\Theta(k^0)$ since a Lorentz transformation cannot change the sign of k^0. That completes the proof.

It is nicer to have Lorentz invariant measures, which motivates us to make the following (scaled) definitions for the creation and annihilation operators in the continuum case[4]

$$\hat{a}_{\mathbf{k}} \to V^{-1/2}(2\omega_{\mathrm{k}})^{-1/2}\hat{a}(\mathbf{k}) \,. \tag{3.2.25}$$

With this scaling, the mode expansion of the field in terms of creation and annihilation operators becomes

$$\hat{\varphi}(x) = \int (dk) \left(\hat{a}(\mathbf{k})e^{-ik\cdot x} + \hat{a}^\dagger(\mathbf{k})e^{ik\cdot x}\right) \,, \tag{3.2.26}$$

and the commutation relation for the creation and annihilation operators becomes

$$\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{q})\right] = 2\omega_{\mathrm{k}}\bar{\delta}^{(3)}(\mathbf{k} - \mathbf{q}) \,. \tag{3.2.27}$$

[4]Note that this scaling relation is different from the scaling relation between an arbitrary function $f_{\mathbf{k}}$ and $f(\mathbf{k})$ we have been using, because of our insistence on using a Lorentz invariant measure this time around.

The Hamiltonian in terms of the creation and annihilation operators

$$\hat{H} = \int (dk)\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\omega_\mathrm{k} + V\int (dk)\omega_\mathrm{k}^2 \,. \tag{3.2.28}$$

The second term is the vacuum energy. Note that its infinity arises not just from the infinite volume up front. The energy density

$$\frac{E_\mathrm{vac}}{V} = \int (dk)\omega_\mathrm{k}^2 = \int \frac{d^3k}{(2\pi)^3}\omega_\mathrm{k} \,. \tag{3.2.29}$$

is also formally infinite. In reality, we should expect that our theory is only valid up to some large momentum, which provides an upper bound for the integral. Nevertheless, such a large vacuum energy plays no role for us (no gravity in this course) if we promise to only care about differences above this energy.

We can build up eigenstates of the Hamiltonian with $\hat{a}^\dagger(\mathbf{k})$. For example, the single particle and two particle states

$$|\mathbf{k}\rangle = \hat{a}^\dagger(\mathbf{k})|0\rangle \qquad \text{and} \qquad |\mathbf{k}_1, \mathbf{k}_2\rangle \quad = \hat{a}^\dagger(\mathbf{k}_1)\hat{a}^\dagger(\mathbf{k}_2)|0\rangle \,. \tag{3.2.30}$$

The vacuum is defined by $\hat{a}(\mathbf{k})|0\rangle = 0$, and we normalize it $\langle 0|0\rangle = 1$. Note, that apart from vacuum our eigenstates are not normalized. In particular, the single particle state

$$\langle \mathbf{k}|\mathbf{k}\rangle = 2\omega_\mathrm{k}\bar{\delta}^{(3)}(\mathbf{k}-\mathbf{k})\langle 0|0\rangle \to \infty \,. \tag{3.2.31}$$

The Dirac-delta function up front comes from a volume factor. The infinity is a result of our completely delocalized state. This is an inconvenience that we can live with. One can always construct normalized states by taking a superposition of many such eigenstates.

We have defined multi-particle states of definite momenta. What about our usual "localized particle" ? Consider the action of $\hat{\varphi}(x)$ on the vacuum $|0\rangle$. To reduce clutter, let $x^0 = 0$. Using eq. (3.2.26), we have

$$\hat{\varphi}(\mathbf{x})|0\rangle = \int (dk)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger(\mathbf{k})|0\rangle = \int (dk)e^{-i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}\rangle = \frac{1}{2m}\int_{|\mathbf{k}|\ll m}\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}\rangle + \ldots \propto |\mathbf{x}\rangle + \ldots \tag{3.2.32}$$

where $|\mathbf{x}\rangle$ is what we would have referred to as describing a localized single particle at $\mathbf{x}$ from our old-school quantum mechanics. Hence we interpret $\hat{\varphi}(\mathbf{x})|0\rangle$ as creating (or destroying) a localized particle at the position $\mathbf{x}$.[5]

3.2.2 Propagation Amplitudes and Positive and Negative Frequency Solutions

For future convenience, let us split our mode expansion given in eq. (3.2.26) into positive and negative frequency parts

$$\hat{\varphi}(x) = \hat{\varphi}^+(x) + \hat{\varphi}^-(x) \,, \tag{3.2.33}$$

where

$$\begin{aligned} \hat{\varphi}^+(x) &\equiv \int (dk)\hat{a}(\mathbf{k})e^{-ik\cdot x} \qquad \text{positive frequency part}\,, \\ \hat{\varphi}^-(x) &\equiv \int (dk)\hat{a}^\dagger(\mathbf{k})e^{ik\cdot x} \qquad \text{negative frequency part}\,. \end{aligned} \tag{3.2.34}$$

[5] Admittedly this seems a bit unsatisfactory, since we cut-off the higher momentum bits. But there are no localized single particle states in field theory at high energies! This heuristic picture is further complicated when we move to interacting fields.

It is $\hat{\varphi}^-$ that is responsible for creating a particle out of the vacuum. Let us consider the amplitude for a particle to propagate from y to x:[6]

$$\begin{aligned}
\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle &= \langle 0|\hat{\varphi}^+(x)\hat{\varphi}^-(y)|0\rangle\,, \\
&= \int (dk)(dq)e^{-ik\cdot x+iq\cdot y}\langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{q})|0\rangle\,, \\
&= \int (dq)e^{-iq\cdot(x-y)}\,, \qquad\qquad \text{because } \left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{q})\right] = 2\omega_{\mathrm{k}}\tilde{\delta}^{(3)}(\mathbf{k}-\mathbf{q})\,, \\
&\equiv \Delta^+(x-y)\,.
\end{aligned} \tag{3.2.35}$$

Similarly,

$$\langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle = \langle 0|\hat{\varphi}^+(y)\hat{\varphi}^-(x)|0\rangle = \int (dq)e^{-iq\cdot(y-x)} = \Delta^+(y-x) \equiv -\Delta^-(x-y)\,. \tag{3.2.36}$$

Note that $\Delta^\pm(x-y)$ behave like numbers, not operators. They multiply the identity operator. Putting these amplitudes together, the commutator

$$\begin{aligned}
\langle 0|\left[\hat{\varphi}(x),\hat{\varphi}(y)\right]|0\rangle &= \Delta^+(x-y) + \Delta^-(x-y)\,, \\
&= \Delta^+(x-y) - \Delta^+(y-x)\,, \\
&= \int (dq)\left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)\,.
\end{aligned} \tag{3.2.37}$$

Note that instead of amplitude, we could have directly evaluated following commutators $[\hat{\varphi}^+(x),\hat{\varphi}^-(y)] = \Delta^+(x-y)$ and $[\hat{\varphi}^-(x),\hat{\varphi}^+(y)] = -\Delta^-(x-y)$, and put them together to get

$$[\hat{\varphi}(x),\hat{\varphi}(y)] = \Delta^+(x-y) + \Delta^-(x-y) = \int (dq)\left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)\,, \tag{3.2.38}$$

since the commutator yields a number, and is the same as $\langle 0|\left[\hat{\varphi}(x),\hat{\varphi}(y)\right]|0\rangle$ if the vacuum is normalized $\langle 0|0\rangle = 1$. Hence the commutator can be though of as the amplitude of a particle to propagate from y to x minus the amplitude of a particle to travel from x to y. We will have use for these expression of the commutator in the discussion on causality and when we later consider interacting fields.

3.2.3 Green's functions

Let us consider the case where $x^0 > y^0$ and define[7] $i\Delta_R(x-y) \equiv \theta(x^0-y^0)\langle 0|\left[\hat{\varphi}(x),\hat{\varphi}(y)\right]|0\rangle$. Then, we can show that

$$(\partial_\mu\partial^\mu + m^2)\Delta_R(x-y) = -\delta^{(4)}(x-y)\,. \tag{3.2.39}$$

where $\partial_\mu = \partial/\partial x^\mu$. Thus $-\Delta_R(x-y)$ is a Green's function for the free Klein-Gordon equation! It is called the *Retarded* Green's function because it vanishes for $x^0 < y^0$. The lesson here is that our commutator is related to the propagation of signals from one point to another.

We will come across the *Feynman* Green's function later in the course, when we carry out scattering calculations. For the moment, let me just write down the definition:

$$\begin{aligned}
i\Delta_F(x-y) &= \theta(x^0-y^0)\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle + \theta(y^0-x^0)\langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle\,, \\
&= \theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^+(y-x)\,.
\end{aligned} \tag{3.2.40}$$

As you can check, it is also a Green's function for the Klein-Gordon equation.

Exercise 3.2.6 : Verify eq. (3.2.39). It might be useful to first show that $(\partial^\mu\partial_\mu + m^2)\Delta^\pm(x-y) = 0$.

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[6] A word of caution: The language used to describe the mathematical objects here is imprecise. Do not take them as cannon, read different references to get a more complete picture.

[7] $\theta(x) = 1$ for $x > 0$ and 0 for $x < 0$

3.2.4 Causality

The fact that a "measurement" at x cannot influence a measurement at y for space-like separations $(x-y)^2 < 0$ should be built into our theory. Else, signals are propagating faster than light, and that is bad for a theory we claimed to be consistent with Special Relativity. In QFT, this statement of *Causality* is formally written as

$$\left[\hat{\mathcal{O}}_1(x), \hat{\mathcal{O}}_2(y)\right]_{(x-y)^2<0} = 0\,. \tag{3.2.41}$$

where $\hat{\mathcal{O}}_1(x)$ and $\hat{\mathcal{O}}_2(x)$ are Hermitian operators corresponding to some observables. For our scalar field theory, since the $\hat{\varphi}$ field is all there is, such operators are constructed from functions of $\hat{\varphi}$ (and their conjugate momenta). The simplest example of such operators is the field $\hat{\varphi}(x)$ itself. Hence, it better be true that:

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_{(x-y)^2<0} = 0\,. \tag{3.2.42}$$

To verify this in our free field theory, recall that any space-like separated events can be made simultaneous by a Lorentz transformation. Since the commutator above (see eq. (3.2.38)) is manifestly Lorentz invariant, it is sufficient to show that $[\hat{\varphi}(x), \hat{\varphi}(y)]_{x^0=y^0} = 0$ (at equal times). Writing the commutator in eq. (3.2.38) for the equal-time case, we immediately have

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_{x^0=y^0} = \int (dq) \left(e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\right) = 0\,. \tag{3.2.43}$$

The last equality follows from the fact that the integral is odd in $\mathbf{q}$. Having proved that the commutator vanishes for space-like separations, you should convince yourself that in general, it does not vanish for time-like separations.

Recall that in the previous chapter, we found that $\mathcal{A} = \langle \mathbf{x}|e^{-i\sqrt{\mathbf{p}^2+m^2}t}|\mathbf{x}_0\rangle \neq 0$ for spacelike separations, which we took to be the death of single particle quantum mechanics. What about an equivalent expression in field theory? As you can check, $\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle = \Delta^+(x-y) \neq 0$ for spacelike separations again! So what have we really gained by moving to field theory? Let us delve a little bit deeper. In single particle quantum mechanics, we state that we have one particle throughout, and no other excitations. So a non-zero overlap of states over spacelike intervals does violate causality. However, in a non-single particle theory (field theory): $\langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle \neq 0$ could mean that there are correlations different excitations not necessarily related to propagation of any signals from one point to another. The appropriate thing to calculate is whether measurement at spacelike separated points can affect each other. That operation is indeed the commutator, which fortunately is zero.[8] A connection between causality and existence of antiparticles can be better appreciated after we discuss complex fields.

3.2.5 Free Complex Scalar Fields

So far we have been dealing with real valued scalar fields. Let us consider a Lagrangian for a free, classical complex field $\varphi(x)$:[9]

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi^* - m^2\varphi^*\varphi \tag{3.2.44}$$

where φ^* is the complex conjugate of φ. We will think about φ and φ^* as being independent fields (we can also chose the real and imaginary parts of φ). In this case the conjugate momentum (density) corresponding to these fields is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}^*(x) \qquad \text{and} \qquad \pi^*(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}^*} = \dot{\varphi}(x)\,. \tag{3.2.45}$$

[8]Thanks to Daniel Green for a discussion on this, though we are both still a little uneasy about the details of the interpretation.

[9]While we consider free scalar fields, the manipulations on this page can be easily generalized to more general potentials $V(\varphi^*\varphi) = \sum_{n=1}^{N}(1/n!)\lambda_n(\varphi^*\varphi)^n$.

The Hamiltonian is

$$H = \int d^3x \, (\pi\dot{\varphi} + \pi^*\dot{\varphi}^* + \mathcal{L}) = \int d^3x \left(\pi^*\pi + \nabla\varphi^* \cdot \nabla\varphi + m^2\varphi^*\varphi\right) . \tag{3.2.46}$$

It is possible to define a "charge" $Q = i\int d^3x[(\pi\varphi)^* - \varphi\pi]$, such that $dQ/dt = \{Q, H\} = 0$, that is the charge is conserved. Here, I seem to have pulled Q out of the hat. When we learn about Noether's theorem in the second half of the course, this definition of Q will seem natural. The conserved Q is a consequence of the fact that $\varphi(x) \to e^{i\alpha}\varphi(x)$ leaves the Lagrangian invariant. We note that this result would hold even when we have an arbitrary potential of the form $V(\varphi^*\varphi)$ (instead of the just the free field case studied here).

Canonical Quantization

Let us postulate the usual commutation relations:

$$[\hat{\varphi}(x), \hat{\pi}(y)]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \qquad \text{and} \qquad \left[\hat{\varphi}^\dagger(x), \hat{\pi}^\dagger(y)\right]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \, , \tag{3.2.47}$$

with $[\hat{\varphi}(x), \hat{\varphi}(y)] = \left[\hat{\varphi}^\dagger(x), \hat{\varphi}^\dagger(y)\right] = \ldots = 0$, and the "$\dagger$" denotes the Hermitian conjugate. The Hamiltonian becomes

$$\hat{H} = \int d^3x \left(\hat{\pi}^\dagger\hat{\pi} + \nabla\hat{\varphi}^\dagger \cdot \nabla\hat{\varphi} + m^2\hat{\varphi}^\dagger\hat{\varphi}\right) , \tag{3.2.48}$$

with the equations of motion given by

$$\begin{aligned} \frac{d\hat{\varphi}}{dt} &= -i\left[\hat{\varphi}, \hat{H}\right] = \hat{\pi}^\dagger \qquad &\text{and}& \qquad \frac{d\hat{\pi}}{dt} = -i\left[\hat{\pi}, \hat{H}\right] = \nabla^2\hat{\varphi}^\dagger - m^2\hat{\varphi}^\dagger \, , \\ \frac{d\hat{\varphi}^\dagger}{dt} &= -i\left[\hat{\varphi}^\dagger, \hat{H}\right] = \hat{\pi} \qquad &\text{and}& \qquad \frac{d\hat{\pi}^\dagger}{dt} = -i\left[\hat{\pi}^\dagger, \hat{H}\right] = \nabla^2\hat{\varphi} - m^2\hat{\varphi} \, . \end{aligned} \tag{3.2.49}$$

Take note of the location of the $\dagger$ s in the above equations. In terms of second-order in time equations, we have

$$\frac{d^2\hat{\varphi}}{dt^2} - \nabla^2\hat{\varphi} + m^2\hat{\varphi} = 0 \, , \tag{3.2.50}$$

and its Hermitian conjugate. As in the classical field case, we have also have a conserved charge:

$$\hat{Q} \equiv i\int d^3x \left(\hat{\varphi}^\dagger\hat{\pi}^\dagger - \hat{\varphi}\hat{\pi}\right) , \qquad \text{such that} \qquad \frac{d\hat{Q}}{dt} = -i\left[\hat{Q}, \hat{H}\right] = 0 \, . \tag{3.2.51}$$

Exercise 3.2.7 : Derive the rightmost sides of eq. (3.2.49). Then, using the definition of $\hat{Q}$ in eq. (3.2.51), show that $[\hat{Q}, \hat{H}] = 0$. (Hint: You will need to use integration by parts over the spatial volume; assumed fields die sufficiently fast at spatial infinity.)

Mode expansion

I claim that the mode expansions for our fields are given by

$$\hat{\varphi}(x) = \int (dk) \left(\hat{b}(\mathbf{k})e^{-ik\cdot x} + \hat{d}^\dagger(\mathbf{k})e^{ik\cdot x}\right) \qquad \text{and} \qquad \hat{\varphi}^\dagger(x) = \int (dk) \left(\hat{d}(\mathbf{k})e^{-ik\cdot x} + \hat{b}^\dagger(\mathbf{k})e^{ik\cdot x}\right) . \tag{3.2.52}$$

Note that this is reasonable. For a real scalar field, $\hat{\varphi}(x) = \hat{\varphi}^\dagger(x) = \int (dk) \left(\hat{a}(\mathbf{k})e^{-ik\cdot x} + \hat{a}^\dagger(\mathbf{k})e^{ik\cdot x}\right)$, and hence the second operator in the mode expansion $\hat{a}^\dagger$ ended up being a hermitian conjugate of the first operator ($\hat{a}$). But for a complex field, $\hat{\varphi}^\dagger(x) \neq \hat{\varphi}(x)$, hence we need two sets of creation and annihilation operators, thus indicating the existence of two distinct particles. The creation and annihilation operators are written in a manner so that upon taking the Hermitian conjugate of the right hand side in the mode expansion of $\hat{\varphi}$, we get the right hand side in the mode expansion of $\hat{\varphi}^\dagger$ above. Similarly, the mode expansions of conjugate momentum densities are:

$$\pi(x) = -i\int (dk)\omega_{\mathrm{k}}\left(\hat{d}(\mathbf{k})e^{-ik\cdot x} - \hat{b}^\dagger(\mathbf{k})e^{ik\cdot x}\right) \qquad \text{and} \qquad \pi^\dagger(x) = -i\int (dk)\omega_{\mathrm{k}}\left(\hat{b}(\mathbf{k})e^{-ik\cdot x} - \hat{d}^\dagger(\mathbf{k})e^{ik\cdot x}\right) . \tag{3.2.53}$$

The commutation relations satisfied by the creation and annihilation operators are

$$\left[\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{q})\right] = 2\omega_{\mathrm{k}}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,, \qquad \text{and} \qquad \left[\hat{d}(\mathbf{k}), \hat{d}^\dagger(\mathbf{q})\right] = 2\omega_{\mathrm{k}}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,, \tag{3.2.54}$$

with all others being zero.

Charge

The Hamiltonian and the charge $\hat{Q}$ in terms of the creation and annihilation operators become

$$\begin{aligned} \hat{H} &= \int (dk)\left(\hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k}) + \hat{d}^\dagger(\mathbf{k})\hat{d}(\mathbf{k})\right)\omega_{\mathrm{k}} + \text{const.} \\ \hat{Q} &= \int (dk)\left(\hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k}) - \hat{d}^\dagger(\mathbf{k})\hat{d}(\mathbf{k})\right) . \end{aligned} \tag{3.2.55}$$

I will derive these expressions so that we can get some practice with the manipulations of creation and annihilation operators, delta functions and facility with changing dummy variables. But before we do that, let us digress to understand the physical interpretation of these expressions. This meaning is clearer when we put these fields in a box with finite volume. In this case

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}}\left(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{k}}\right)\omega_{\mathrm{k}} + \text{const.} \\ \hat{Q} &= \sum_{\mathbf{k}}\left(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} - \hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{k}}\right) . \end{aligned} \tag{3.2.56}$$

From the expression for the Hamiltonian we see that, we have two sets of particles, each has the same mass, and they contribute equally to the Hamiltonian. Since $\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ and $\hat{d}^\dagger_{\mathbf{k}}\hat{d}_{\mathbf{k}}$ simply count the number of b and d particles with momentum $\mathbf{k}$, the conservation of $\hat{Q}$ is a statement about conservation of a difference between the number of these particles. The sign difference corresponding to the number operator allows us to interpret one set of particles as "positively" charged and the other as "negatively" charged with all else being equal. Think of these as particles and their anti-particles. [10] [11]

As promised, let us derive the expressions for $\hat{Q}$ in eq. (3.2.55). By using the mode expansions of $\hat{\varphi}$ and $\hat{\varphi}^\dagger$ in eq. (3.2.52) and $\hat{\pi} = d\hat{\varphi}^\dagger/dt$ and $\hat{\pi}^\dagger = d\hat{\varphi}/dt$ in eq. (3.2.53) we get

$$\begin{aligned} \hat{Q} &= \underbrace{i\int d^3x\,\hat{\varphi}^\dagger\hat{\pi}^\dagger}_{1} - \underbrace{i\int d^3x\,\hat{\varphi}\hat{\pi}}_{2}\,, \\ 1 &= \int d^3x(dk)(dq)\omega_{\mathrm{q}}\left(\hat{d}(\mathbf{k})e^{-ik\cdot x} + \hat{b}^\dagger(\mathbf{k})e^{ik\cdot x}\right)\left(\hat{b}(\mathbf{q})e^{-iq\cdot x} - \hat{d}^\dagger(\mathbf{q})e^{iq\cdot x}\right) , \\ 2 &= \int d^3x(dk)(dq)\omega_{\mathrm{q}}\left(\hat{b}(\mathbf{q})e^{-iq\cdot x} + \hat{d}^\dagger(\mathbf{q})e^{iq\cdot x}\right)\left(\hat{d}(\mathbf{k})e^{-ik\cdot x} - \hat{b}^\dagger(\mathbf{k})e^{ik\cdot x}\right) . \end{aligned} \tag{3.2.57}$$

[10] Note these particles are not electrons/positrons etc. which are quanta of spin 1/2 fields, not scalar fields. A reasonable (but approximate) real life example of particles described by scalar fields with our usual electric charge would be pions. The charge here need not be electric charge.

[11] There was some ambiguity in how we chose to order operators in the Hamiltonian and the charge in terms of fields and their conjugate momenta. Ultimately, we always write these as a sum of a finite part and an infinite constant (related to the vacuum), which renders this ambiguity inconsequential. Also the sign of the charge is convention dependent.

For (2) we have interchanged $\mathbf{k}$ and $\mathbf{q}$ since they are dummy variables. You might have noticed that I did not change the label for ω_{k}. To understand this, note that $i(k \pm q) \cdot x = (\omega_{\mathrm{k}} \pm \omega_{\mathrm{q}})t - (\mathbf{k} \pm \mathbf{q}) \cdot \mathbf{x}$ and, $\int d^3x\, e^{-i(\mathbf{k}\pm\mathbf{q})\cdot\mathbf{x}} = \bar{\delta}^{(3)}(\mathbf{k} \pm \mathbf{q})$, which always sets $\omega_{\mathrm{k}} = \omega_{\mathrm{q}}$.

For the difference, (1) − (2), first focus on the db and bd terms. Since b and d commute these terms cancel each other. The same is true for $b^\dagger d^\dagger$ and $d^\dagger b^\dagger$ terms. Hence we are left with

$$\hat{Q} = \frac{1}{2} \int (dk) \left(\hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k}) + \hat{b}(\mathbf{k})\hat{b}^\dagger(\mathbf{k}) - \hat{d}^\dagger(\mathbf{k})\hat{d}(\mathbf{k}) - \hat{d}(\mathbf{k})\hat{d}^\dagger(\mathbf{k}) \right) , \tag{3.2.58}$$

where we used $(dq)\omega_{\mathbf{q}} = d^3q/(2(2\pi)^3)$ along with the $\bar{\delta}(\mathbf{k} - \mathbf{q})$. Finally, using the commutation relations (see eq. (3.2.54)), we arrive at

$$\hat{Q} = \int (dk) \left(\hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k}) - \hat{d}^\dagger(\mathbf{k})\hat{d}(\mathbf{k}) \right) . \tag{3.2.59}$$

Exercise 3.2.8 : Derive the expression for $\hat{Q}$ in eq. (3.2.56) using a finite box size. This should be follows immediately from our now familiar rules for going between the finite and infinite volume cases: $(2\omega_k V) \int (dk) \leftrightarrow \sum_{\mathbf{k}}$ and $V^{-1/2}(2\omega_{\mathrm{k}})^{-1/2}\hat{a}(\mathbf{k}) \leftrightarrow \hat{a}_{\mathbf{k}}$. For practice, you should also (i) re-write the mode functions and definition for $\hat{Q}$ in the finite box case, and (ii) work through the manipulations at the end of this subsection on *Charge* to re-derive the expression for $\hat{Q}$ in terms of the creation and annihilation operators.

Causality, again

Given the above mode expansions in eq. (3.2.52), you can again check that

$$[\hat{\varphi}(x), \hat{\varphi}^\dagger(y)] = \Delta^+(x - y) - \Delta^+(y - x) , \qquad \text{and} \qquad [\hat{\varphi}(x), \hat{\varphi}^\dagger(y)]_{(x-y)^2<0} = 0 . \tag{3.2.60}$$

Recall that we have conserved charge and a notion of positive and negative charged particles. Picking a convention for the sign of the charge, we can say that $\varphi^\dagger|0\rangle$ creates negatively charged particles out of vacuum (and annihilates positively charged ones), whereas $\hat{\varphi}|0\rangle$ creates positively charged particles (and annihilates negatively charged ones).

The vanishing of the commutator outside on space-like separations can be interpreted as follows. The $\Delta^+(x - y)$ represents the amplitude of propagation of a negatively charged particle from y to x whereas $\Delta^+(y - x)$ represents the propagation of a positively charged particle from x to y. Each individually has non-zero contributions outside the light-cone. However, for the commutator to vanish, they must be equal to each other outside the light-cone. This of course is only possible because they have the same mass. Thus in a way, causality requires the existence of antiparticles (opposite charge, same mass!). In the case of the real scalar field earlier, particles are their own antiparticles.

For further discussion, see for example, section 2.1 and 2.4 in Peskin and Schroeder. Another short discussion can be found in section 2.1 of *Modern QFT, A Concise Introduction* by Banks. For a detailed discussion of conceptual issues related to causality, also see *The Conceptual Framework for QFT*, by Duncan.[12]

Non-relativistic Fields

Before we move on to interactions, let me make a brief digression to cold-atom systems. Bose-Einstein condensates of cold atoms are well described by a non-linear Schrödinger equation. We can get to this equation by considering a multi-particle wavefunction with interactions, or by taking the non-relativistic

[12]Thanks to D. Baumann for this reference.

limit of the our relativistic Klein-Gordon equation. While we do not have the time to go through this in detail, you can work through the problem below to get a bit of the flavor.

Exercise 3.2.9 : Consider the Lagrangian density $\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - m^2|\varphi|^2 - \lambda|\varphi|^4$ where φ is a complex scalar field. Derive the Euler-Lagrange equation for φ. Then change variables $\varphi(t, \mathbf{x}) = \exp[-imt]\psi(t, \mathbf{x})$, and derive an equation for ψ (a complex scalar field as well) assuming the time-scale and length-scale of variation in ψ is much larger and longer that m^{-1}. This is the non-relativistic limit of our theory. You should arrive at the (non-linear) Schrodinger equation. But be careful here: φ should not be interpreted as a single-particle wave function. Define a conserved charge for this system.

CHAPTER 4

WEAKLY INTERACTING FIELDS

In sections 3.2.1 and 3.2.5 we dealt with free scalar fields. For such cases, the classical and quantum field equations are linear in the fields. This means that each Fourier mode evolves independently, and the problem essentially reduces to that of quantizing a harmonic oscillator for each Fourier mode. The dynamics is simple, but also boring. There are no interactions – no scattering and no decays.[1] In this chapter we introduce interactions, which will allow for non-trivial scattering and decays. We will concern overselves with perturbative calculations, where the interactions introduced are in some sense weak. Non-perturbative field theory is fascinating, but beyond the scope of this course (for the most part). The formalism we develop in this chapter will lead us to Feynman Diagrams.

4.1 Adding Interactions

To make our scalar field theory more interesting, and somewhat more realistic, we need to introduce interactions. At the level of the Lagrangian density, this means adding nonlinear terms in the fields, or coupling different fields. Let us look at a couple of examples:
Massive φ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4\,, \tag{4.1.1}$$

where $(\partial\varphi)^2 = g_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi$ and we will drop the "hats" from the fields. We are dealing with quantum fields from now onwards. The φ^4 makes the equations of motion nonlinear. We will also learn that it allows for processes like two φ quanta scattering of each other.

Similarly, we can write down a Lagrangian density with both a real and complex scalar field. We will now denote the complex field as ψ:
Scalar version of Quantum Electrodynamics:

$$\mathcal{L} = |\partial\psi|^2 - M^2\psi^\dagger\psi + \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - g\varphi\psi^\dagger\psi\,. \tag{4.1.2}$$

The interaction term $g\varphi\psi^\dagger\psi$ allows for processes like the decay of a φ quantum, into a ψ particle and anti-particle. It also allows for scattering of ψ particles and anti-particles via an exchange of φ quanta and so on. In this toy example, you can think of φ as representing a photon for $m \to 0$ and ψ and $\psi^\dagger$ for electrons.[2]

[1] Classically speaking, ripples in the field just pass through each other, without any changes.

[2] Note photons are quanta of gauge fields, and electrons of fermionic fields. They are definitely not represented by scalar fields.

4.1.1 Perturbative Control

Free theory, without the interaction term was simple and solvable. We want to make use of it as much as possible, but include effects from the interaction terms so that interesting processes become allowed. We humbly start by thinking about including the effects of the interaction perturbatively. It is reasonable that this imposes a restriction on the *coupling constants* λ and g. However, saying that they are small, and that their effects will be small is not as trivial as it seems.

Note that λ must be dimensionless, whereas g must have dimensions. To see this, recall that in our $c = \hbar = 1$ system of units, energy, mass, momentum can be measured with the same units. It is convenient to define a "mass-dimension" denoted by [...] such that [mass] = [energy] = [momentum] = 1 and correspondingly, [length] = [time] = −1. The action always has mass-dimension 0. Now, since $[d^4x] = -4$, we must have $[\mathcal{L}] = 4$. By looking at the $(\partial\varphi)^2$ or $|\partial\psi|^2$ terms, $[\mathcal{L}] = 4$ and $[\partial] = 1$ implies that $[\varphi] = [\psi] = 1$. Moving back to the interaction terms, we can see that $[\lambda] = 0$ and $[g] = 1$. Since λ is indeed dimensionless, saying that $|\lambda| \ll 1$ is reasonable. But saying that g is small is not possible without constructing a dimensionless ratio with some other mass or energy scale.

To understand the kind of problems that might arise, first suppose we want to calculate the amplitude of some process, say the scattering of two φ particles in the φ^4 theory. For $|\lambda| \ll 1$, a perturbative calculation of the Amplitude (in terms of λ) can be expected to have the form

$$\mathcal{A}(\varphi\varphi \to \varphi\varphi) = \sum_{n=0} \lambda^n f_n(\{p^\mu\}) = 1 + \lambda f_1(\{p^\mu\}) + \lambda^2 f_2(\{p^\mu\}) + \dots \tag{4.1.3}$$

with a good chance that first few terms yield a reliable answer. Here $\{p^\mu\}$ stand for the momenta of the incoming and out-going particles, and f_n are dimensionless functions of the external momenta of the incoming and outgoing particles.

Now, suppose we want to calculate the amplitude the scattering of ψ particles: $\mathcal{M}(\psi\psi \to \psi\psi)$. This time, we follow our nose, and write

$$\mathcal{A}(\psi\psi \to \psi\psi) = \sum_n \left(\frac{g}{E}\right)^n f_n(\{p^\mu\})\,. \tag{4.1.4}$$

For some g, if $E \gg g$, then we can get away with calculating the first few terms. However, if our experiment is very low energy ($E \ll g$), the above expansion is useless. Can $E \ll g$ be avoided? For this problem, yes! Since our particles have mass M, our energy scale $E \gtrsim M$. Hence, we can get away with the "small g" expansion if $g \ll M$.

To summarize, if we want to do perturbative calculations using λ or g to organize our expansion (for arbitrary E), then we should at least make sure that $\lambda \ll 1$ and $g \ll M$ (if $M \to 0$, at least at this heuristic level, we cannot use this small g expansion at low energies).

There are some general lessons to be learnt. For the real scalar field example, consider general interaction terms of the form $(\lambda_n/n!)\varphi^n$ (where $n > 2$). Then for perturbative control we need $|\lambda_n|/E^{4-n} \ll 1$. For massive fields, we expect $E \gtrsim m$. Hence, it is sufficient to have $|\lambda_n| \ll m^{4-n}$. For fields with different mass particles, we have to make sure that the coupling constants g_n are smaller than the appropriate powers of the lightest mass involved: m_l.

These considerations are meant as a guidance, not proof of what actually happens in explicit calculations. Some $f_n's$ might be zero or formally infinite (say at resonances, or from higher order contributions), complicating our simple arguments. Moreover, by insisting on "renormalizability" of the theory, we can severely limit the kind of interaction terms that can be added. More on this, later.[3]

[3] You might also want to read pg 47-50 of David Tong's lecture notes to get a broader picture of the structure of lagrangians. I also recommend reading section 4.1 of Peskin and Schroeder to get an overview of the what principles we typically follow in writing down interaction terms.

4.2 Time Evolution in the Interaction Picture

In the previous chapter we discussed the *Heisenberg* and *Schrödinger Pictures* as being equivalent way of capturing time evolution. For weakly interacting fields, yet another picture is useful: the *Interaction Picture* which is hybrid on the Heisenberg and Schrödinger pictures. Consider a Hamiltonian[4]

$$H = H_0 + H_{\text{int}}\,, \tag{4.2.1}$$

where H_0 is the free Hamiltonian and $H = \int d^3x \mathcal{H}_{\text{int}} = -\int d^3x \mathcal{L}_{\text{int}}$. We will be thinking about H_{int} as being a small correction to H. In terms of examples of interactions mentioned earlier, $\mathcal{H}_{\text{int}} = (\lambda/4!)\varphi^4$ or $g\varphi\psi^\dagger\psi$.

Recall that in the Heisenberg picture, the operators evolve according to $f(t) = e^{iH(t-t_0)}f(t_0)e^{-iH(t-t_0)}$ but states $|\alpha\rangle$ do not, whereas in the Schrödinger picture, states evolve according to $|\alpha(t)\rangle_{\text{s}} = e^{-iH(t-t_0)}|\alpha(t_0)\rangle_{\text{s}}$, whereas operators f_{s} do not. Operators and states in different pictures agree at $t = t_0$: $f(t_0) = f_{\text{s}}$ and $|\alpha\rangle = |\alpha(t)\rangle_{\text{s}}$. Note that the evolution is determined by the full Hamiltonian H. The expectation value $\langle\alpha|f(t)|\alpha\rangle = {}_{\text{s}}\langle\alpha(t)|f_{\text{s}}|\alpha(t)\rangle_{\text{s}}$ have to be the same in either picture since it is an observable.

In the Interaction picture, we will evolve operators using the free part of the Hamiltonian H_0: $f_{\text{I}}(t) = e^{iH_0(t-t_0)}f(t_0)e^{-iH_0(t-t_0)}$. How must the states $|\alpha(t)\rangle_{\text{I}}$ evolve in this picture ? We know that the expectation values must agree with those in the Schrödinger pictures. Hence,

$$\begin{aligned}
{}_{\text{I}}\langle\alpha(t)|f_{\text{I}}(t)|\alpha(t)\rangle_{\text{I}} &= {}_{\text{s}}\langle\alpha(t)|f_{\text{s}}|\alpha(t)\rangle_{\text{s}}\,,\\
{}_{\text{I}}\langle\alpha(t)|e^{iH_0(t-t_0)}f_{\text{s}}e^{-iH_0(t-t_0)}|\alpha(t)\rangle_{\text{I}} &= {}_{\text{I}}\langle\alpha(t_0)|e^{iH(t-t_0)}f_{\text{s}}e^{-iH(t-t_0)}|\alpha(t_0)\rangle_{\text{I}}\,.
\end{aligned} \tag{4.2.2}$$

where we used $f_{\text{I}}(t_0) = f_{\text{s}}$ on the left-hand-side and and $|\alpha(t_0)\rangle_{\text{s}} = |\alpha(t_0)\rangle_{\text{I}}$ on the right-hand-side. Comparing the two sides, we have

$$|\alpha(t)\rangle_{\text{I}} = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}|\alpha(t_0)\rangle_{\text{I}} \equiv U(t,t_0)|\alpha(t_0)\rangle_{\text{I}} \tag{4.2.3}$$

where we have defined the time evolution operator

$$U(t,t_0) \equiv e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\,, \tag{4.2.4}$$

for evolving states in the Interaction picture. Note that $e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \neq e^{-i(H-H_0)(t-t_0)}$ since the operators do not necessarily commute. Let us understand how the time evolution operator itself evolves with time:

$$\begin{aligned}
\frac{d}{dt}U(t,t_0) &= i\left[e^{iH_0(t-t_0)}H_0e^{-iH(t-t_0)} - e^{iH_0(t-t_0)}He^{-iH(t-t_0)}\right],\\
&= i\left[e^{iH_0(t-t_0)}H_0\underbrace{e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}}_{1}e^{-iH(t-t_0)} - e^{iH_0(t-t_0)}H\underbrace{e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}}_{1}e^{-iH(t-t_0)}\right],\\
&= i\left[e^{iH_0(t-t_0)}H_0e^{-iH_0(t-t_0)}\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{U} - e^{iH_0(t-t_0)}He^{-iH_0(t-t_0)}\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{U}\right],\\
&= i\left[e^{iH_0(t-t_0)}\left(H_0-H\right)e^{-iH_0(t-t_0)}U(t,t_0)\right],\\
&= -i\left[e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}U(t,t_0)\right],\\
&= -iH_{\text{I}}(t)U(t,t_0)\,,
\end{aligned}$$

where, in the last line we defined the interaction picture version of the operator H_{int}: $H_{\text{I}}(t) = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}$. Note that H_{I} has explicit time dependence now even if H_{int} did not. If H_{I} was a number rather than an

[4] Split in the Schrödinger picture.

operator, the $(d/dt)U = -iH_{\rm I}U$ has a formal solution $U(t,t_0) = e^{-i\int_{t_0}^t d\tau H_{\rm I}(\tau)}$. However, since $H_{\rm I}$ is an operator, and it does not commute with itself at different times, we need to do a bit more work.

It is possible to write down a compact, formal solution for the time-evolution operator:

$$U(t,t_0) = T\left\{\exp\left(-i\int_{t_0}^t H_{\rm I}(\tau)d\tau\right)\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}(-i)^n\int_{t_0}^t d\tau_1\ldots\int_{t_0}^t d\tau_n T\left\{H_{\rm I}(\tau_1)H_{\rm I}(\tau_2)\ldots H_{\rm I}(\tau_n)\right\}\,. \tag{4.2.5}$$

This is the *Dyson*-series expansion. The symbol $T\{\ldots\}$ is for *time ordering*; it means

$$T\left\{H_{\rm I}(\tau_1)H_{\rm I}(\tau_2)\right\} = \begin{cases} H_{\rm I}(\tau_1)H_{\rm I}(\tau_2) & \tau_1 < \tau_2\,, \\ H_{\rm I}(\tau_2)H_{\rm I}(\tau_1) & \tau_2 < \tau_1\,. \end{cases} \tag{4.2.6}$$

Let us verify that U above satisfies $dU/dt = -iH_{\rm I}U$. To this end, let us write out the first few terms of the expansion for U:

$$\begin{aligned} U(t,t_0) &= 1 + (-i)\int_{t_0}^t d\tau_1 H_{\rm I}(\tau_1) + \frac{1}{2!}(-i)^2\int_{t_0}^t d\tau_1\int_{t_0}^t d\tau_2 T\{H_{\rm I}(\tau_1)H_{\rm I}(\tau_2)\} + \ldots\,, \\ &= 1 + (-i)\int_{t_0}^t d\tau_1 H_{\rm I}(\tau_1) + (-i)^2\int_{t_0}^t d\tau_2 H_{\rm I}(\tau_2)\int_{t_0}^{\tau_2} d\tau_1 H_{\rm I}(\tau_1) + \ldots\,, \end{aligned} \tag{4.2.7}$$

Let us understand the changes from the first to second line for the third term on the right-hand side. Using the definition of the time-ordering symbol:

$$\frac{1}{2!}\int_{t_0}^t d\tau_1\int_{t_0}^t d\tau_2 T\{H_{\rm I}(\tau_1)H_{\rm I}(\tau_2)\} = \frac{1}{2!}\left(\int_{t_0}^t d\tau_1\int_{t_0}^{\tau_1} d\tau_2 H_{\rm I}(\tau_1)H_{\rm I}(\tau_2) + \int_{t_0}^t d\tau_2\int_{t_0}^{\tau_2} d\tau_1 H_{\rm I}(\tau_2)H_{\rm I}(\tau_1)\right)\,, \tag{4.2.8}$$

where you should pay attention to the change in limit of integration based on whether $\tau_1 < \tau_2$ or not. We can now interchange the dummy integration variables in the first term on the right hand side to yield $2\times\int d\tau_2\ldots$. This 2 precisely cancels the 2! in front of the brackets. This behavior carries over to the higher order terms: the $n!$ gets cancelled each time when we open up the time ordering symbol and change the limits of integration.

Now, let us differentiate both side of eq. (4.2.7):

$$\begin{aligned} \frac{d}{dt}U(t,t_0) &= 0 + (-i)\,H_{\rm I}(t) + (-i)^2\,H_{\rm I}(t)\int_{t_0}^t d\tau_1 H_{\rm I}(\tau_1) + \ldots \\ &= (-i)\,H_{\rm I}(t)\left[1 + (-i)\int_{t_0}^t d\tau_1 H_{\rm I}(\tau_1) + \ldots\right]\,, \\ &= -iH_{\rm I}(t)U(t,t_0)\,, \end{aligned} \tag{4.2.9}$$

which is what we needed to show. We only considered the first couple of terms, you might want to convince yourself of the general result using induction.

Let us generalize the definition of $U(t,t_0)$ from our particular time t_0 (where the different pictures agreed) to an arbitrary time $t' \le t$ as

$$U(t,t') \equiv T\left\{\exp\left(-i\int_{t'}^t H_{\rm I}(\tau)d\tau\right)\right\}\,.$$

As you can check, this general $U(t,t')$ satisfies $\partial_t U(t,t') = -iH_I(t)U(t,t')$. Moreover, you can prove based on the integral expression above that $U(t,t') = U(t,t'')U(t'',t')$ where $t \ge t'' \ge t'$. In particular, $U(t,t_0) = U(t,t')U(t',t_0)$ for $t \ge t' \ge t_0$. This immediately yields

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}\,.$$

which is manifestly Unitary. This expression of course agrees with $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ when $t' \to t_0$. On the other hand, note that we have $U(t_0,t) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}$ which follows from defining $U(t_0,t)$ as the time evolution operator evolving states from t to t_0 and using the fact that $U(t,t_0)$ is unitary.

It is worthwhile knowing the following identities for the time-evolution operator

1. $U(t,t) = 1$.
2. $U^\dagger(t,t')U(t,t') = U(t,t')U^\dagger(t,t') = 1$. This is the property of *Unitarity.*
3. $U(t,t') = U(t,t'')U(t'',t')$ with $t \geq t'' \geq t'$.
4. $U(t,t') = U^{-1}(t',t)$.

Exercise 4.2.1 : Verify the above identities of the time evolution operator.

4.2.1 The S-matrix

We will often be interested in evolving a state from far back in time $t' = -\infty$ to the far future $t = \infty$. This motivates the definition of the S-*matrix*:

$$\begin{aligned} S &\equiv U(\infty,-\infty)\,, \\ &= T\left\{\exp\left[-i\int_{-\infty}^{\infty} d\tau H_\mathrm{I}(\tau)\right]\right\} = T\left\{\exp\left[-i\int_{-\infty}^{\infty} d^4x\mathcal{H}_\mathrm{I}\right]\right\} = T\left\{\exp\left[i\int_{-\infty}^{\infty} d^4x\mathcal{L}_\mathrm{I}\right]\right\}\,. \end{aligned} \tag{4.2.10}$$

where, for example, $\mathcal{L}_\mathrm{I} = -(\lambda/4!)\varphi_\mathrm{I}^4$, with $\varphi_\mathrm{I}(t,\mathbf{x}) = e^{iH_0(t-t_0)}\varphi(t_0,\mathbf{x})e^{-iH_0(t-t_0)}$. Note the following properties of the S-matrix:

1. S is Lorentz invariant.
2. S is Unitary.
3. $S = 1 - i\bar{\delta}^{(4)}(p_i - p_f)\mathcal{M}$ where p_i and p_f are the 4-momenta of initial and final state respectively, 1 indicates "nothing-happened" between the initial and final states, and $\bar{\delta}^{(4)}(p_i - p_f)$ guarantees energy-momentum conservation.

Exercise 4.2.2 : Verify that S is Lorentz invariant. Assume that the interaction Hamiltonian density is constructed out of some polynomial in the fields. Hint: This needs no calculation. One of the main things you need to argue is why time ordering does not spoil Lorentz invariance.

Why use the Interaction Picture?

First, note that S matrix is written in terms of the interaction part of the interaction term only. If the interaction term is controlled by a small coupling constant, we can expand S in that coupling constant systematically (this is essentially what the Dyson expansion does).

Second, the φ_I appearing S are solutions to the free-field Hamiltonian H_0. To see this, note that at some fixed time t_0, we can always expand an arbitrary field in terms of our creation and annihilation operators associated with the free field

$$\varphi(t_0,\mathbf{x}) = \int (dk)\left(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{4.2.11}$$

The time evolution of the field in the interaction picture

$$\varphi_{\rm I}(t,\mathbf{x}) = e^{iH_0(t-t_0)}\varphi(t_0,\mathbf{x})e^{-iH_0(t-t_0)} = \int (dk)\left(a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right) \tag{4.2.12}$$

which is just the free field! Note that we used the fact that the time evolution of the creation and annihilation operators for the free theory: $e^{iH_0(t-t_0)}a(\mathbf{k})e^{-iH_0(t-t_0)} = a(\mathbf{k})e^{-i\omega_{\mathbf{k}}t}$. What this means is that the S-matrix (or more generally, U) is made up of free fields, in particular, it is just a string of free-field creation and annihilation operators. We will drop the "I" from the fields, and it is to be understood that we are talking about interaction picture fields.

What about single or multi-particle states? Can we produce them by acting with our free-field creation operators on the free field vacuum? The answer, in general, is no. Interactions do not "turn off" in the asymptotic past/future and the vacuum of the interacting theory is not the same as that of the free theory. However, we will trick ourselves for the moment into thinking that the answer is "yes". We will eventually pay a price for this trickery and have to come back to address it.

4.2.2 Normal Ordering and Wick's theorem

In the asymptotic past and future, let us imagine that the initial and final states are given by, for example, two particle states $|i\rangle_{\rm free} = a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle = |\mathbf{k}_1,\mathbf{k}_2\rangle$ and $|f\rangle_{\rm free} = |\mathbf{k}_3,\mathbf{k}_4\rangle$. Then the matrix element:

$$\begin{aligned} S_{if} &= {}_{\rm free}\langle f|U(\infty,-\infty)|i\rangle_{\rm free} \\ &= \langle 0|a(\mathbf{k}_3)a(\mathbf{k}_4)T\{1 - i\int d^4x\mathcal{H}_{\rm I} + \ldots\}\}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle \\ &= \langle 0|T\{a(\mathbf{k}_3)a(\mathbf{k}_4)\{1 - i\int d^4x\mathcal{H}_{\rm I} + \ldots\}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\}|0\rangle \end{aligned} \tag{4.2.13}$$

where in the second line, $T\{\ldots\}$ is also made up of a string of creation and annihilation operators of the free field. We enveloped the initial and final state creation and annihilation operators inside the $T\{\ldots\}$ as well, since they are in the asymptotic past and future, hence already time ordered. In this way, the matrix elements are completely calculated by creation and annihilation operators acting on the free-field vacuum.

We know that annihilation operators annihilate the vacuum to the right, whereas creation operators annihilate the vacuum to the left. Wouldn't it be nice if somehow we could move all the creation operators to the left and all the annihilation operators to the right, while picking up delta functions from the commutation relations for the creation and annihilation operators. The formal procedure for doing this is through *Wick's Theorem*. Before getting to Wick's theorem, let us start with some preliminaries.

Normal Ordering

A *Normal Ordered* of a product operators in $O_1O_2\ldots O_n$ (each operator is constructed from strings of creation and annihilation operators), denoted by the same operators between two colons $:\ \ldots\ :$, is such that $:O_1O_2\ldots O_n:$ has all the creation operators to the left, and annihilation operators to the right. For example

$$:a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2): \; = a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)\,. \tag{4.2.14}$$

An important property of Normal ordered products is that they have a zero vacuum expectation value:

$$\langle 0|:O_1O_2\ldots O_n:|0\rangle = 0\,. \tag{4.2.15}$$

In the above example, this is manifest:

$$\langle 0|:a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2):|0\rangle \;= \langle 0|a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)|0\rangle = 0\,. \tag{4.2.16}$$

Contractions and Wick's Theorems

Consider a collection of fields $\varphi_{\rm a}(x), \varphi_{\rm b}(y)\ldots$ in the interaction picture (hence free) constructed out of creation and annihilation operators. Recall that

$$\begin{aligned}\varphi_{\rm a}(x) &= \int (dk)\left(a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right)\,, \\ &= \varphi^+_{\rm a}(x) + \varphi^-_{\rm a}(x)\,, \\ &\sim a + a^\dagger\,,\end{aligned} \tag{4.2.17}$$

where in the second line we used the definition of the positive and negative frequency parts of the field (see section 3.2.2) $\varphi^+_{\rm a}(x) = \int (dk) a(\mathbf{k})e^{-ik\cdot x}$ and $\varphi^-_{\rm a}(x) = \int (dk) a^\dagger(\mathbf{k})e^{ik\cdot x}$. The third line is just short hand to remind us of the relevant operator structure. For example $\varphi_{\rm b} \sim b + b^\dagger$ and so on. Keep in mind that $\varphi^+_{\rm b} \sim b$ whereas $\varphi^-_{\rm b} \sim b^\dagger$ and so on for each field. Finally, from the section 3.2.2, recall that $[\varphi^+_{\rm a}(x), \varphi^-_{\rm b}(y)] = \delta_{\rm ab}\int (dk) e^{-ik\cdot(x-y)} = \delta_{\rm ab}\Delta^+(x-y)$.

As a warm-up, let us first consider the the product of two fields:

$$\begin{aligned}\varphi_{\rm a}(x)\varphi_{\rm b}(y) &= \left(\varphi^+_{\rm a}(x) + \varphi^-_{\rm a}(x)\right)\left(\varphi^+_{\rm b}(y) + \varphi^-_{\rm b}(y)\right)\,, \\ &= \underbrace{\varphi^+_{\rm a}(x)\varphi^+_{\rm b}(y) + \varphi^-_{\rm a}(x)\varphi^-_{\rm b}(y) + \varphi^-_{\rm a}(x)\varphi^+_{\rm b}(y)}_{\text{normal ordered}} + \underbrace{\varphi^+_{\rm a}(x)\varphi^-_{\rm b}(y)}_{\text{not normal ordered}}\,, \\ &= \underbrace{:\varphi_{\rm a}(x)\varphi_{\rm b}(y):}_{\text{normal ordered}} + \underbrace{[\varphi^+_{\rm a}(x), \varphi^-_{\rm b}(y)]}_{\text{a c-number}}\,, \\ &= :\varphi_{\rm a}(x)\varphi_{\rm b}(y): + \,\delta_{\rm ab}\Delta^+(x-y)\,.\end{aligned} \tag{4.2.18}$$

We define the *Wick Contraction* of two fields:

$$\wick{\c\varphi_{\rm a}(x)\c\varphi_{\rm b}(y)} \equiv \delta_{\rm ab}\Delta^+(x-y) = [\varphi^+_{\rm a}(x), \varphi^-_{\rm b}(y)]\,. \tag{4.2.19}$$

The Wick contraction of fields satisfies the following useful properties: It is a c-number,

$$\wick{\c\varphi_{\rm a}(\ldots)\c\varphi_{\rm b}} = \wick{\c\varphi_{\rm a}\c\varphi_{\rm b}}(\ldots)\,, \qquad \text{and} \qquad \wick{\c\varphi_{\rm a}(x)\c\varphi_{\rm b}(y)} \neq \wick{\c\varphi_{\rm b}(y)\c\varphi_{\rm a}(x)}\,. \tag{4.2.20}$$

Thus, for two fields we can write an ordinary product in terms of a Normal product and a Wick contraction.

$$\varphi_{\rm a}(x)\varphi_{\rm b}(y) = \,:\varphi_{\rm a}(x)\varphi_{\rm b}(y): + \,\wick{\c\varphi_{\rm a}(x)\c\varphi_{\rm b}(y)}\,. \tag{4.2.21}$$

3-fields: Let us move two a three field example $\varphi_{\rm a}(x)\varphi_{\rm b}(y)\varphi_{\rm c}(z)$. This will be a bit of work, but once we do it, the pattern will become obvious. To derive an expression for the ordinary product of three fields in terms of normal ordered product and Wick contractions, start with the two field result.

$$\varphi_{\rm a}\varphi_{\rm b} = \,:\varphi_{\rm a}\varphi_{\rm b}: + \,\wick{\c\varphi_{\rm a}\c\varphi_{\rm b}}\,. \tag{4.2.22}$$

Multiplying *both* sides of eq. (4.2.22) by $\varphi^+_{\rm c}$ on the right, we get

$$\varphi_{\rm a}\varphi_{\rm b}\varphi^+_{\rm c} = \,:\varphi_{\rm a}\varphi_{\rm b}:\varphi^+_{\rm c} + \,\wick{\c\varphi_{\rm a}\c\varphi_{\rm b}}\varphi^+_{\rm c}\,. \tag{4.2.23}$$

Now multiply both sides of eq. (4.2.22) by $\varphi^-_{\rm c}$ on the left, to get

$$\varphi^-_{\rm c}\varphi_{\rm a}\varphi_{\rm b} = \,\varphi^-_{\rm c}:\varphi_{\rm a}\varphi_{\rm b}: + \,\varphi^-_{\rm c}\wick{\c\varphi_{\rm a}\c\varphi_{\rm b}}\,. \tag{4.2.24}$$

Now, rewrite the left hand side of eq. (4.2.24) $\varphi^-_{\rm c}\varphi_{\rm a}\varphi_{\rm b}$, by commuting $\varphi^-_{\rm c}$ past $\varphi_{\rm a}$ to get

$$\varphi^-_{\rm c}\varphi_{\rm a}\varphi_{\rm b} = \varphi_{\rm a}\varphi^-_{\rm c}\varphi_{\rm b} + \,[\varphi^-_{\rm c}, \varphi_{\rm a}]\varphi_{\rm b} = \varphi_{\rm a}\varphi^-_{\rm c}\varphi_{\rm b} - \,\wick{\c\varphi_{\rm a}\c\varphi_{\rm c}}\,. \tag{4.2.25}$$

In the last equality we used $[\varphi_c^-, \varphi_a] = [\varphi_c^-, \varphi_a^+] = -[\varphi_a^+, \varphi_c^-] = -\bcontraction{}{\varphi_a}{}{\varphi_c}\varphi_a\varphi_c$ Now, commute φ_c^- past φ_b on the rightmost expression in eq. (4.2.25), to get

$$\varphi_c^- \varphi_a \varphi_b = \varphi_a \varphi_b \varphi_c^- - \varphi_a \bcontraction{}{\varphi_b}{}{\varphi_c}\varphi_b\varphi_c - \bcontraction{}{\varphi_a}{}{\varphi_c}\varphi_a\varphi_c \varphi_b \, . \tag{4.2.26}$$

Substituting eq. (4.2.26) in eq. (4.2.24) and then adding the resulting equation to eq. (4.2.23), we get

$$\underbrace{\varphi_a \varphi_b \varphi_c^+ + \varphi_a \varphi_b \varphi_c^-}_{\varphi_a\varphi_b\varphi_c} - \varphi_a \bcontraction{}{\varphi_b}{}{\varphi_c}\varphi_b\varphi_c - \bcontraction{}{\varphi_a}{}{\varphi_c}\varphi_a\varphi_c\varphi_b = \, : \varphi_a \varphi_b : \varphi_c^+ + \bcontraction{}{\varphi_a}{}{\varphi_b}\varphi_a\varphi_b \varphi_c^+ + \varphi_c^- : \varphi_a \varphi_b : + \varphi_c^- \bcontraction{}{\varphi_a}{}{\varphi_b}\varphi_a\varphi_b \, . \tag{4.2.27}$$

On the right hand side, combine terms the two terms with normal ordering, and the two terms with Wick contractions separately, to yield

$$\varphi_a \varphi_b \varphi_c - \varphi_a \bcontraction{}{\varphi_b}{}{\varphi_c}\varphi_b\varphi_c - \bcontraction{}{\varphi_a}{}{\varphi_c}\varphi_a\varphi_c\varphi_b = \, : \varphi_a \varphi_b \varphi_c : + \bcontraction{}{\varphi_a}{}{\varphi_b}\varphi_a\varphi_b \varphi_c \, , \tag{4.2.28}$$

which upon re-arranging yield the desired result:

$$\varphi_a \varphi_b \varphi_c = \, : \varphi_a \varphi_b \varphi_c : + \bcontraction{}{\varphi_a}{}{\varphi_b}\varphi_a\varphi_b \varphi_c + \bcontraction{}{\varphi_a}{\varphi_b}{\varphi_c}\varphi_a\varphi_b\varphi_c + \varphi_a \bcontraction{}{\varphi_b}{}{\varphi_c}\varphi_b\varphi_c \, . \tag{4.2.29}$$

Exercise 4.2.3 : Derive the following result for four fields starting with the three field result.

$$\begin{aligned} \varphi_a \varphi_b \varphi_c \varphi_d = \, & : \varphi_a \varphi_b \varphi_c \varphi_d : \\ & + : \bcontraction{}{\varphi_a}{}{\varphi_b}\varphi_a\varphi_b\varphi_c\varphi_d : + : \bcontraction{}{\varphi_a}{\varphi_b}{\varphi_c}\varphi_a\varphi_b\varphi_c\varphi_d : + : \bcontraction{}{\varphi_a}{\varphi_b\varphi_c}{\varphi_d}\varphi_a\varphi_b\varphi_c\varphi_d : + : \bcontraction{\varphi_a}{\varphi_b}{}{\varphi_c}\varphi_a\varphi_b\varphi_c\varphi_d : + : \bcontraction{\varphi_a}{\varphi_b}{\varphi_c}{\varphi_d}\varphi_a\varphi_b\varphi_c\varphi_d : + : \bcontraction{\varphi_a\varphi_b}{\varphi_c}{}{\varphi_d}\varphi_a\varphi_b\varphi_c\varphi_d : \\ & + \bcontraction{}{\varphi_a}{}{\varphi_b}\bcontraction{\varphi_a\varphi_b}{\varphi_c}{}{\varphi_d}\varphi_a\varphi_b\varphi_c\varphi_d + \bcontraction{}{\varphi_a}{\varphi_b}{\varphi_c}\bcontraction[2ex]{\varphi_a}{\varphi_b}{\varphi_c}{\varphi_d}\varphi_a\varphi_b\varphi_c\varphi_d + \bcontraction[2ex]{}{\varphi_a}{\varphi_b\varphi_c}{\varphi_d}\bcontraction{\varphi_a}{\varphi_b}{}{\varphi_c}\varphi_a\varphi_b\varphi_c\varphi_d \end{aligned}$$

There is a pattern here. Wick's Theorem is just the generalized version of the above examples, and can be proved by induction.

Wick's Theorem for Ordinary Products

$$\begin{aligned} \varphi_a \varphi_b \ldots \varphi_z = \, & : \varphi_a \varphi_b \varphi_c \ldots \varphi_z : \\ & + \sum_{\text{single Wick cont.}} : \bcontraction{}{\varphi_a}{}{\varphi_b}\varphi_a\varphi_b \ldots \varphi_z : + \ldots \\ & + \sum_{\text{double Wick cont.}} : \bcontraction{}{\varphi_a}{}{\varphi_b}\bcontraction{\varphi_a\varphi_b}{\varphi_c}{\ldots}{\varphi_z}\varphi_a\varphi_b\varphi_c \ldots \varphi_z : + \ldots \\ & + \sum_{\text{triple Wick cont.}} \ldots \\ & \qquad \vdots \end{aligned} \tag{4.2.30}$$

In words, Wick's theorem for ordinary products states that an ordinary product can be written the sum of all possible pairings $\bcontraction{}{\varphi}{}{\varphi}\varphi\varphi$ within normal products (including no pairings). To preserve your sanity, always write the fields in the same order as the ordinary product.

Why is this going to be useful? Note that since the normal ordered product always has annihilation operators on the right and creation operators on the left, the vacuum expectation value of all the normal ordered products is zero, unless there are no fields left after the Wick contractions (this means we need an even number of fields). For the example with three fields, the vacuum expectation value $\langle 0|\varphi_a \varphi_b \varphi_c|0\rangle = 0$, whereas for the two field and four field examples, the fully contracted expressions are the only terms that contribute to the vev (vacuum expectation value). Explicitly,

$$\langle 0|\varphi_{\rm a}\varphi_{\rm b}|0\rangle = \wick{\c1{\varphi}_{\rm a}\c1{\varphi}_{\rm b}}\,, \qquad \langle 0|\varphi_{\rm a}\varphi_{\rm b}\varphi_{\rm c}|0\rangle = 0\,, \qquad \text{and} \qquad \langle 0|\varphi_{\rm a}\varphi_{\rm b}\varphi_{\rm c}\varphi_{\rm d}|0\rangle = \wick{\c1{\varphi}_{\rm a}\c1{\varphi}_{\rm b}\c2{\varphi}_{\rm c}\c2{\varphi}_{\rm d}} + \wick{\c1{\varphi}_{\rm a}\c2{\varphi}_{\rm b}\c1{\varphi}_{\rm c}\c2{\varphi}_{\rm d}} + \wick{\c1{\varphi}_{\rm a}\c2{\varphi}_{\rm b}\c2{\varphi}_{\rm c}\c1{\varphi}_{\rm d}}\,,$$

and so on. This is nice, but remember that we actually needed vev's for time ordered products of fields. So, we need a Wick's theorem for time ordered products.

Wick's Theorem for Time-Ordered Products

In words, Wick's theorem for time-ordered products states that a time-ordered product can be written the sum of all possible pairings $\wick{\c1{\varphi}\c1{\varphi}}$ within normal products (including no pairings). Explicitly

$$\begin{aligned} T\{\varphi_{\rm a}\varphi_{\rm b}\dots\varphi_{\rm z}\} = &\ :\varphi_{\rm a}\varphi_{\rm b}\varphi_{\rm c}\dots\varphi_{\rm z}: \\ &+ \sum_{\text{single Feynman cont.}} :\wick{\c1{\varphi}_{\rm a}\c1{\varphi}_{\rm b}}\dots\varphi_{\rm z}: + \dots \\ &+ \sum_{\text{double Feynman cont.}} :\wick{\c1{\varphi}_{\rm a}\c1{\varphi}_{\rm b}\c2{\varphi}_{\rm c}\dots\c2{\varphi}_{\rm z}}: + \dots \\ &+ \sum_{\text{triple Feynman cont.}} \dots \\ &\ \ \vdots \end{aligned} \tag{4.2.31}$$

where the *Feynman* contraction is defined in terms of the Wick contraction as follows:

$$\wick{\c1{\varphi}_{\rm a}(x)\c1{\varphi}_{\rm b}}(y) = \begin{cases} \wick{\c1{\varphi}_{\rm a}(x)\c1{\varphi}_{\rm b}}(y) & x^0 > y^0\,, \\ \wick{\c1{\varphi}_{\rm b}(y)\c1{\varphi}_{\rm b}}(x) & y^0 > x^0\,. \end{cases} \tag{4.2.32}$$

This definition follows immediately from writing down $T\{\varphi_{\rm a}(x), \varphi_{\rm b}(y)\}$ in terms of Normal ordered products and Wick contractions for the $x^0 > y^0$ and the $x^0 < y^0$ cases separately. Unlike the Wick contraction case where $\wick{\c1{\varphi}_{\rm a}(x)\c1{\varphi}_{\rm b}}(y) \neq \wick{\c1{\varphi}_{\rm b}(y)\c1{\varphi}_{\rm a}}(x)$, for the Feynman contraction, $\wick{\c1{\varphi}_{\rm a}(x)\c1{\varphi}_{\rm b}}(y) = \wick{\c1{\varphi}_{\rm b}(y)\c1{\varphi}_{\rm a}}(x)$.

4.2.3 Feynman Propagator

Note that $\wick{\c1{\varphi}_{\rm a}(x)\c1{\varphi}_{\rm b}}(y) = \delta_{\rm ab}\Delta^+(x-y)$ and $\wick{\c1{\varphi}_{\rm b}(y)\c1{\varphi}_{\rm b}}(x) = \delta_{\rm ab}\Delta^+(y-x) = \delta_{\rm ab}\Delta^-(x-y)$ where $\Delta^\pm(x) = \int(dk)e^{\mp ik\cdot x}$. With this in mind, let us consider the vev of the time-ordered product of two identical fields (drop the a and b labels):

$$\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = \wick{\c1{\varphi}(x)\c1{\varphi}}(y) = \theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^-(x-y)\,, \tag{4.2.33}$$

where we have combined the two "branches" of the Feynman contraction using Heavyside functions, and used the fact that vevs of normal ordered products are zero. The *Feynman Propagator* is defined in terms of this time ordered product

$$i\Delta_F(x-y) \equiv \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = \theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^-(x-y)\,. \tag{4.2.34}$$

The Feynman propagator is a Green's function of the Klein-Gordon equation, that is

$$(\partial^2 + m^2)\Delta_F(x-y) = -\delta^{(4)}(x-y)\,. \tag{4.2.35}$$

where $\partial^2 = \partial_\mu\partial^\mu$, with $\partial_\mu = \partial/\partial x^\mu$ and $\delta(x-y)$ is a four dimensional Dirac-delta function. [5]. Note that $i\Delta_F(x-y) = i\Delta_F(y-x)$.

[5] Recall that in section 3.2.3, we had come across the Retarded Green's function $i\Delta_R(x-y) = \theta(x^0-y^0)\langle 0|[\varphi(x),\varphi(y)]|0\rangle$ which also satisfied the above equation. Think about what the difference between Δ_F and Δ_R is

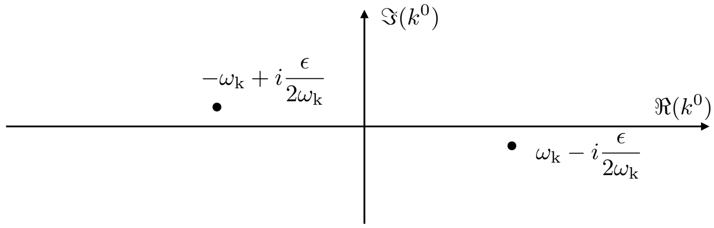


Figure 4.1

Feynman Propagator in Momentum Space

The Feynman Propagator will play an essential role in our calculations of scattering/decay processes. The 4-dimensional Fourier transform of the Feynman propagator $i\Delta_F(k)$ is simpler to deal with than the $i\Delta_F(x)$. Let us calculate an explicit form for $i\Delta_F(k)$.

Let us remind ourselves of the Fourier transform definitions, and the definition of $\Delta_F(x)$:

$$\begin{aligned} &\Delta_F(x) = \int d^4\bar{k} e^{-ik\cdot x}\Delta_F(k) \qquad \text{and} \qquad \Delta_F(k) = \int d^4x e^{ik\cdot x}\Delta_F(x)\,, \\ &\Delta_F(x) = -i\left\{\theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^-(x)\right\}\,, \end{aligned} \tag{4.2.36}$$

where recall that $d^n\bar{k} = d^nk/(2\pi)^n$. My claim is that

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon} \qquad\qquad \epsilon \to 0^+\,. \tag{4.2.37}$$

To prove that this is the correct result, we will check that its Fourier transform yields $\Delta_F(x)$. To this end

$$\begin{aligned} \int d^4\bar{k}\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} &= \int d^3\bar{k} e^{i\mathbf{k}\cdot\mathbf{x}} \int d\bar{k}^0 \frac{e^{-ik^0x^0}}{(k^0)^2 - \omega_{\mathrm{k}}^2 + i\epsilon}\,, \\ &= \int d^3\bar{k} e^{-ik\cdot x} \int d\bar{k}^0 \frac{e^{-ik^0x^0}}{\left[k^0 - \left(\omega_{\mathrm{k}} - i\frac{\epsilon}{2\omega_{\mathrm{k}}}\right)\right]\left[k^0 + \left(\omega_{\mathrm{k}} - i\frac{\epsilon}{2\omega_{\mathrm{k}}}\right)\right]}\,, \end{aligned} \tag{4.2.38}$$

where we have used $\epsilon \to 0^+$ while factoring the denominator. The integral over k^0 has poles at $k^0 = \pm\left(\omega_{\mathrm{k}} - i\frac{\epsilon}{2\omega_{\mathrm{k}}}\right)$ in the k^0-complex plane. See Fig. 4.1

If $x^0 > 0$, then we can close the contour in the lower half of the complex plane (since $e^{-ik^0x^0} \to 0$ as the radius of the contour goes to infinity). The only pole within this closed contour is $k^0 = \omega_{\mathrm{k}} - i\epsilon/2\omega_k$. Using the Residue theorem,

$$\begin{aligned} \int d^4\bar{k}\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} &= \int d^3\bar{k} e^{i\mathbf{k}\cdot\mathbf{x}}\frac{1}{2\pi}(-2\pi i) \times \left(\text{ Res at } k^0 = \omega_k\right)\,, \\ &= \int d^3\bar{k} e^{i\mathbf{k}\cdot\mathbf{x}}\frac{-i}{2\omega_{\mathrm{k}}}e^{-i\omega_{\mathrm{k}}x^0}\,, \\ &= -i\int (dk)e^{-ik\cdot x}\,, \\ &= -i\Delta^+(x)\,. \end{aligned} \tag{4.2.39}$$

The minus sign in the application of the Residue theorem arose because of the counter-clockwise contour. The extra 2π in the denominator came from $d\bar{k}^0 = dk^0/2\pi$.

For the case of $x^0 < 0$, we have to close the contour in the upper half plane, and we pick up the pole at $k^0 = -(\omega_k - i\epsilon/2\omega_k)$. This yields

$$\int d^4\bar{k}\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} = -i\Delta^-(x)\,. \tag{4.2.40}$$

Putting the $x^0 > 0$ and $x^0 < 0$ results together, we have

$$\int d^4\!\!\!\bar{}k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} = -i\{\theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^-(x)\} = \Delta_F(x)\,. \tag{4.2.41}$$

This completes our proof. Note that the point of the $i\epsilon$ was to guide us in the choice of poles to yield the correct Fourier transform of the Feynman propagator.

Exercise 4.2.4 Consider a general(G) Green's function of the Klein-Gordon equation: $(\partial^2 + m^2)\Delta_G(x - y) = -\delta^{(4)}(x - y)$. Using the 4-d Fourier transform, show that $\Delta_G(k) = 1/(k^2 - m^2)$. Now, start with $\Delta_G(k) = 1/(k^2 - m^2)$, and try to get an explicit expression for $\Delta_G(x - y)$ using the inverse Fourier transform. You will be faced with a choice on how to evaluate the contour integral. The contour/pole prescription we chose in Fig. 4.1 yields the Feynman Green's function $\Delta_F(x-y)$. What is the contour/pole-prescription that is needed to recover the Retarded Green's function $\Delta_R(x - y)$?

4.3 Perturbative Calculations in A Toy Model

Let us now put all of this technology to work in a toy example:

$$\mathcal{L} = |\partial\psi|^2 - M^2\psi^\dagger\psi + \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - g\varphi\psi^\dagger\psi\,. \tag{4.3.1}$$

where $\mathcal{H}_{\text{int}} = g\varphi\psi^\dagger\psi$, φ is a Hermitian field, and ψ a non-Hermitian one. Recall that for perturbative calculations, we want $g \ll m, M$. For convenience, I am going to write down the mode expansions for these fields in the interaction picture (we have dropped the "I" denoting the interaction picture.)

$$\begin{aligned}
\varphi(x) &= \int (dk)\left(a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right)\,,\\
\psi(x) &= \int (dk)\left(b(\mathbf{k})e^{-ik\cdot x} + d^\dagger(\mathbf{k})e^{ik\cdot x}\right)\,,\\
\psi^\dagger(x) &= \int (dk)\left(d(\mathbf{k})e^{-ik\cdot x} + b^\dagger(\mathbf{k})e^{ik\cdot x}\right)\,.
\end{aligned} \tag{4.3.2}$$

While far from reality, if it helps, you can think of φ particles as toy photons (γ) (they are really "toys", we will even allow them to have mass m), and ψ particles as toy electrons (e^-) and positrons (e^+). Again, I want to stress that this is a toy example. Real world photons are quanta of spin 1, massless gauge fields and electrons/positrons are quanta of spin 1/2 fermionic fields. Nevertheless, the essentials of perturbative calculations will be present in our toy example without distractions (and more constraining structure) from the higher spin fields.

Recall the following shorthand of the mode expansions, and their take-away:

- $\varphi \sim a + a^\dagger$: $\quad a^\dagger$ creates a γ, and a annihilates it.
- $\psi \sim b + d^\dagger$, $\quad \psi^\dagger \sim d + b^\dagger$: $\quad d^\dagger$ creates an e^+, and d annihilates it. Whereas $b^\dagger$ creates a e^-, and b annihilates it.

Let us consider the amplitude for the following processes: (1) $\gamma \to e^- + e^+$ (2) $e^- + e^- \to e^- + e^-$ (3) $e^- + e^+ \to e^- + e^+$ (4) $e^- + \gamma \to e^- + \gamma$.

4.3.1 Decay: $\gamma \to e^- + e^+$

We will go through this calculation of the decay amplitude step by step, in excruciating detail. But having done so once, the rest will hopefully be quicker.

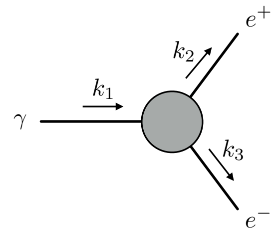


Figure 4.2: Decay of our massive scalar "photon" into a scalar "electron-positron" pair.

Initial and Final States

The first step is to write down the initial and final states.

$$|i\rangle = a^\dagger(\mathbf{k}_1)|0\rangle = |\underbrace{\mathbf{k}_1}_{\gamma}\rangle, \qquad \text{and} \qquad |f\rangle = b^\dagger(\mathbf{k}_3)d^\dagger(\mathbf{k}_2)|0\rangle = |\underbrace{\mathbf{k}_3}_{e^-}\underbrace{\mathbf{k}_2}_{e^+}\rangle\,. \tag{4.3.3}$$

Here we are considering initial and final states as momentum eigenstates (of the free theory). You can superpose a bunch of these to get a wavepacket if you want.

Dyson Expansion

We wish to calculate

$$\begin{aligned}
\langle f|S|i\rangle &= \langle f|T\{\exp\left(-i\int d^4x\mathcal{H}_I(x)\right)\}|i\rangle\,,\\
&= \langle f|T\{1 + (-i)\int d^4x\mathcal{H}_\text{I}(x) + \frac{(-i)^2}{2!}\int d^4xd^4y\mathcal{H}_I(x)\mathcal{H}_I(y) + \ldots\}|i\rangle\,,\\
&= \langle f|1 + (-ig)T\{\int d^4x(\psi^\dagger\psi\varphi)_x + \frac{(-ig)^2}{2!}T\{\int d^4xd^4y(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y\} + \ldots|i\rangle\,.
\end{aligned} \tag{4.3.4}$$

Let us look at this term by term, organized by orders in g.

- 0 order in g: On physical grounds, $\langle f|1|i\rangle$ should be zero because if there are no interactions, how can γ decay into e^- and e^+? For the sake of practice, let us compute this overlap and confirm:

  $$\langle f|1|i\rangle = \langle 0|b(\mathbf{k}_3)d(\mathbf{k}_2)a^\dagger(\mathbf{k}_1)|0\rangle = \langle 0|a^\dagger(\mathbf{k}_1)b(\mathbf{k}_3)d(\mathbf{k}_2)|0\rangle = 0\,. \tag{4.3.5}$$

  In the second equality, since $a^\dagger$ commutes with bd, there is nothing preventing up from moving it through to the left. As a result either b or d can act on the right vacuum to give 0 or we can have $a^\dagger$ acting on the left vacuum state to yield zero.

- 1st order in g. The term we wish to know is

  $$(-ig)\langle f|T\{\int d^4x(\psi^\dagger\psi\varphi)_x\}|i\rangle = (-ig)\int d^4x\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_2)(\psi^\dagger\psi\varphi)_x a^\dagger(\mathbf{k}_1)\}|0\rangle\,. \tag{4.3.6}$$

  Note that the time ordering symbol is innocuous for this term, which is already time ordered. The interaction in the "middle" in terms of time. Moreover the interaction part are all evaluated at the same x.[6]

[6] For next order in g terms, time ordering will play a more significant role.

Contractions

Let us evaluate the following time-ordered vev using Wick's theorem:

$$\begin{aligned}\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1)\}|0\rangle &= b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1) + b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1) \\ &\quad + \text{all possible contractions (with no non-contracted fields)}\,.\end{aligned}$$

where we have used Wick's theorem for time ordered fields, and the fact that vevs of normal ordered fields are 0. The types of contractions we have to deal with include

$$\begin{aligned}\varphi(x)a^\dagger(\mathbf{k}_1) &= \langle 0|T\{\varphi(x)a^\dagger(\mathbf{k}_1)\}|0\rangle\,, \\ &= \langle 0|\varphi(x)a^\dagger(\mathbf{k}_1)|0\rangle, \\ &= \int (dq)e^{-iq\cdot x}\langle 0|a(\mathbf{q})a^\dagger(\mathbf{k}_1)|0\rangle\,, \\ &= \int (dq)e^{-iq\cdot x}\langle 0|a^\dagger(\mathbf{k}_1)a(\mathbf{q}) + [a(\mathbf{q}),a^\dagger(\mathbf{k}_1)]|0\rangle\,, \\ &= e^{-ik_1\cdot x}\,.\end{aligned} \tag{4.3.7}$$

The second line is the innocuousness of time ordering for this term. The third line results from writing down the mode expansion for $\varphi(x)$; $\langle 0|a^\dagger(\mathbf{q}) = 0$ is why $a^\dagger(\mathbf{q})$ does not appear in the expansion. In the fourth line, we commuted a to the right past the $a^\dagger$, paying the price of doing so with the commutator. Finally, in the fifth line, we used the commutation relation $[a(\mathbf{q}),a^\dagger(\mathbf{k}_1)] = 2\omega_{\mathrm{k}_1}\bar{\delta}^{(3)}(\mathbf{q}-\mathbf{k}_1)$ and integrated w.r.t $\int(dq)$. Note that final result is in terms of the initial momentum k_1 of the incoming particle.

Now note that contractions $\langle 0|T\{\varphi_{\mathrm{a}}\varphi_{\mathrm{b}}\}|0\rangle = \varphi_{\mathrm{a}}\varphi_{\mathrm{b}} \propto \delta_{\mathrm{ab}}$. Hence, if a $\neq$ b (i.e different field), we get no contribution. This implies that all contractions : ba, $b\varphi$, $d\varphi$, da, $\varphi\psi$ and $\psi^\dagger\varphi$ vanish. Moreover, since $\psi \sim b + d^\dagger$ and $\psi^\dagger \sim d + b^\dagger$, only $b\psi^\dagger$ and $d\psi$ survive. Thus the only surviving contributions are

$$\begin{aligned}\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1)\}|0\rangle &= b(\mathbf{k}_3)d(\mathbf{k}_2)\psi^\dagger(x)\psi(x)\varphi(x)a^\dagger(\mathbf{k}_1)\,, \\ &= \langle 0|b(\mathbf{k}_3)\psi^\dagger(x)|0\rangle \times \langle 0|d(\mathbf{k}_2)\psi(x)|0\rangle \times \langle 0|\varphi(x)a^\dagger(\mathbf{k}_1)|0\rangle\,, \\ &= e^{ik_3\cdot x}e^{ik_2\cdot x}e^{-ik_1\cdot x}\,.\end{aligned} \tag{4.3.8}$$

Notice the sign difference between the incoming and outgoing momenta: k_2, k_3 are outgoing, whereas k_1 is incoming.

The Matrix Elements

Thus we have arrived at

$$(-ig)\langle f|T\int d^4x(\psi^\dagger\psi\varphi)_x|i\rangle = (-ig)\int d^4x e^{-i(k_3+k_2-k_1)\cdot x} = (-ig)\bar{\delta}^{(4)}(k_1-k_2-k_3)\,. \tag{4.3.9}$$

Let us stop at this first nontrivial result which appears at first order in g (for dragons lurk at higher orders). Combining this result with the 0th order result, we arrive at

$$\langle f|S-1|i\rangle = (-ig)\bar{\delta}^{(4)}(k_1-k_2-k_3) + \mathcal{O}[g^2]\,. \tag{4.3.10}$$

Recall from section 4.2.1 where we discussed properties of the S-matrix, that $S = 1 - i\bar{\delta}^{(4)}(p_i - p_f)\mathcal{M}$ where p_i and p_f are the 4-momenta of the initial and final states. The 4-dimensional delta function which was meant to impose energy-momentum conservation in the process is precisely what we found in our explicit calculation: $\bar{\delta}^{(4)}(k_1-k_2-k_3)$. Thus, at leading order in g, we have

$$\langle f|\mathcal{M}|i\rangle = g + \mathcal{O}[g^2]\,. \tag{4.3.11}$$

An exceptionally simple result! The factors of i and signs were all put in (with hindsight) to make the final results look nice.

4.3.2 Scattering: $e^- + e^- \to e^- + e^-$

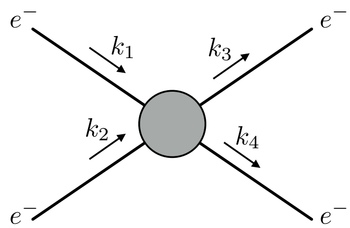


Figure 4.3: Scattering of two "electrons" off of each other.

Let us now calculate the amplitude for the following scattering process at the leading non-trivial order in g. As with our decay calculation, we will proceed systematically, but now without loitering around for all the details.

Initial and Final States

$$|i\rangle = b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)|0\rangle = |\mathbf{k}_1\mathbf{k}_2\rangle \qquad \text{and} \qquad |f\rangle = b^\dagger(\mathbf{k}_3)b^\dagger(\mathbf{k}_4)|0\rangle = |\mathbf{k}_3\mathbf{k}_4\rangle\,, \tag{4.3.12}$$

where we will assume that $\mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{k}_3, \mathbf{k}_4$.

Dyson Expansion

Since $\mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{k}_3, \mathbf{k}_4$, we immediately have $\langle \mathbf{k}_3\mathbf{k}_4|\mathbf{k}_1\mathbf{k}_2\rangle = 0$. Hence we can directly write down the parts of the scattering amplitude with g dependence:

$$\begin{aligned}
&\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle \\
&\qquad = (-ig)\int d^4x \langle 0|T\{b(\mathbf{k}_3)b(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)\}|0\rangle \\
&\qquad + \frac{(-ig)^2}{2!}\int d^4x d^4y \langle 0|T\{b(\mathbf{k}_3)b(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)\}|0\rangle\,, \\
&\qquad + \mathcal{O}[g^3]\,.
\end{aligned} \tag{4.3.13}$$

Contractions

Consider the $\mathcal{O}[g]$ term. It has an odd number of operators. Which means the time ordered vev. of these operators will be zero. We must calculate the $\mathcal{O}[g^2]$ term. Consider $\langle 0|T\{b^\dagger(\mathbf{k}_3)b^\dagger(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b(\mathbf{k}_1)b(\mathbf{k}_2)\}|0\rangle$. The number of possible complete contractions are enormous, but most will have vanishing contributions. What sorts of contractions have non-vanishing contributions?

Note that any complete contraction with a non-vanishing contribution must include $\varphi(x)\varphi(y)$ because none of the other fields (including the initial and final states) contain any part of the φ field (recall that $\varphi_\mathrm{a}\varphi_\mathrm{b} \propto \delta_\mathrm{ab}$). Moreover, since $\psi \sim b + d^\dagger$ and $\psi^\dagger \sim d + b^\dagger$, any b must contract with $\psi^\dagger$ and $b^\dagger$ with ψ. There are four possible ways of doing this:

1.
$$b(\mathbf{k}_3)b(\mathbf{k}_4)\psi^\dagger(x)\psi(x)\varphi(x)\psi^\dagger(y)\psi(y)\varphi(y)b^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)\,. \tag{4.3.14}$$

2. Same as 1., but with $b(\mathbf{k}_3)\psi^\dagger(x) \to b(\mathbf{k}_3)\psi^\dagger(y)$, and $b(\mathbf{k}_4)\psi^\dagger(y) \to b(\mathbf{k}_4)\psi^\dagger(x)$.

3. Same as 1., with $x \leftrightarrow y$.

4. Same as 2., with $x \leftrightarrow y$.

Let us take a closer look at individual two field contractions that appear in the above expressions.

- The $\varphi\varphi$ contraction is nothing but the Feynman propagator:

$$\overbrace{\varphi(x)\varphi}(y) = \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = i\Delta_\varphi(x-y) \qquad \text{Feynman Propagator!} \tag{4.3.15}$$

- The ψ contraction with any (incoming) $b^\dagger$ yields:

$$\overbrace{\psi(x)b}^\dagger(\mathbf{k}_j) = e^{-ik_j\cdot x} \qquad \text{incoming} \tag{4.3.16}$$

- The $\psi^\dagger$ contraction with any (outgoing) b yields:

$$\overbrace{b(\mathbf{k}_j)\psi}^\dagger(x) = e^{ik_j\cdot x} \qquad \text{outgoing} \tag{4.3.17}$$

Matrix Element

Using the above calculated building blocks,

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = \frac{(-ig)^2}{2!}\int d^4xd^4y\, i\Delta_\varphi(x-y)\left(e^{ik_3\cdot x}e^{ik_4\cdot y}e^{-ik_1\cdot x}e^{ik_2\cdot y} + e^{ik_3\cdot y}e^{ik_4\cdot x}e^{-ik_1\cdot x}e^{ik_2\cdot y} + x\leftrightarrow y\right)$$

The $x \leftrightarrow y$ simply doubles the contribution, cancelling the 2! (this kind of stuff happens a lot). We end up with

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4xd^4y\, i\Delta_\varphi(x-y)\left(e^{-i(k_1-k_3)\cdot x}e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x}e^{-i(k_2-k_3)\cdot y}\right). \tag{4.3.18}$$

This is the nontrivial part of the scattering amplitude at order $\mathcal{O}[g^2]$; we will get to higher order contributions later.

I will take this opportunity to introduce one of the more elegant tools of QFT: *Feynman Diagrams*[7] Julian Schwinger, one of the most prominent contributors to the development of QFT said:

"*Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses.*"

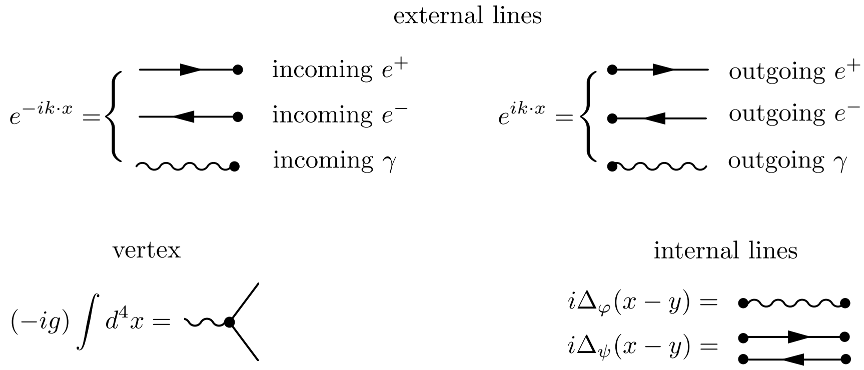
Feynman Rules in x-space

There is powerful graphical way of representing the matrix element in eq. (4.3.18). Let me write down *x-space Feynman Rules* for our theory in Fig. 4.4 Using these rules, we now depict the two terms in eq. (4.3.18) as shown below in Fig. 4.5. Note that there is no accumulation of charge at the vertices. Let us try to (heuristically) say in words what is going on in the process, and connect them to various parts of the mathematical expression for the amplitude.

- "Electrons" with momenta k_1 and k_2 came in and exchanged a "photon", and came out with momenta k_3 and k_4.

- The integrals $\int d^4x$ and $\int d^4y$ sums over all the locations x and y where the "photon" was exchanged.

[7] For a gentle introduction to Feynman diagrams (suitable even if you have never taken field theory course), can be found at http://www.quantumdiaries.org/2010/02/14/lets-draw-feynman-diagams/. If you are interested in the relevance of Feynman diagrams in post WW2 era physics, see the book *Drawing Theories Apart* by D. Kaiser.



Time always flows from left to right. Arrows indicate the flow of positive charge.

Figure 4.4: $x-$space Feynman rules for our theory with $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\varphi$.

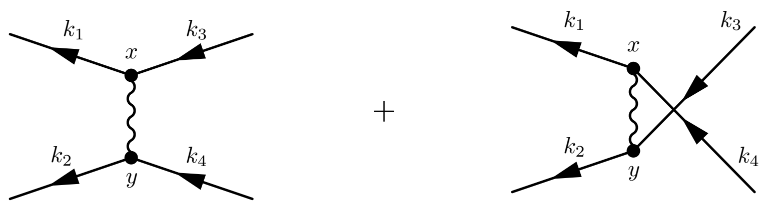


Figure 4.5: The leading order $\mathcal{O}[g^2]$ contribution to the $e^-+e^- \to e^-+e^-$ scattering amplitude: $\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4x d^4y\, i\Delta_\varphi(x-y)\left(e^{-i(k_1-k_3)\cdot x}e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x}e^{-i(k_2-k_3)\cdot y}\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in x-space shown in Fig. 4.4 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the amplitude.

- The propagator $\Delta_\varphi(x-y)$ is symmetric in x and y and should be interpreted at the amplitude for the "photon" going from x to y and y to x. This is internal (the "photon" is virtual), and cannot be observed directly.
- The two diagrams are *topologically distinct*. You cannot twist and bent the lines to get from one to another.

Doing calculations in x-space is fine, and sometimes necessary. Nevertheless, it is in Fourier space that the calculations are the simplest.

Feynman Rules in k-space

We can simplify the expression for the matrix element significantly by writing down the Feynman propagator in Fourier space: $\Delta_\varphi(x-y) = \int d^4\bar{k} e^{-ik\cdot(x-y)}\Delta_\varphi(k)$ to get

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4x d^4y d^4\bar{k}\, i\Delta_\varphi(k)e^{-ik\cdot(x-y)}\left(e^{-i(k_1-k_3)\cdot x}e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x}e^{-i(k_2-k_3)\cdot y}\right) .$$

Now integrate over d^4x and d^4y to get a bunch of delta functions:

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4\bar{k}\, i\Delta_\varphi(k)\{\bar{\delta}^{(4)}(k_1-k_3+k)\bar{\delta}^{(4)}(k_2-k_4-k) + \bar{\delta}^{(4)}(k_1-k_4+k)\bar{\delta}^{(4)}(k_2-k_3-k)\} , \tag{4.3.19}$$

Once again, this expression can be depicted graphically using Feynman diagrams. Since the expressions are in Fourier space, let me now provide the Feynman rules for our theory in k-space. Using these rules,

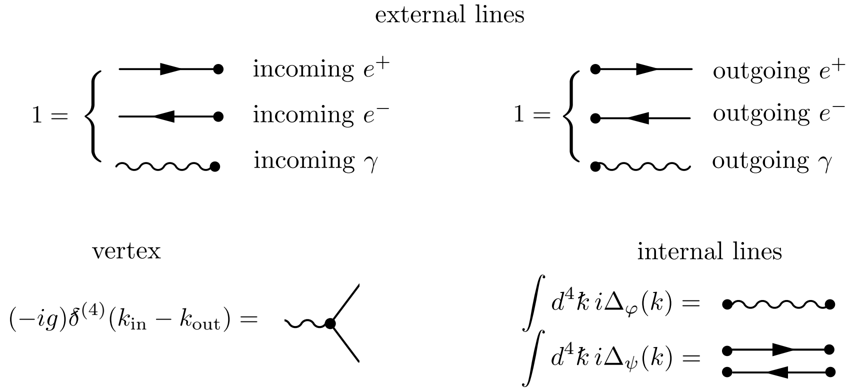


Figure 4.6: $k-$space Feynman Rules for our theory with $\mathcal{L}_{\text{int}} = -g\psi^\dagger\psi\varphi$.

the matrix element in eq. (4.3.19) can be represented again in terms of Feynman diagrams (see Fig. 4.7). Two diagrams corresponds to the two terms in the expression for the matrix element. You can make the correspondence by carefully looking at the delta functions.

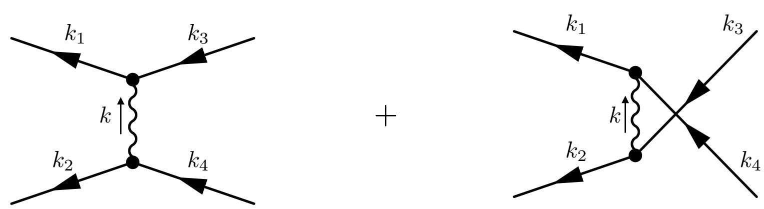


Figure 4.7: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^- \to e^- + e^-$ matrix element: $\langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4x d^4y d^4 d^4\bar{k}\, i\Delta_\varphi(k) e^{-ip\cdot(x-y)} \left(e^{-i(k_1-k_3)\cdot x} e^{-i(k_2-k_4)\cdot y} + e^{-i(k_1-k_4)\cdot x} e^{-i(k_2-k_3)\cdot y}\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in k-space shown in Fig. 4.6 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the matrix element. Note that the choice of direction of the internal momentum k is arbitrary, you just have to be consistent at both vertices once the direction is chosen.

We can even go further here. First note that the products of Dirac-delta functions appearing in eq. (4.3.19) can be combined to yield a momentum conserving delta function that can be moved out of the intergals:

$$\langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\bar{\delta}^4(k_1 + k_2 - k_3 - k_4) \int d^4\bar{k}\, i\Delta_\varphi(k)\{\bar{\delta}^{(4)}(k_1 - k_3 + k) + \bar{\delta}^{(4)}(k_2 - k_3 - k)\}\,, \tag{4.3.20}$$

Now, recall that $-i\bar{\delta}^{(4)}(k_1 + k_2 - k_3 - k_4)\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle = \langle \mathbf{k}_3\mathbf{k}_4|S - 1|\mathbf{k}_1\mathbf{k}_2\rangle$, hence (finally!) the interesting part of our matrix element for this scattering process at order g^2 is

$$-i\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2 \int d^4\bar{k}\, i\Delta_\varphi(k)\{\bar{\delta}^{(4)}(k_1 - k_3 + k) + \bar{\delta}^{(4)}(k_2 - k_3 - k)\}\,. \tag{4.3.21}$$

Furthermore, since we know the form of the momentum-space propagator $i\Delta_\varphi(k) = i/(k^2 - m^2)$. The martrix element then becomes

$$\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle = g^2\left[\frac{1}{(k_1 - k_3)^2 - m^2} + \frac{1}{(k_2 - k_3)^2 - m^2}\right]\,. \tag{4.3.22}$$

We will come back and relate this matrix element to cross section of scattering later. For the moment let us continue calculating matrix elements for different types of scattering.

4.3.3 Scattering: $e^+ + e^- \to e^+ + e^-$

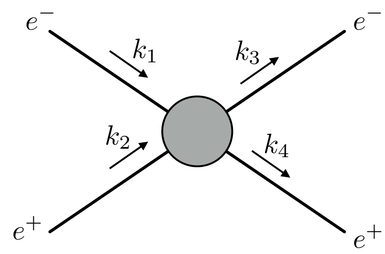


Figure 4.8: "electron"-"positron" scattering.

We will once again carry out the following steps. (1) Write down the initial and final states in terms of creation and annihilation operators of the free fields. (2) Write down the Dyson expansion for the relevant matrix element, and expand to the required non-trivial order in g. (3) Use Wick's theorem; write down the relevant contractions. (4) Write down the expression for the matrix element in position and Fourier space.

Initial and Final States

$$|i\rangle = b^\dagger(\mathbf{k}_1)d^\dagger(\mathbf{k}_2)|0\rangle = |\underbrace{\mathbf{k}_1}_{e^-}\underbrace{\mathbf{k}_2}_{e^+}\rangle \qquad \text{and} \qquad |f\rangle = b^\dagger(\mathbf{k}_3)d^\dagger(\mathbf{k}_4)|0\rangle = |\underbrace{\mathbf{k}_3}_{e^-}\underbrace{\mathbf{k}_4}_{e^+}\rangle\,, \tag{4.3.23}$$

where for simplicity, we will assume $\mathbf{k}_1, \mathbf{k}_2 \neq \mathbf{k}_3, \mathbf{k}_4$.

Dyson Expansion

The part of the scattering amplitude with g dependence (note that the $\mathcal{O}[g]$ term is zero):

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = \frac{(-ig)^2}{2!}\int d^4x d^4y \langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b^\dagger(\mathbf{k}_1)d^\dagger(\mathbf{k}_2)\}|0\rangle \,+\mathcal{O}[g^3]\,. \tag{4.3.24}$$

Contractions

$$\begin{aligned}
&\langle 0|T\{b(\mathbf{k}_3)d(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b^\dagger(\mathbf{k}_1)d^\dagger(\mathbf{k}_2)\}|0\rangle \\
&= b(\mathbf{k}_3)d(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b^\dagger(\mathbf{k}_1)d^\dagger(\mathbf{k}_2) + x \leftrightarrow y \\
&\quad + b(\mathbf{k}_3)d(\mathbf{k}_4)(\psi^\dagger\psi\varphi)_x(\psi^\dagger\psi\varphi)_y b^\dagger(\mathbf{k}_1)d^\dagger(\mathbf{k}_2) + x \leftrightarrow y
\end{aligned} \tag{4.3.25}$$

Note that $\varphi\varphi$ contraction is essential since the $|i\rangle$ and $|f\rangle$ do not contain and $a, a^\dagger$. We cannot have any $\psi^\dagger\psi$ or $\psi\psi$ contractions because the operators in $|i\rangle$ and $|f\rangle$ do not yield any non-zero contractions for $|i\rangle \neq |f\rangle$.[8]

[8] If writing down these contractions gives you a bit of a headaches, be patient for a bit. We will soon get away from its clutches.

Matrix Element and Feynman Diagrams

Evaluating the expressions for the surviving contractions, we have

$$\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4x d^4y \; i\Delta_\varphi(x-y)\left(e^{-i(k_1+k_2)\cdot y}e^{i(k_3+k_4)\cdot x}+e^{-i(k_1-k_3)\cdot y}e^{-i(k_2-k_4)\cdot x}\right) . \tag{4.3.26}$$

Using our Feynman rules, we find that the two terms can be expressed graphically as follows: We can

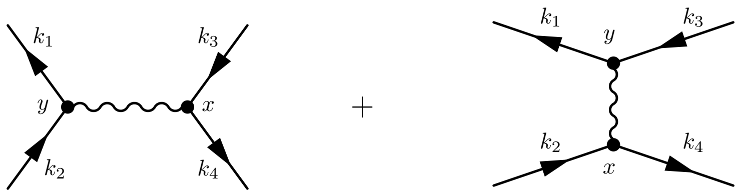


Figure 4.9: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^+ \to e^- + e^+$ scattering amplitude: $\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4xd^4y \; i\Delta_\varphi(x-y)\left(e^{-i(k_1+k_2)\cdot y}e^{i(k_3+k_4)\cdot x}+e^{-i(k_1-k_3)\cdot y}e^{-i(k_2-k_4)\cdot x}\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in x-space shown in Fig. 4.4 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the amplitude.

repeat the calculation in momentum space, to get

$$\begin{aligned}&\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle \\ &= (-ig)^2\int d^4\bar{k} \; i\Delta_\varphi(k)\left(\bar{\delta}^{(4)}(k_1+k_2-k)\bar{\delta}^{(4)}(k-k_3-k_4)+\bar{\delta}^{(4)}(k_1+k-k_3)\bar{\delta}^{(4)}(k_2-k-k_4)\right) .\end{aligned} \tag{4.3.27}$$

Once again, we may represent these terms in terms of Feynman diagrams in momentum space as shown in Fig. 4.10.

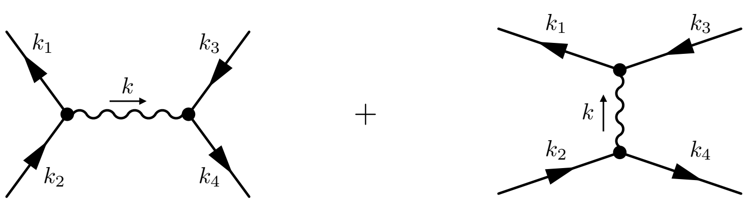


Figure 4.10: The leading order $\mathcal{O}[g^2]$ contribution to the $e^- + e^+ \to e^- + e^+$ matrix element: $\langle \mathbf{k}_3\mathbf{k}_4|S-1|\mathbf{k}_1\mathbf{k}_2\rangle = (-ig)^2\int d^4\bar{k} \; i\Delta_\varphi(k)\left(\bar{\delta}^{(4)}(k_1+k_2-k)\bar{\delta}^{(4)}(k-k_3-k_4)+\bar{\delta}^{(4)}(k_1-k_3-k)\bar{\delta}^{(4)}(k_2-k-k_4)\right)$ represented in terms of Feynman diagrams. Refer to the Feynman rules in k-space shown in Fig. 4.6 to see how the elements of the diagrams correspond to different parts of the mathematical expression for the matrix element. Note that the choice of direction of the internal momentum k is arbitrary.

The expression for the scattering amplitude, can be simplified further. First, note that the products of delta functions yield the usual momentum conserving delta function for the process. Using $S = 1 - i\bar{\delta}^{(4)}(p_i - p_f)\mathcal{M}$, we have

$$\begin{aligned}\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle &= g^2\int d^4\bar{k} \; \Delta_\varphi(k)\left(\bar{\delta}^{(4)}(k-k_1-k_2)+\bar{\delta}^{(4)}(k_1-k_3-k)\right) , \\ &= g^2\left(\frac{1}{(k_1+k_2)^2-m^2}+\frac{1}{(k_1-k_3)^2-m^2}\right) .\end{aligned} \tag{4.3.28}$$

Exercise 4.3.1 : Consider the following scattering process: $\gamma + e^- \to \gamma + e^-$. Following the same route as in the examples above: (1) Write down the initial and final states in terms of creation and annihilation operators of the free fields. (2) Write down the Dyson expansion for the relevant matrix element, and expand to the required non-trivial order in g. (3) Use Wick's theorem; write down the relevant contractions. (4) Write down the expression for the matrix element in position and Fourier space. Draw and label the relevant Feynman diagrams in position and momentum space.

4.3.4 The Diagrammar's way

So far, we have just noted that the final expressions for the scattering amplitude can be nicely represented in terms of Feynman diagrams, but we have really not taken advantage of the graphical representation. Indeed the power of Feynman diagrams lies in going the other way: draw diagrams, and the diagrams tell you how to organize your calculations and compute amplitudes.[9]

Let me carry out this process for the theory under consideration: $\mathcal{L} = |\partial\psi|^2 - M^2\psi^\dagger\psi + (1/2)(\partial\varphi)^2 - (1/2)m^2\varphi^2 - g\varphi\psi^\dagger\psi$. with $\mathcal{L}_{\text{int}} = -g\varphi\psi^\dagger\psi$. It is (typically) easiest to work in momentum space.

1. We will associate squiggly lines with φ and solid lines with $\psi, \psi^\dagger$ (see Fig. 4.6). When they are external, they contribute 1 to the amplitude.

2. Consider the free part of the theory (without $\mathcal{L}_{\text{int}} = 0$). The information about this theory is contained in it's Green's functions, or the transition amplitude. Calculate these, to get:

$$i\Delta_\varphi(k) = i/(k^2 - m^2 + i\epsilon) \qquad \text{and} \qquad i\Delta_\psi(k) = i/(k^2 - M^2 + i\epsilon) \tag{4.3.29}$$

   For internal line of φ contributes $\int d^4\bar{k}\, i\Delta_\varphi(k)$, whereas of $\psi, \psi^\dagger$ contributes $\int d^4\bar{k}\, i\Delta_\psi(k)$.

3. The interaction part of the Lagrangian density gives you the strength of the vertex: the three fields represent the coming together of three different lines $(\varphi, \psi, \psi^\dagger)$. This is your fundamental vertex. This is the only possible way in which different lines in your diagram can meet. Construct the contribution to the

$$\text{3-point vertex} = i\frac{\partial^3\mathcal{L}_{\text{int}}}{\partial\psi\partial\psi^\dagger\partial\varphi} \times \bar{\delta}^{(4)}(\Sigma k) = (-ig)\bar{\delta}^{(4)}(\Sigma k) \tag{4.3.30}$$

   where the sum is over the momenta meeting at the vertex (by convention, incoming are given a positive sign). Each vertex in the diagram will contribute a factor of g.

4. Now consider the process you are interested in. Say, for example $e^-(k_1) + e^+(k_2) \to e^-(k_3) + e^+(k_4)$ with all momenta being distinct. To calculated the amplitude, draw all possible *topologically distinct* diagrams (up to the order of g you care about, which will fix the number of vertices allowed) with the external momenta fixed (endpoints pinned down). If you can get from one diagram to another by twisting, but not cutting lines, then the diagrams are the same. (Note: There are annoying combinatorial factors that come with the diagrams in many cases, but we are safe here for the model under consideration). Now, we will draw all diagrams for the process of interest using the above rules. Up to $\mathcal{O}[g^2]$, we can have the following diagrams shown in Fig. 4.11. Once you have drawn the diagrams, write down the integral expressions corresponding to the diagrams, and you are done calculating the amplitude up to that order in g.

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[9]For a defense of "diagrams first" approach, read the introduction to *Diagrammar* by t'Hooft and Veltman. A recent treatment along these lines is nicely presented in the introduction(s) of *QFT I* notes by Mojzis.

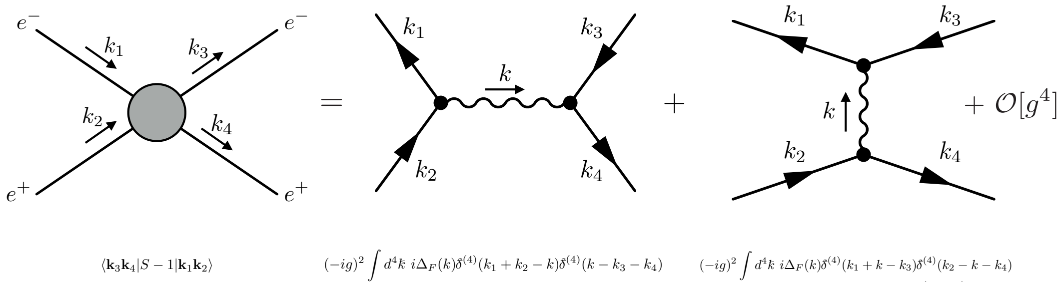


Figure 4.11: Feynman diagrams and corresponding expressions for the scattering amplitude for the $e^-(k_1)+e^+(k_2)\to e^-(k_3)+e^+(k_4)$ process. Note that there are no diagrams you can draw with 1 or 3 vertices.

Exercise 4.3.2 : Draw the equivalent figure(s) to the one in Fig. 4.11, along with the corresponding expressions for the following processes at the leading non-trivial order in g: (1) $\gamma(k_1)\to e^+(k_2)+e^-(k_3)$ (2) $e^-(k_1)+e^-(k_2)\to e^-(k_3)+e^-(k_4)$ (3) $e^-(k_1)+\gamma(k_2)\to e^-(k_3)+\gamma(k_4)$ with none of the (individual) initial and final momenta being equal.

Exercise 4.3.3 : Consider the Lagrangian density $\mathcal{L}=(1/2)(\partial\varphi)^2-(1/2)m^2\varphi^2-(\lambda/4!)\varphi^4$ where φ is a real scalar field. Write the down the Feynman rules for this theory in momentum space, which should include (1) rules for incoming and outgoing external lines, (2) propagator for internal lines and (3) the contribution from the vertex. "Follow your nose" in terms of defining the strength of the vertex by referring to the examples we have already considered. (4) Now draw the Feynman diagrams and write the corresponding expression for a $\varphi(k_1)+\varphi(k_2)\to\varphi(k_3)+\varphi(k_4)$ process. Again assume all momenta are distinct, and only consider the leading order diagrams in λ.

4.3.5 Amplitudes to Observables

While we have been pretending otherwise, amplitudes are not "directly" observable. Let us now compute observables like decay rates, scattering cross sections etc. using our scattering amplitudes. The interesting part of the processes like the strength of interaction, momentum dependence etc. are contained in the amplitude, what remains is to be done in putting in kinematics related to the final states (the "phase space" factors). This can be done in a nice and formal way. We will skip the derivation, and just state the answers here for the processes of interest: two-body decay and two-to-two scattering. [10]

Decay rate

For the decay of a particle with mass m_1 (at rest) into two particles with masses m_2 and m_3, the decay rate

$$\Gamma_{1\to 2+3}=S\frac{|\mathbf{k}|}{8\pi m_1^2}|\langle f|\mathcal{M}|i\rangle|^2\,. \tag{4.3.31}$$

where $\mathbf{k}=(2m_1)^{-1}\sqrt{m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2}$ is the momentum of either outgoing particle. The factor S is related to the number of identical particles in the final state. For our $\varphi\psi^\dagger\psi$

[10] You can find the derivation, for example in Paul Stevenson's notes or in any textbook on QFT or Particle Physics. For a nice introduction, with detailed examples, see chapter 6 in *Introduction to Elementary Particles* by the master of undergraduate textbooks, David J. Griffiths.

theory and the case of $\gamma \to e^+ + e^-$, we have $S = 1$ (if they were identical $S = 1/2!$). The outgoing momenta $|\mathbf{k}| = \sqrt{m^2 - 4M^2}$. Hence

$$\Gamma_{\gamma \to e^+ e^-} = \frac{g^2}{16\pi m}\sqrt{1 - \frac{4M^2}{m^2}} \tag{4.3.32}$$

Note that $m > 2M$ is necessary for decay, as expected (φ clearly is not the electromagnetic field!). In the limit that $M \ll m$, we have $\Gamma \approx g^2/8\pi m$.

Differential cross-section

Now let us consider a $1 + 2 \to 3 + 4$ scattering in the center of momentum frame ($\mathbf{k}_1 = -\mathbf{k}_2, \mathbf{k}_3 = -\mathbf{k}_4$). The differential cross-section (cross-section per unit solid-angle for the outgoing particles),

$$\left(\frac{d\sigma}{d\Omega}\right)_{1+2\to 3+4} = \frac{S}{(8\pi)^2}\frac{|\langle f|\mathcal{M}|i\rangle|^2}{(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})^2}\frac{|\mathbf{k}_f|}{|\mathbf{k}_i|} \tag{4.3.33}$$

where $\omega_{\mathbf{k}_1}$ and $\omega_{\mathbf{k}_2}$ are the energies of the incoming particles, $|\mathbf{k}_f|$ is the magnitude of the 3-momentum of either outgoing particle and $|\mathbf{k}_i|$ is the magnitude of either incoming particle. For our $\varphi\psi^\dagger\psi$ theory, and for the particular case of $e^+ + e^- \to e^+ + e^-$ scattering, we have $|\mathbf{k}_f| = |\mathbf{k}_i|$, and the formula simplifies to

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to e^+e^-} = \frac{1}{64\pi^2}\frac{|\langle f|\mathcal{M}|i\rangle|^2}{E_{\mathrm{cm}}^2} \tag{4.3.34}$$

where we used $(k_1 + k_2)^2 = (\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})^2 = E_{\mathrm{cm}}^2$ which is (the square of) the energy of the system in the center of momentum frame. Recall, that we calculated $\langle f|\mathcal{M}|i\rangle$ for this process in eq. (4.3.28). Thus we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to e^+e^-} = \frac{g^4}{64\pi^2 E_{\mathrm{cm}}^2}\left(\frac{1}{E_{\mathrm{cm}}^2 - m^2} - \frac{1}{2|\mathbf{k}_f|^2(1-\cos\theta) + m^2}\right)^2 . \tag{4.3.35}$$

The angle θ is the angle between the initial and final (back to back) particle trajectories in the center of momentum frame. Notice the first term inside the brackets has a denominator which goes to zero at $E_{\mathrm{cm}} \to m$. Thus the cross section diverges (in reality, it will rise and fall fast) near $E_{\mathrm{cm}} = m$ as we scan through E_{cm}. This sharp rise and fall reveals the presence of a particle of mass m as the means by which the interaction between our e^+ and e^- (mass M each) takes place! Note that if $2M > m$, then there is no "resonance" possible.

As we have seen from the calculation of amplitudes, the Lorentz invariant combinations $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$ and $u = (k_1 - k_4)^2$ appear quite naturally. These are called *Mandelstam* variables. They have nice interpretations. For example, $\sqrt{s}$ is the center of momentum energy and t and u are related to momentum transfer. In terms of the Mandelstam variables, that the amplitude for the $e^+e^- \to e^+e^-$ scattering (see eq. (4.3.28)) can be written as

$$\langle \mathbf{k}_3\mathbf{k}_4|\mathcal{M}|\mathbf{k}_1\mathbf{k}_2\rangle = g^2\left(\frac{1}{s - m^2} + \frac{1}{t - m^2}\right) \tag{4.3.36}$$

The process related to the diagrams contributing these pieces would be called s-channel (first diagram in Fig. 4.10), and t-channel (second diagram in Fig. 4.10) process respectively.

Amplitudes and cross-sections to forces

Consider the $e^-(k_1) + e^+(k_2) \to e^-(k_3) + e^+(k_4)$ process; the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{64\pi^2 s}\left(\frac{1}{s - m^2} + \frac{1}{t - m^2}\right)^2 . \tag{4.3.37}$$

Let us focus on the non-relativistic limit (that is, $|\mathbf{k}_j| \ll M$) and restrict ourselves to the case where $m \ll M$. In this case $k_j \approx (M, \mathbf{k}_j)$. Hence, $s \approx 4M^2$, $t \approx -|\mathbf{k}_1 - \mathbf{k}_3|^2 \equiv -|\mathbf{q}|^2$ where $\mathbf{q}$ is the momentum transfer and

$$\frac{d\sigma}{d\Omega} \approx \frac{g^4}{256\pi^2 M^2}\left(\frac{1}{|\mathbf{q}|^2 + m^2}\right)^2 . \tag{4.3.38}$$

Let us now consider the same problem $e^-(k_1) + e^+(k_2) \to e^-(k_3) + e^+(k_4)$ in non-relativistic quantum mechanics. We imagine that e^+ and e^- interact via a potential $V(\mathbf{x})$ where $\mathbf{x}$ is their separation vector. The Born-approximation, then tells us that the scattering amplitude from a momentum eigenstate $|\mathbf{k}_i\rangle$ to $|\mathbf{k}_f\rangle$ is $\langle \mathbf{k}_f|V(\mathbf{x})|\mathbf{k}_i\rangle \propto \int d^3\mathbf{x} V(\mathbf{x}) e^{-i(\mathbf{k}_i - \mathbf{k}_f)\cdot\mathbf{x}}$, and the differential cross section

$$\frac{d\sigma}{d\Omega} \propto \left|\int d^3\mathbf{x} V(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}\right|^2 = \left|\tilde{V}(\mathbf{q})\right|^2 , \tag{4.3.39}$$

where $\mathbf{q}$ is $\mathbf{k}_i - \mathbf{k}_f$, ie. it is the momentum transfer and $\tilde{V}(\mathbf{q})$ is the Fourier transform of $V(\mathbf{x})$. Comparing eq. (4.3.39) and eq. (4.3.37), we see that

$$\tilde{V}(\mathbf{q}) \propto \frac{1}{|\mathbf{q}|^2 + m^2} \Longrightarrow V(\mathbf{x}) \propto \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} . \tag{4.3.40}$$

If this is the interaction potential, the force will be $\mathbf{F} = -\nabla V(\mathbf{x})$. Thus we have discovered that the force between our "electron" and "positron" due to the exchange of a scalar particle of mass m is given by

$$|\mathbf{F}| \propto \begin{cases} 1/|\mathbf{x}|^2 & |\mathbf{x}| \ll m^{-1} \\ e^{-m|\mathbf{x}|}/|\mathbf{x}| & |\mathbf{x}| \gg m^{-1} \end{cases} \tag{4.3.41}$$

That is, we have a "$1/r^2$" force for distances small compared to m^{-1} and a "Yukawa screening" for distances large compared to m^{-1}![11]

Exercise 4.3.4 (i) Verify that in the non-relativistic limit, and with $m \ll M$, eq. (4.3.37) leads to eq. (4.3.38). (ii) Then verify the implication in eq. (4.3.40) (ie. calculate the Fourier transform). (ii) Sketch $V(\mathbf{x})$ as a function of $|\mathbf{x}|$ on a log-log plot. (iv) From the dynamics of the inner planets in our solar system[12], one finds that force law is "$1/r^2$" (or more correctly, consistent with general relativity which has massless gravitons) at order $\mathcal{O}(10^{-8})$. Using our Yukawa type potential to parametrize the effect of a massive graviton, what is constraint on the mass of the graviton m_g? (give your answer in eV). Order of magnitude is fine, but explain your reasoning. What is the length scale in meters corresponding to this mass?

4.3.6 Beyond Leading Order

Connected and Amputated Contributions

So far we have always calculated amplitudes at the leading order in the coupling constant. For example, the amplitude for $e^- + e^- \to e^- + e^-$ scattering, we calculated the matrix element $\langle f|S - 1|i\rangle$ to order g^2. What happens if we want to calculate amplitudes beyond the leading order. Following the "Diagrammer's

[11] It is worth noting that if we kept track of signs, we would find an attractive force regardless of whether we consider e^+e^- or e^-e^- scattering. Scalar exchange in our theory yields an attractive force. We have to go to Quantum Electrodynamics, the correct theory for electrons, positrons and photons to see that like charges repel, and unlike charges attract. The spin of the force carrier matters! Electrodynamics has a (massless) spin-1 carrier, and gravity, a (massless) spin-2 carrier which determine whether we can have attractive/and or repulsive forces.

[12] Note that 1AU $\approx 1.5 \times 10^{11}$ meters, which you can take as the typical size of orbits of inner planets

way", let us try to draw the diagrams at next order in g, making use of the fundamental vertex. I will be a bit sloppy, and ignore the arrows on the $\psi, \psi^\dagger$ lines, not label external momenta and also not draw diagrams that can be obtained by permutations of external legs, rotation from one to the other etc. We have the following diagrams (shown in Fig. 4.12). Each diagram will have an expression associated with (which can be written down using our Feynman rules), which when added together should yield the total amplitude. Take a close look. Some parts of the diagrams (for example the one inside $(\ldots)$) are disconnected from the external lines. They are "vacuum bubbles". You can convince yourself that you can factor the diagrams into a product of connected diagrams and disconnected vacuum bubbles.[13]

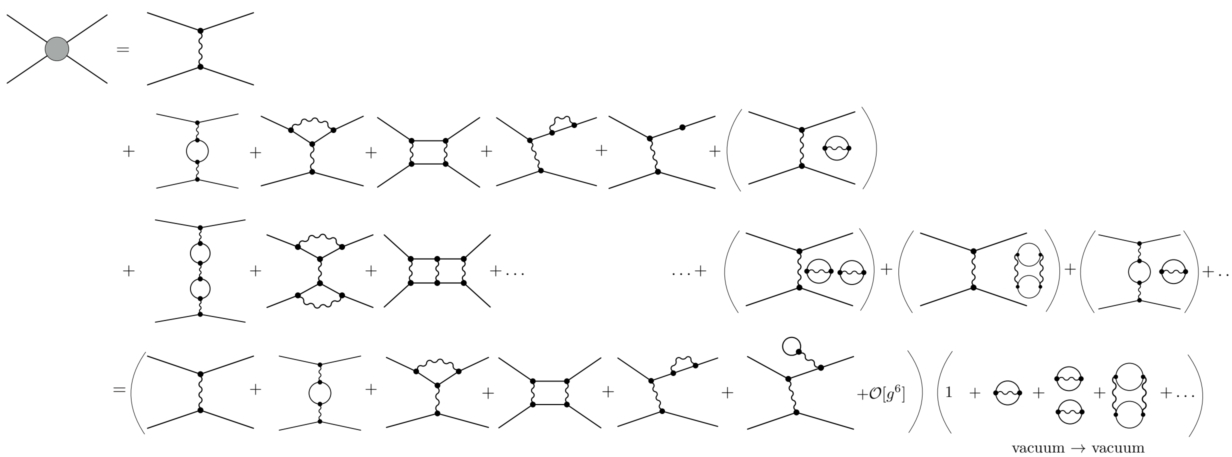


Figure 4.12: The diagrammatic expansion of the amplitude for $e^-e^- \to e^-e^-$. Note the factorization of the vacuum bubbles in the last line. Permutations of external legs has been ignored for simplicity.

The pieces with the vacuum bubbles can be thought of as contributing to the vacuum-to-vacuum transition. Since we normalize the vacuum, this means that at best contribution from the bubbles is a phase $e^{i\theta}$, and does not influence the overall scattering probability density ($|e^{i\theta}|^2 = 1$).[14]

Now look at diagrams with "loops" on external legs. It turns out that we can amputate these legs (for example, cut before the loop begins on the outgoing external legs, or cut after the loop for the incoming external legs). Why is this physically reasonable? This is ok, because as you might recall, we used free-field initial and final states. The external legs with loops is just correcting for the initial and final states in the full theory, and do not have anything to do with the scattering itself. So, with these two rules, we can simplify our scattering calculation as follows:

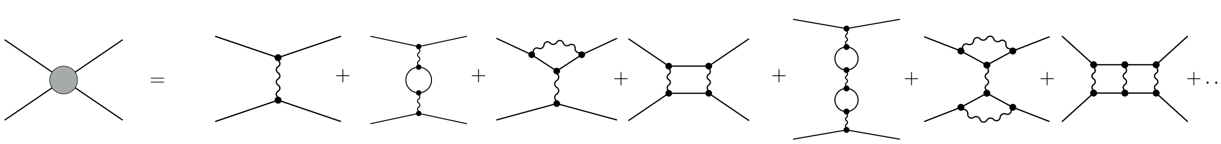


Figure 4.13: The diagrammatic expansion, including only the connected amputated diagrams, of the amplitude for $e^-e^- \to e^-e^-$ scattering.

$$\langle f|S-1|i\rangle = \sum (\text{All connected, amputated Feynman diagrams}) \tag{4.3.42}$$

[13]Our simplifying assumption that the external momenta are not equal precludes disconnected diagrams with external legs.

[14]I am being a bit sloppy here about whether I am talking about the interacting vacuum or the non-interacting vacuum and such. You can look at, for example, Ch. 22 of *QFT for the Gifted Amateur*

Divergences at higher orders

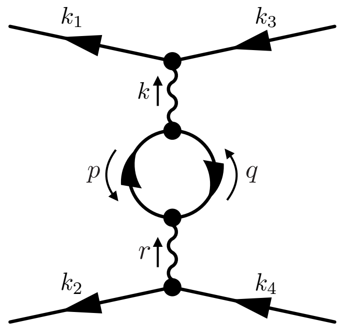


Figure 4.14: The diagrammatic expansion, including only the connected amputated diagrams, of the amplitude for $e^-e^- \to e^-e^-$ scattering.

Let us focus on one of the diagrams at $\mathcal{O}[g^4]$ in the for $e^-e^- \to e^-e^-$ process discussed above (see Fig. 4.14, and try to calculate the integral corresponding to this diagram. Following the Feynman rules in Fig. 4.6, we have

$$\begin{aligned}\langle f|S-1|i\rangle =&(-ig)^4 \int đ^4k\, đ^4p\, đ^4q\, đ^4r\, i\Delta_\varphi(k)\, i\Delta_\varphi(r)\, i\Delta_\psi(p)\, i\Delta_\psi(q) \\ &\times \delta\!\!\bar{}^{(4)}(k_1+k-k_3)\delta\!\!\bar{}^{(4)}(q-p-k)\delta\!\!\bar{}^{(4)}(r-q+p)\delta\!\!\bar{}^{(4)}(k_2-r-k_4)\,, \\ =&(-ig)^4\delta\!\!\bar{}^{(4)}(k_1+k_2-k_3-k_4)\, i\Delta_\varphi(k_1-k_3)\, i\Delta_\varphi(k_2-k_4)\, \underbrace{\int đ^4q\, i\Delta_\psi(q)\, i\Delta_\psi(k_1-k_3+q)}_{\text{internal loop}}\,.\end{aligned} \tag{4.3.43}$$

Using our expressions for the propagators in (4.3.29), we can write the contribution from the "internal loop" as

$$\int đ^4q\, \frac{i}{q^2-M^2}\, \frac{i}{(k_1-k_3+q)^2-M^2} \tag{4.3.44}$$

Roughly speaking, the integral measure yields q^4, whereas the two propagators also yield q^{-4} as $q \to \infty$. This scaling at large momenta indicates that the integral will diverge logarithmically at large q. This is terrible news! The same is true for other higher order diagrams. So the scattering amplitudes have the form $\langle f|S-1|i\rangle \sim g^2(\text{finite}) + g^4(\infty?) + \ldots$ This means that the contributions to the amplitude diverges at higher orders in g, and puts our entire scheme of perturbation in small g into question. What did we miss ?

At this point we need to turn to the important idea of renormalization, which provides an algorithm as well as physical reasoning to remove these infinites. Powerful ideas such as the renormalization group, and the scale dependence of parameters make an appearance. In this course we do not have time to discuss this in detail, you can learn about this in a later course or on your own. [15].

Exercise 4.3.5 : Consider the process $\gamma(k_1) + e^+(k_2) \to \gamma(k_3) + e^+(k_4)$. (a) Draw all *connected and amputated* Feynman diagrams up to 4-th order in g. You can ignore permutations of external legs (and no need to label the external momenta for this part). (b) Draw a diagram with a disconnected piece at order g^4. (c) Write down the expression corresponding to the "box" diagram $\mathcal{O}[g^4]$ in (Similar, but not identical to the 4-th diagram from the left after the = sign in Fig. 4.13) and simplify as much as you can.

[15] In class, we will have a hand-wavy discussion of renormalization

Exercise 4.3.6 : Consider the process $\varphi(k_1)+\varphi(k_2) \to \varphi(k_3)+\varphi(k_4)$ in the $\mathcal{L} = (1/2)(\partial\varphi)^2 - (1/2)m^2\varphi^2 - (\eta/3!)\varphi^3$ theory. (a) Write down the fundamental vertex in this theory in momentum space. (b) Draw all *connected and amputated* Feynman diagrams up to 4rth order in η. You can ignore permutations of external legs (and no need to label the external momenta for this part). (c) Draw a diagram corresponding to an s-channel process at $\mathcal{O}[\eta^2]$ and write down the expression corresponding to this diagram. Simplify as far as possible. (d) Draw a diagram corresponding to an s-channel process at $\mathcal{O}[\eta^4]$, with a bubble on the inner propagator and write down the expression corresponding to this diagram. Simplify as far as possible.

CHAPTER 5

SYMMETRY AND FIELDS

5.1 Spacetime and Internal Symmetries

5.1.1 Noether's Theorem and its Consequences

5.2 Spin 0

5.3 Spin 1

5.3.1 Massive Vector Fields

5.3.2 Massless Vector Fields

5.4 Spin 1/2

5.4.1 Dirac Fermions

5.5 Local Gauge Invariance

5.5.1 Quantum Electrodynamics

5.5.2 Yang-Mills Theories

5.6 Symmetry Breaking

5.6.1 Preliminaries – Nambu-Goldstone Theorem

5.6.2 Anderson-Higgs Mechanism and Applications

5.6.3 Defects

Topological Solitons

APPENDIX A

MATHEMATICAL PRELIMINARIES

A.1 Fourier Transforms

Consider a function $f(\mathbf{x})$ which satisfied periodic boundary conditions in a box of size L (volume $V = L^3$). Then

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}} \\ f_{\mathbf{k}} &= \frac{1}{\sqrt{V}} \int_V d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \\ \mathbf{k} &= \frac{2\pi}{L}(n_x, n_y, n_z) \end{aligned} \tag{A.1.1}$$

If $L \to \infty$,

$$\begin{aligned} f(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}) \\ f(\mathbf{k}) &= \int_V d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \end{aligned} \tag{A.1.2}$$

The finite and infinite box cases are related by

$$\begin{aligned} \sum_{\mathbf{k}} &\to V \int \frac{d^3k}{(2\pi)^3} \\ f_{\mathbf{k}} &\to \frac{1}{\sqrt{V}} f(\mathbf{k}) \end{aligned} \tag{A.1.3}$$

In $3+1$ spacetime dimensions $x = x^\mu = (x^0, \mathbf{x})$ and $k = k^\mu = (k^0, \mathbf{k})$, with $k \cdot x = k^0 x^0 - \mathbf{k}\cdot\mathbf{x}$ (also see the review of Special Relativity).

$$\begin{aligned} f(x) &= \int d^4\bar{k} e^{-ik\cdot x} f(k) \\ f(k) &= \int d^4x e^{ik\cdot x} f(x) \end{aligned} \tag{A.1.4}$$

A.2 Delta Functions

Kroneckar Delta: The Kroneckar Delta function is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j\,, \\ 0 & i \neq j\,. \end{cases} \tag{A.2.1}$$

For Fourier Transforms in a finite box,

$$\delta_{\mathbf{q},\mathbf{k}} = \begin{cases} 1 & \mathbf{q} = \mathbf{k}\,, \\ 0 & \mathbf{q} \neq \mathbf{k}\,. \end{cases} \tag{A.2.2}$$

A useful representation of the Kroneckar Delta function is

$$\delta_{\mathbf{q},\mathbf{k}} = \frac{1}{V}\int_V d^3x e^{-i(\mathbf{q}-\mathbf{k})\cdot\mathbf{x}}\,. \tag{A.2.3}$$

<u>Dirac Delta</u>: The Dirac Delta function is defined by

$$\int d^3x \delta(\mathbf{x}-\mathbf{y}) f(\mathbf{x}) = f(\mathbf{y})\,, \tag{A.2.4}$$

for sufficiently well behaved $f(\mathbf{x})$. A useful representation of the Dirac Delta function is

$$\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}}\,. \tag{A.2.5}$$

The Dirac Delta "function" is an example of a distribution. Two distributions D_1 and D_2 are equal if

$$\int dx D_1(x) f(x) = \int dx D_2(x) f(x)\,. \tag{A.2.6}$$

Some useful identities related to the Dirac-Delta function (which can be appropriately generalized to arbitrary number of dimensions), are listed below:

$$\begin{aligned} \delta(ax) &= \frac{1}{|a|}\delta(x)\,, \\ \delta(f(x)) &= \sum_i \frac{1}{|f'(x_i)|}\delta(x-x_i) \qquad \text{where } f(x_i)=0\,, \\ \frac{d}{dx}\delta(x) &= -\delta(x)\frac{d}{dx}\,. \end{aligned} \tag{A.2.7}$$

The last property can be checked using integration by parts: $\int dx f(x)\delta'(x) = -\int dx f'(x)\delta(x)$.

We will see the combination $(2\pi)^3\delta(\mathbf{k})$ as well as $d^3k/(2\pi)^3$ often, hence it is useful to define:

$$\begin{aligned} d^3\not{k} &\equiv \frac{d^3k}{(2\pi)^3}\,, \\ \not{\delta}(\mathbf{k}) &\equiv (2\pi)^3\delta(\mathbf{k})\,. \end{aligned} \tag{A.2.8}$$

These generalize to an arbitrary number of dimensions as well.

A.3 Functionals

We can think of a function $f(x)$ as taking an argument x and returning a number $f(x)$. A functional $F[f]$, takes an entire function f and returns a number $F[f]$. For example, let $f(x) = x^2$ and

$$F[f] = \int_{-1}^{1} dx f(x) = \int_{-1}^{1} dx x^2 = \frac{2}{3}\,. \tag{A.3.1}$$

You have been using functionals all along, without calling them by that name. The function $f(x)$ can also be though of as a specific functional:

$$F[f] = \int dy \delta(x-y) f(y) = f(x)\,. \tag{A.3.2}$$

though we will call such "local" functionals, just functions. Thinking in terms of functionals is quite useful for variational problems (extremizing the action), as well as in, unsurprisingly, the functional formulation of QFT. Let us learn some rules for dealing with functionals. The derivative of a functional at a location $x = y$ is

$$\frac{\delta F}{\delta f(y)} \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ F[f(x) + \epsilon\delta(x-y)] - F[f(x)] \right\} . \tag{A.3.3}$$

This definition captures the following. What is the rate of change of the functional $F[f]$ as we perturb $f(x)$ at the location y. The usual rules of derivative such as the chain rule apply to functional derivatives as well.

As a very relevant application, consider the *action functional*:

$$S = \int_{t_i}^{t_f} dt L(q, \dot{q}, t) \,, \tag{A.3.4}$$

Extremizing the action corresponds to finding the path $q(t)$, such that for $q(t) \to q(t) + \delta q(t)$, $\delta S/\delta q = 0$ for all $\delta q(t)$ such that $\delta q(t_i) = \delta q(t_f) = 0$. Let us calculate $\delta S/\delta q(t)$.

$$\begin{aligned}
\frac{\delta S}{\delta q(t)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ S[q(\tau) + \epsilon\delta(\tau - t)] - S[q(\tau)] \right\} , \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_i}^{t_f} d\tau \left\{ L(q(\tau) + \epsilon\delta(\tau - t), \dot{q}(\tau) + \epsilon\dot{\delta}(\tau - t)) - L(q(\tau), \dot{q}(\tau)) \right\} , \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_i}^{t_f} d\tau \left\{ \epsilon \frac{\partial L(q, \dot{q})}{\partial q} \delta(\tau - t) + \epsilon \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \dot{\delta}(\tau - t) \right\} , \\
&= \frac{\partial L(q, \dot{q})}{\partial q} - \frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) .
\end{aligned} \tag{A.3.5}$$

The first line is just the definition of a functional derivative. In the third line, we treat L as a usual function (takes $\{q(t), \dot{q}(t)\}$ and spits out a number) and use the chain rule. In going from the penultimate line to the last line, used integration by parts. Extremizing the action

$$\frac{\delta S}{\delta q} = 0 \Longrightarrow \frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) = \frac{\partial L(q, \dot{q})}{\partial q} . \tag{A.3.6}$$

which are our usual Euler-Lagrange equations.

Let us now consider the action functional for a relativistic scalar field $\varphi(x) = \varphi(t, \mathbf{x})$:

$$S = \int d^4y \mathcal{L}(\varphi(y), \partial_\mu \varphi(y)) . \tag{A.3.7}$$

where $\mathcal{L}$ is the Lagrangian density. Repeating the steps we took in the previous case, (and assuming the boundary at infinity),

$$\begin{aligned}
\frac{\delta S}{\delta \varphi(x)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ S[\varphi(y) + \epsilon\delta(y - x)] - S[\varphi(y)] \right\} , \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y \left\{ \mathcal{L}(\varphi(y) + \epsilon\delta(y - x), \partial_\mu\varphi(y) + \epsilon\partial_\mu\delta(y - x)) - \mathcal{L}(\varphi(x), \partial_\mu\varphi(x)) \right\} , \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y \left\{ \epsilon \frac{\partial \mathcal{L}(\varphi, \partial_\mu\varphi)}{\partial \varphi} \delta(y - x) + \epsilon \frac{\partial \mathcal{L}(\varphi, \partial_\mu\varphi)}{\partial \varphi} \partial_\mu \delta(y - x) \right\} , \\
&= \frac{\partial \mathcal{L}(\varphi, \partial_\mu\varphi)}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial_\mu\varphi)}{\partial_\mu \varphi} \right) .
\end{aligned} \tag{A.3.8}$$

where in going from the penultimate to the last line, we have used the 3+1 dimensional version of the divergence theorem along with integration by parts. Extremizing the action

$$\frac{\delta S}{\delta \varphi} = 0 \Longrightarrow \partial_\mu \left(\frac{\partial \mathcal{L}(\varphi, \partial_\mu\varphi)}{\partial_\mu \varphi} \right) = \frac{\partial \mathcal{L}(\varphi, \partial_\mu\varphi)}{\partial \varphi} . \tag{A.3.9}$$

Further Reading: For a quick and gentle introduction to functionals, see for example, section 1.3 in *QFT for the Gifted Amateur* by Lancaster and Blundell.

A.4 Green's Functions

Consider a linear differential equation

$$\mathcal{D}\varphi(x) = f(x)\,, \tag{A.4.1}$$

where $\mathcal{D}$ is some linear differential operator, say for example $\mathcal{D} = \partial^\mu \partial_\mu$. The general solution for the above equation can be written as

$$\varphi(x) = \varphi_h(x) + \int d^4y G(x,y) f(y)\,, \tag{A.4.2}$$

where

$$\begin{aligned} &\mathcal{D}G(x,y) = \delta(x-y)\,, \\ &\mathcal{D}\varphi_h(x) = 0\,. \end{aligned} \tag{A.4.3}$$

where $\varphi_h(x)$ is the *homogeneous* solution. The function $G(x,y)$ is the *Green's function* for the system. One needs to specify initial/boundary conditions to determine it. Once $G(x,y)$ is determined, our problem is solved for arbitrary $f(x)$! This is why Green's functions are useful.

Further Reading: See for example, https://en.wikipedia.org/wiki/Green's_function.

A.5 Contour Integration

Let $f(z)$ be a function in the complex plane, and C a closed curve in the complex plane. Then

$$\oint_C dx f(z) = 2\pi i \sum_j \left(\text{Residue of } f \text{ at } z_j \text{ inside } C\right)\,, \tag{A.5.1}$$

where the integration is done in the anti-clockwise direction, and

$$\text{Residue of } f \text{ at } z_j = \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_j)^n f(z)\right]\,, \tag{A.5.2}$$

The largest n for which Residue of f at $z_j \neq 0$, determined the nature of the "Pole" in the function f at z_j. For example, if $n = 1$, we have a "Simple Pole" at $z = z_j$, whereas if $n = 2$, we have a "Double Pole". Said another way, recall that the series expansion of a function $f(z)$ in the complex plane (Laurent Series)

$$f(z) = b_0 + \frac{b_1}{(z-z_j)} + \frac{b_2}{(z-z_j)^2} + \ldots + a_1(z-z_j) + a_2(z-z_j)^2 + \ldots \tag{A.5.3}$$

Then if $b_n = 0$ for all $n \geq 2$, $f(z)$ is said to have a simple pole at $z = z_j$.

We will often use this "Residue Theorem" to evaluate integrals along the real line. As an example, consider the following integral (which is related to an integral we will encounter in the discussion of "Propagators")

$$I = \int_{-\infty}^{\infty} dx \frac{e^{-ixy}}{x^2 - a^2 + i\epsilon}\,, \tag{A.5.4}$$

where $a > 0$ and $y < 0$ and $\epsilon \to 0^+$. The above integral is along the real line. To evaluate this integral, consider the following integral which is to be done over a closed curve in the complex plane.

$$\mathcal{I} = \oint dz \frac{e^{-izy}}{z^2 - a^2 + i\epsilon} = 2\pi i \sum_j \left(\text{Residue of } f \text{ at } z_j \text{ inside } C\right)\,, \tag{A.5.5}$$

where the curve will be chosen in a way so that upon evaluation, this integral is equal to I. The chosen curve runs along the real axis and closes with a large semi-circle in the upper half of the complex plane. Why upper half? To see this, let us move to polar co-ordinates $z = re^{i\theta}$. Then the integral becomes

$$\mathcal{I} = \lim_{r\to\infty} \int_{-r}^{r} dx \frac{e^{-ixy}}{x^2 - a^2 + i\epsilon} + \lim_{r\to\infty} \int_0^{\pi} d\theta r \frac{e^{ry\sin\theta} e^{-iry\cos\theta}}{r^2 e^{i2\theta} - a^2 + i\epsilon} . \tag{A.5.6}$$

Note that since $y < 0$, the exponent drives the second term to 0 as $r \to \infty$. Note that if $y > 0$, we would have chosen to close the integral in the lower half plane (along with an associated minus sign since the curve would now be clockwise). Thus we have shown that $\mathcal{I} = I$ with this chosen contour C.

Now let us use the residue theorem. Thanks to $\epsilon > 0$, there is only one simple pole in the upper-half plane at $z = -a + i\epsilon/(2a)$. To see this, note that $z^2 - a^2 + i\epsilon = (z - (a - i\epsilon/2a))(z - (-a + i\epsilon/2a))$. Thus

$$(\text{Residue of } f \text{ at } z_* = -a + i\epsilon/(2a)) = (z - z_*) f(z_*) \to \frac{e^{iay}}{-2a} . \tag{A.5.7}$$

Putting this all together, we have

$$I = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dx \frac{e^{-ixy}}{x^2 - a^2 + i\epsilon} = -i\pi \frac{e^{iay}}{a} . \tag{A.5.8}$$

Further Reading: For a quick review of useful complex analysis, see the Appendix B in *QFT for the Gifted Amateur* by Lancaster and Blundell.

A.6 Groups and Representation Theory

It is not essential for you to know about Lie Groups and Representation Theory inside out, but a bit of familiarity will go a long way in helping you make sense of the second part of the course. You may want to read chapter 9 of *QFT for the Gifted Amateur*.

Lecture 13

Symmetries :

$\boxed{\text{Noether's theorem}}$

Every continuous symmetry of the action yields a current j^μ which is conserved via the equations of motion

EOM $\Rightarrow$ $\partial_\mu j^\mu = 0$.

[More explicitly $\partial_0 j^0 + \nabla \cdot \vec{j} = 0$]

Proof:

Consider $S = \int d^4x \, \mathcal{L}(\varphi_a, \partial_\mu \varphi_a)$ $= 1, 2 \ldots N$

A symmetry transformation is a transformation that leaves the action invariant;

$$\mathcal{L} \longrightarrow \mathcal{L} + \partial_\mu F^\mu$$

ie. it can change $\mathcal{L}$ by at most a total derivative. (F^μ are arbitrary).

Continuous symmetry $\Rightarrow$ we are allowed to work with infinitesimal transformations.

Consider :

$$\varphi_a \longrightarrow \varphi_a + \delta\varphi_a$$

Then

$$\delta\mathcal{L} = \sum_a \left(\frac{\partial \mathcal{L}}{\partial \varphi_a} \delta\varphi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial_\mu(\delta\varphi_a) \right)$$

$$= \overbrace{\sum_a \left[\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \right] \delta\varphi_a}^{①} + \overbrace{\sum_a \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a \right]}^{②}$$

If φ_a satisfies the equation of motion then ① $= 0$.

$$\therefore \quad \delta\mathcal{L} = \sum_a \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a \right] \qquad (a)$$

For a symmetry transformation

$$\delta\mathcal{L} = \partial_\mu F^\mu \qquad (b)$$

. Hence from (a) & (b)

$$\Rightarrow \quad \partial_\mu \left[\sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} - F^\mu \right] = 0$$

Thus, there exists $j^\mu \equiv \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta \varphi_a - F^\mu$

such that $\partial_\mu j^\mu = 0$ on the EOM.

Notes :

(1)

$$\partial_\mu j^\mu = 0$$

$$\Rightarrow \quad \partial_0 j^0 = -\nabla \cdot \vec{j}$$

$$\Rightarrow \quad \partial_0 \int d^3x \, j^0 = -\int \nabla \cdot \vec{j} \, d^3x \qquad (\textstyle\int = \text{all space})$$

$$= -\int \vec{j} \cdot d\vec{s}$$

$$= 0$$

$$\Rightarrow \quad \partial_0 Q = 0 \quad \text{where} \quad Q \equiv \int j^0 d^3x .$$

Thus Noether's theorem implies the existence of a conserved "charge". $\frac{dQ}{dt} = 0$

(2) $j^\mu = \sum_a \Pi_a^\mu \delta \varphi_a - F^\mu \qquad \Pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}$

(3) In quantum mechanics (QFT), $\hat{Q}$ becomes a conserved operator, ie it commutes with the hamiltonian

$$\frac{d\hat{Q}}{dt} = i[H, \hat{Q}] = 0$$

" $\hat{Q}$ is a generator of this symmetry."

What does that mean?

$$i[\hat{Q}, \varphi_a] = i\left[\int d^3x \sum\left(\hat{\Pi}^0_b \, \delta\hat{\varphi}_b - \hat{F}^0\right), \hat{\varphi}_a\right]$$

Assuming $\delta\varphi_a$ & F^0 only depend on φ_a & not $\dot{\varphi}_a$,

$$i[\hat{Q}, \varphi_a] = i\int d^3x \, \delta\hat{\varphi}_b \, [\hat{\Pi}^0_b, \hat{\varphi}_a]$$

$$= \delta\varphi_a$$

$$\therefore \quad \delta\hat{\varphi}_a = i[\hat{Q}, \hat{\varphi}_a]$$

Thus $\hat{Q}$ generates the symmetry transformation.

Two types of symmetries.

1. Internal Symmetries
2. Spacetime symmetries.

Lecture 14

Internal symmetries.

example

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \psi^* - M^2 \psi^* \psi$$

Lagrangian is invariant under the symmetry $\psi \to \psi e^{i\alpha}$

* infinitesimal version $\psi \to \psi + \underbrace{i\alpha \psi}_{\delta\psi}$

Note that $\delta\mathcal{L} = \partial_\mu F^\mu = 0$ (we can ignore the F piece, conserved independently)

The conserved current is

$$\therefore \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \delta\psi^*$$

$$= i\alpha \, \partial^\mu \psi^* \psi - i\alpha \, \partial^\mu \psi \, \psi^*$$

$$= i\alpha \left(\psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi \right)$$

[Check that $\partial_\mu j^\mu = 0$!]

$$j^0 = i\alpha(\psi\dot{\psi}^* - \psi^*\dot{\psi})$$

$$Q = i\alpha\int d^3x \left(\psi\dot{\psi}^* - \psi^*\dot{\psi}\right)$$

Is Q a generator of symmetries?

$$i[Q, \psi] = -\alpha\int d^3x\, \psi[\dot{\psi}^*, \psi]$$

$$= i\alpha\psi$$

$$= \delta\psi$$

$\therefore$ Q indeed generates the appropriate shift in ψ

The same analysis can be repeated with

$$\psi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi_1\partial^\mu\varphi_1 - m^2\varphi_1^2\right) + \frac{1}{2}\left(\partial_\mu\varphi_2\partial^\mu\varphi_2 - m_2\varphi_2^2\right)$$

$$\frac{j^\mu}{\theta} = \left(\varphi_2\partial^\mu\varphi_1 - \varphi_1\partial^\mu\varphi_2\right)$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$$Q = \int d^3x \left(\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1\right)$$

$$\simeq \begin{pmatrix}1 & -\theta\\ +\theta & 1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

Spacetime symmetries.

Special Relativity ⇔ theory should be invariant under translations, boosts & rotations.

Poincare group : translations ⊗ Lorentz groups
↑
boosts + rotations.

Actions should be invariant under Poincare transformations

1a. translations $x^\mu \to x^\mu + \varepsilon a^\mu$ $\varepsilon << 1$, $\varepsilon > 0$

Let $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$.

$$\therefore \quad \varphi(x^\nu - \varepsilon a^\nu) = \varphi(x) - \underbrace{\varepsilon a^\nu \partial_\nu \varphi(x)}_{\delta\varphi}$$

(∵ active transf. so -ve sign)

$$\delta \mathcal{L} = -\varepsilon a^\nu \partial_\nu \mathcal{L} = \partial_\nu F^\nu$$

$$\therefore F^\nu = -\varepsilon a^\nu \mathcal{L}.$$

Note sign diff from P. Stevenson notes)

$$\therefore \quad j^\mu = \overbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}}^{\pi^\mu} \delta\varphi - F^\mu = -\varepsilon \overbrace{\partial^\mu \varphi}^{\pi^\mu} a^\nu \partial_\nu \varphi - \varepsilon a^\mu \mathcal{L}$$

$$= -\varepsilon a^\nu \left[\pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \right]$$

$$\therefore \quad \partial_\mu \left\{ \Pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \right\} = 0$$

$$T^\mu{}_\nu \equiv \Pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}$$ (energy-momentum tensor).

$$\partial_\mu T^\mu{}_\nu = 0$$

* usual to divide by ε in defining currents.

There are 4 conserved current "j^μ", one for each ν. (4 spacetime translations).

Conserved charges.

$$P_\nu \equiv \int T^0{}_\nu \, d^3x .$$ (= 4 momentum)

$$P_0 = \int T^0{}_0 \, d^3x = \int d^3x \left(\Pi^0 \partial_0 \varphi - \mathcal{L} \right) = \int d^3x \, \mathcal{H} = H.$$

↑ energy or Hamiltonian.

$P_0 = H$ generates translations in time.

$$i [H, \varphi] = \frac{d\varphi}{dt}$$

In general

$$\partial_\mu \varphi = i [P_\mu, \varphi] \qquad \text{ie } P_\mu \text{ is a generator of translations.}$$

[Note: P^μ is constructed out of fields integrated over space].

[Note: I have been sloppy about infinitesimals. Typically when defining j^μ, we "divide" by the infinitesimal parameters]

[Note: Generalize to many component fields
$\vec{\varphi} = \{\varphi_r\} = \{\varphi_1, \varphi_2 \ldots \varphi_N\}$]

$$T^\mu{}_\nu \equiv \sum_r \left(\Pi^\mu_r \partial_\nu \varphi_r \right) - g^\mu_\nu \mathcal{L}.$$

$$\partial_\mu \varphi_r = i [P_\mu, \varphi_r]$$

[$T^\mu{}_\nu$, P^μ have sum over all fields]

We have already discussed translations. Let us now turn to boosts and rotations.

Boosts and rotations are include in Lorentz transformation (which form a group under multiplication) : $\Lambda^{\mu}{}_{\nu}$

$$\tilde{x}^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

The infinitesimal version

$$\tilde{x}^{\mu} = \underbrace{\left(\delta^{\mu}_{\nu} + \varepsilon a^{\mu}{}_{\nu}\right)}_{\Lambda^{\mu}{}_{\nu}} x^{\nu} \qquad \varepsilon \ll 1$$

Since the defining property of Lorentz transformations is

$$\Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} g^{\sigma\tau} = g^{\mu\nu}$$

$$\Rightarrow \qquad a^{\mu\nu} = -a^{\nu\mu} \qquad \leftarrow \text{check this}$$

ie $a^{\mu\nu}$ is anti-symmetric.

A bit more concretely

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$$

Consider a boost in the x-direction with velocity v.

$$\tilde{t} = \gamma(t - vx)$$
$$\tilde{x} = \gamma(x - vt)$$

$$\therefore \begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^\mu{}_\nu} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Similarly, a rotation in the x-y plane.

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

θ = rotation around the z-axis

Note that $\theta \ll 1$

$$\Lambda \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \approx \mathbb{1} + \theta \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{antisymmetric}}$$

Be careful about $\Lambda^{\mu}{}_{\nu}$ & $\Lambda^{\mu\nu}$

$$\Lambda^{\mu\nu} = \eta^{\mu\nu} + a^{\mu\nu}$$

antisymmetric, not $a^{\mu}{}_{\nu}$

Let us consider the transformation of a scalar field under the Lorentz transformation.

$$\varphi_{New}(x) = \varphi_{old}(\Lambda^{-1}x) \qquad \text{(active trans)}$$

ie $\varphi(x) \longrightarrow \varphi(\Lambda^{-1}x)$ $\qquad \Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \varepsilon a^{\mu}{}_{\nu}$

$= \varphi(\delta^{\mu}_{\nu} x^{\nu} - \varepsilon a^{\mu}{}_{\nu} x^{\nu})$ $\qquad (\Lambda^{\mu}{}_{\nu})^{-1} = \delta^{\mu}_{\nu} - \varepsilon a^{\mu}{}_{\nu}$

$= \varphi(x) - \varepsilon a^{\mu}{}_{\nu} x^{\nu} \partial_{\mu}\varphi$

$$\therefore \quad \delta\varphi = -\varepsilon a^{\mu\nu} x_{\nu} \partial_{\mu}\varphi = -\varepsilon \frac{a^{\mu\nu}}{2}(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu})\varphi$$

because $a^{\mu\nu}$ is antisymmetric.

[Note $a^{\mu\nu}$ (here) $= -a^{\mu\nu}$ in P. Stevenson's notes]

Similarly $\quad \delta\mathcal{L} = -\varepsilon a^{\mu}{}_{\nu} x^{\nu} \partial_{\mu}\mathcal{L} = -\varepsilon \partial_{\mu}(a^{\mu}{}_{\nu} x^{\nu} \mathcal{L})$

use $a^{\mu\nu}\eta_{\mu\nu} = 0$

$= -\varepsilon \partial_{\mu} F^{\mu}$ where $\quad$ check! $F^{\mu} = -\varepsilon a^{\mu}{}_{\nu} x^{\nu} \mathcal{L}$

$\therefore$ From Noether's theorem

$$j^{\mu} = \Pi^{\mu}\delta\varphi - F^{\mu} = -\varepsilon[\Pi^{\mu} a^{\sigma}{}_{\nu} x^{\nu} \partial_{\sigma}\varphi - a^{\mu}{}_{\nu} x^{\nu} \mathcal{L}]$$

$$= -\varepsilon a^{\sigma}{}_{\nu} [\Pi^{\mu} x^{\nu} \partial_{\sigma}\varphi - \delta^{\mu}_{\sigma} x^{\nu} \mathcal{L}]$$

$$\therefore \quad j^\mu = -\varepsilon\, a^\sigma_{\ \nu} \left[x^\nu \Pi^\mu \partial_\sigma \varphi \; - \; x^\nu \delta^\mu_\sigma \mathcal{L} \right]$$

$$= -\varepsilon\, a^\sigma_{\ \nu}\, x^\nu\, T^\mu_{\ \sigma} \qquad\qquad T^\mu_\rho = \Pi^\mu \partial_\sigma \varphi - \delta^\mu_\sigma \mathcal{L} .$$

$$= -\varepsilon\, a^{\sigma\nu}\; x_\nu\, T^\mu_{\ \sigma}$$

$$= -\frac{\varepsilon}{2}\, a^{\sigma\nu} \left[x_\nu T^\mu_{\ \sigma} - x_\sigma T^\mu_{\ \nu} \right]$$ [antisymmetry of $a^{\mu\nu}$]

Stripping off the infinitesimal transformation $\varepsilon a^{\sigma\nu}$

$$(j^\mu)_{\nu\sigma} \equiv x_\nu T^\mu_{\ \sigma} - x_\sigma T^\mu_{\ \nu}$$

$$\partial_\mu (j^\mu)_{\nu\sigma} = 0$$

$\nu\sigma$ = antisymmetric

= 6 conserved currents

(compared to 4 for translations)

Conserved "charges" is $(j^0)_{\nu\sigma}$

$$J_{\nu\sigma} \equiv \int d^3x \left(x_\nu T^0_{\ \sigma} - x_\sigma T^0_{\ \nu} \right)$$

& $[J_{\nu\sigma}, \varphi] = -i \left(x_\mu \partial_\nu - x_\nu \partial_\mu \right) \varphi$.

Note that J_{ij} are the (ij antisymmetric) total angular momentum of the field. It is the conserved charge corresponding to rotations. (angular momentum conservation).

J_{0i} are the conserved charges corresponding to boosts.

Lecture 15

Spacetime Symmetries = Translations + Lorentz Trans.

$$S = \int d^4x \; \mathcal{L}(\varphi, \partial_\mu \varphi).$$

<u>Translations</u> : $x^\mu \longrightarrow x^\mu + \varepsilon a^\mu$.

Noether $\Rightarrow$ $(j^\mu)_\nu = T^\mu{}_\nu$, $\partial_\mu T^\mu{}_\nu = 0$. 4 currents

where $T^\mu{}_\nu = \Pi^\mu \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L}$; $\Pi^\mu = \dfrac{\partial \mathcal{L}}{\partial_\mu \varphi}$.

charges: $P_\nu = \int d^3x \; T^0{}_\nu$; $i[P_\nu, \varphi] = \partial_\nu \varphi$.

$\hookrightarrow$ 4 charges.

<u>Lorentz Trans</u>. : $x^\mu \longrightarrow x^\mu + \varepsilon a^\mu{}_\nu$ where $\Lambda^\mu{}_\nu = 1 + \varepsilon a^\mu{}_\nu$

Noether $\Rightarrow$ $(j^\mu)_{\sigma\nu} = x_\sigma T^\mu{}_\nu - x_\nu T^\mu{}_\sigma$

charges : $\underbrace{J_{\sigma\nu}}_{\uparrow} = \int d^3x \left(x_\sigma T^0{}_\nu - x_\nu T^0{}_\sigma \right)$

antisymmetric $\Rightarrow$ 6 charges.

$$i[J_{\mu\nu}, \varphi] = (x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi.$$

$J_{0i} \Rightarrow$ boosts (3) ; $J_{ij} \Rightarrow$ rotations. (3)

P_μ = generators of translations (4).

$+$ $J_{\rho\sigma}$ = generators of Lorentz trans. $(6 = \underset{\text{boosts}}{3} + \underset{\text{rot.}}{3})$

form the generators of the Poincaré group.

(translations + $\underset{\text{boosts} \quad \text{rot.}}{\text{Lorentz trans.}}$)

These generators which characterize how the fields transform under the action of the Poincaré group satisfy the following "algebra"

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, J_{\rho\sigma}] = i\,(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho)$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\,(g_{\mu\sigma} J_{\nu\rho} + g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} - g_{\nu\sigma} J_{\mu\rho})$$

Let us see how to verify these commutators using the Jacobi Identity

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$$

and
$$[P_\mu, \Phi] = -i\,\partial_\mu \Phi$$
$$[J_{\mu\nu}, \Phi] = -i\,(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi$$

We will verify $[P_\mu, P_\nu] = 0$ in class, but the rest are part of your homework.

To verify that $[P_\mu, P_\nu] = 0$, first write down the Jacobi identity

$$[P_\mu, [P_\nu, \Phi]] + [\Phi, [P_\mu, P_\nu]] + [P_\nu, [\Phi, P_\mu]] = 0$$

$$\therefore\ [\Phi, [P_\mu, P_\nu]] = -[P_\mu, [P_\nu, \Phi]] - [P_\nu, [\Phi, P_\mu]]$$

$$= -[P_\mu, -i\partial_\nu \Phi] - [P_\nu, i\partial_\mu \Phi]$$
$$= i\,\partial_\nu [P_\mu, \Phi] - i\,\partial_\mu [P_\nu, \Phi]$$
$$= \partial_\nu \partial_\mu \Phi - \partial_\mu \partial_\nu \Phi$$
$$= 0$$

$\therefore \quad [\Phi, [P_\mu, P_\nu]] = 0$, for any field configuration Φ,

$\Rightarrow \quad [P_\mu, P_\nu] = 0.$ QED.

Toward's <u>finite</u> transformations

Note that for our scalar fields (and restricting to translations for simplicity)

$$\delta\Phi \approx -\varepsilon a^\nu \partial_\nu \Phi$$

$$
\begin{aligned}
\text{ie} \quad \Phi(x-a) &\approx \Phi(x) - \varepsilon a^\nu \partial_\nu \Phi \qquad \because \partial_\mu \Phi = i[P_\mu, \Phi] \\
&\approx \Phi(x) - i\varepsilon a^\nu [P_\nu, \Phi] \\
&= \left(1 - i\varepsilon a^\nu P_\nu\right) \Phi(x) \left(1 + i\varepsilon a^\nu P_\nu\right) \\
&= e^{-i\varepsilon a^\nu P_\nu} \Phi(x) e^{i\varepsilon a^\nu P_\nu}
\end{aligned}
$$

simply connected "Lie group".

true for finite transformations!

So knowing the generators & their commutation relations allows us to do finite transformations ie probe the entire group.

Similarly for Lorentz transformations.

$$\phi(\Lambda^{-1}x) \approx \phi(x) - \frac{a^{\mu\nu}}{2}(x_\nu \partial_\mu - x_\mu \partial_\nu)\phi$$

$$\approx \phi(x) + i\frac{a^{\mu\nu}}{2}[J_{\mu\nu}, \phi]$$

$$\approx \left(1 + \frac{ia^{\mu\nu}}{2}J_{\mu\nu}\right)\phi(x)\left(1 - \frac{ia^{\mu\nu}}{2}J_{\mu\nu}\right)$$

$$\rightarrow e^{+i\frac{a^{\mu\nu}}{2}J_{\mu\nu}}\phi(x)e^{-i\frac{a^{\mu\nu}}{2}J_{\mu\nu}}$$

finite.

Another view:

The commutation relations satisfied by $J_{\mu\nu}$ & P_ν are the same as those satisfied by

$\tilde{P}_\mu = -i\partial_\mu$

$\tilde{J}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

In this case, you can also write

$$\phi(x) = e^{-ia^\mu \tilde{P}_\mu}\phi(x) = e^{-a^\mu \partial_\mu}\phi(x) \quad (\approx \phi - a^\mu \partial_\mu \phi)$$

and for Lorentz transformations.

$$\phi(\Lambda^{-1}x) = e^{-i\frac{a^{\mu\nu}}{2}\tilde{J}_{\mu\nu}}\phi(x)$$

$$= e^{\frac{a^{\mu\nu}}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)}\phi(x).$$

$$\Big(\approx \phi(x) - \underbrace{\frac{a^{\mu\nu}}{2}\left(x_\nu\partial_\mu - \partial_\nu x_\mu\right)}_{\text{note sign flip from } \nu\mu \leftrightarrow \mu\nu}\varphi(x)$$

$$= \phi(x) + \delta\varphi(x) \Big)$$

Let us return to Lorentz transformations.

$$\Lambda^T g \Lambda = g. \qquad \text{(think of } \Lambda \text{ as a } 4\times 4 \text{ matrix)}$$

with comp. $\Lambda^\mu{}_\nu$.

Such transformations form a <u>group</u> under multiplication

1) $\underset{\text{Lorentz}}{\Lambda_1}\ \underset{\text{Lorentz}}{\Lambda_2} = \underset{\text{Lorentz}}{\Lambda_3}$ closure.

2) $(\Lambda_1 \Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2 \Lambda_3)$ associativity

3) $\underset{I}{\mathbb{1}}\cdot\Lambda = \Lambda\cdot\mathbb{1} = \Lambda$ identity exists

4). $\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = \mathbb{1}$ inverse exists.

Consider infinitesimal transformations.

$$\Lambda = 1 + i\alpha_A T^A \qquad A = 1, \ldots 6$$

$$T^A = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ldots \right\}$$

As you can check.

$$[T^A, T^B] = i f^{AB}{}_C T^C$$

$$\sim i \text{``}(gT + gT - gT - gT)\text{''} \quad \text{structure}$$

Same as $[\tilde{J}, \tilde{J}] = i\text{``}(g\tilde{J} + g\tilde{J} - g\tilde{J} - g\tilde{J})\text{''}$ structure

seen for our operator.

$$\tilde{J}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

(or eq. our charges in the scalar field case)

Recall where $\tilde{J}_{\mu\nu}$ really came from?

$$\begin{aligned}
\varphi(\Lambda^{-1}x) &= \varphi(x) + \delta\varphi(x) \\
&= \varphi(x) + \frac{\varepsilon}{2} a^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi(x). \\
&= \varphi(x) - i\frac{\varepsilon}{2} a^{\mu\nu} \tilde{J}_{\mu\nu} \varphi.
\end{aligned}$$

Now consider $\Lambda^\mu{}_\nu = 1 + \varepsilon a^\mu{}_\nu$. Then for every $a^\mu{}_\nu$ there is an "object":

$$\mathcal{I} = \left(1 - i\frac{\varepsilon}{2} a^{\mu\nu} \tilde{J}_{\mu\nu}\right) \quad \text{such that}$$

$$\mathcal{I}(\Lambda_1)\mathcal{I}(\Lambda_2) = \mathcal{I}(\Lambda_3) \quad \text{if } \Lambda_1\Lambda_2 = \Lambda_3.$$

$$\mathcal{I}(\Lambda^{-1}) = [\mathcal{I}(\Lambda)]^{-1} \qquad \Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = \mathbb{1}$$

(Same is true for finite trans.)

"$\mathcal{I}$" is a "representation" of the Lorentz group.

It is an "infinite dimensional" representation because it acts on objects like $\varphi(x)$

think of infinite locations.

Are there other representations? What do they mean?

Let us think again about out field φ.
Consider a spacetime point P (event).
Let $x^\mu(P)$ be the co-ordinates of this point in one reference frame and $y^\mu(P)$ be the co-ordinates in another frame related to the x-frame by a Lorentz transformation (imagine running past the event).

The field φ is physical entity that should be defined independent of the frame.

$$\varphi(P) = f(x^\mu(P)) = g(y^\mu(P))$$

But of course $f(x) \neq g(x)$.
If x^μ & y^μ are linked by a Lorentz transformation then this yields a relation between f & g.

$$\varphi(\Lambda^{-1}x) = \varphi(x) + \frac{\varepsilon}{2} a^{\mu\nu} (x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi.$$

is simply a reflection of how f & g are related.

For a little while imagine that we live in a world where fields have no variation in space & time. $\varphi = \text{const}$.

In this case, under a Lorentz transformation.

$\varphi \xrightarrow{\Lambda} \varphi$.

or $\varphi = \mathbb{1}\varphi$.

Consider $M(\Lambda) = \mathbb{1}$.

Is this a representation?

Yes! So a constant scalar field transforms under the trivial rep of the Lorentz group. The dimensionality of the rep. is 1.

In this constant world, now think of a 4-vector field. say A^μ

$$\mathcal{A} = A^\mu \hat{e}_\mu$$

physical object

basis vectors.

oriented along co-ordinate axes.

Now consider a different frame.
related to the original by a Lorentz transformation.

$$\vec{A} = A'^{\alpha} \hat{e}'_{\alpha}$$

oriented along the new axis.

Are A'^{α} & A^{α} the same?

No! $\qquad A^{\alpha} = \Lambda^{\alpha}{}_{\nu} A'^{\nu}$

Hence $\qquad A \xrightarrow{\Lambda} M(\Lambda)\, A$

just Λ itself.

$M(\Lambda)$ is of course a representation of the Lorentz group.

Thus we see that $A_{\alpha} = [M(\Lambda)]_{\alpha}{}^{\beta} A'_{\beta}$

$$= \Lambda_{\alpha}{}^{\beta} A'_{\beta}.$$

The dimensionality of the representation is 4.

In general the dimensionality of a representation is the dimensions of the vector space on which it acts.

Lecture 16

Under a Lorentz transformation, a multicomponent field transforms as.

$$\Phi_\alpha(x) \xmapsto{\Lambda} [M(\Lambda)]_\alpha^{\ \beta}\, \Phi_\beta(\Lambda^{-1}x)$$

① Example : $\Phi_\alpha(x)$ is a (collection of) scalar field.

$$[M(\Lambda)]_\alpha^{\ \beta} = \delta_\alpha^{\ \beta}.$$

$$\Phi_\alpha(x) \xmapsto[\text{scalar fields}]{\Lambda} \Phi_\alpha(\Lambda^{-1}x) \overset{\text{infinitesimal}}{=} \Phi_\alpha(x) + \frac{\varepsilon a^{\mu\nu}}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi_\alpha(x).$$

② Example : $\Phi_\alpha(x)$ is a Lorentz vector. $\alpha = 0,1,2,3.$

$$[M(\Lambda)]^\alpha{}_\beta = \Lambda_\alpha^{\ \beta}$$

$$\Phi_\alpha(x) \xmapsto[\text{vector}]{\Lambda} \Lambda_\alpha^{\ \beta}\, \Phi_\beta(\Lambda^{-1}x)$$

③ Example : $\Phi_\alpha(x)$ is a 4 component dirac spinor

$$[M(\Lambda)]_\alpha^{\ \beta} = ? \quad \text{(coming soon)}.$$

Let us consider a general form of $[M(\Lambda)]^{\beta}_{\alpha}$ and find the Noether current and charges first.

$$\Phi_\alpha(x) \xmapsto{\Lambda} [M(\Lambda)]^{\beta}_{\alpha}\, \Phi_\beta(\Lambda^{-1}x)$$

$$= \underbrace{\left[\delta^{\beta}_{\alpha} - \frac{i\varepsilon}{2} a^{\mu\nu}(\Sigma_{\mu\nu})^{\beta}_{\alpha}\right]}\left[\Phi_\beta(x) + \frac{a^{\mu\nu}}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)\varphi_\beta(x)\right] \text{ infinitesimal Lorentz}$$

<u>Pause</u>: ① $\varepsilon a^{\mu\nu}$ = parameters specifying the Lorentz transformation

IMP: same params in $M[\Lambda]^{\beta}_{\alpha}$ as inside $\varphi_\beta(\Lambda^{-1}x)$, since we are implementing the same Lorentz transformations.

② $(\Sigma_{\mu\nu})^{\beta}_{\alpha}$: are matrices.
$\mu\nu$ label which matrix (6 since $\mu\nu$ is antisymmetric).

: $\alpha\beta$ label entries of matrix.

corresponds to 6 independent Lorentz trans.

Back:

$$\Phi_\alpha(x) \xmapsto{\Lambda} \Phi_\alpha(x) + \underbrace{\frac{\varepsilon a^{\mu\nu}}{2}\left[(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^{\beta}_{\alpha} - \frac{i}{2}(\Sigma_{\mu\nu})^{\beta}_{\alpha}\right]\Phi_\beta(x)}_{\delta\varphi_\alpha(x)}$$

Following Noether, the current

$$j^\mu = (\Pi^\mu)^\alpha \delta\varphi_\alpha - F^\mu .$$

$$\rightarrow (j^\mu)_{\sigma\nu} = x_\sigma T^\mu_\nu - x_\nu T^\mu_\sigma - i(\Pi^\mu)^\alpha (\Sigma_{\sigma\nu})_\alpha^\beta \phi_\beta$$

Noether charges :

$$J_{\mu\nu} = \int d^3x \left[\underbrace{\left(x_\mu T^0_\nu - x_\nu T^0_\mu \right)}_{\substack{\text{orbital angular momentum} \\ (\text{call it } L_{\mu\nu} \text{ now})}} - \underbrace{i(\Pi^0)^\alpha (\Sigma_{\mu\nu})_\alpha^\beta \phi_\beta(x)}_{\text{internal} \rightarrow \text{spin}} \right]$$

If $[\Sigma, \Sigma] = -i\left(g\Sigma + g\Sigma - g\Sigma - g\Sigma\right)$

then $[J, J] = -i\left(gJ + gJ - gJ - gJ\right)$.

Finite transformations

$$\phi_\alpha(x) \xrightarrow{\Lambda} e^{-\frac{i}{2}a^{\mu\nu}(\Sigma_{\mu\nu})_\alpha{}^\beta} e^{\frac{i}{2}a^{\mu\nu}\tilde{L}_{\mu\nu}} \phi_\beta(x)$$

$$= e^{-\frac{i}{2}a^{\mu\nu}(\Sigma_{\mu\nu}-\tilde{L}_{\mu\nu})_\alpha{}^\beta}\phi_\beta(x)$$

* What is the connection between these T^A (generators) of the Lorentz groups and $\tilde{L}_{\mu\nu}$, $\Sigma_{\mu\nu}$ etc. (Recall $\Lambda = 1 + i\alpha_A T^A$)

Note first that for constant fields. ie $\phi_\alpha(\cancel{x}) = \tilde{\Phi}_\alpha$.

$$\tilde{\Phi}_\alpha \longrightarrow [M(\Lambda)]_\alpha^\beta \tilde{\Phi}_\beta$$

$$= \left[e^{-\frac{i}{2}a^{\mu\nu}(\Sigma_{\mu\nu})} \right]_\alpha^\beta \tilde{\Phi}_\beta .$$

$$\Sigma_{\mu\nu} \longleftrightarrow T^A$$

$$-\frac{a^{\mu\nu}}{2} \longleftrightarrow \alpha^A$$

antisymmetry $\Rightarrow$ $A = 1 \ldots 6$. at most.

Different choice of $(\Sigma_{\mu\nu})$ $\longleftrightarrow$ different representation

If $(\Sigma_{\mu\nu})$ is a $N \times N$ matrix, then it is called an N-dimensional representation.

If N is finite $\longleftrightarrow$ "finite dimensional" representation.

Similar arguments hold for $\tilde{L}_{\mu\nu}$ & T^A. But now, the rep. is infinite dimensional.

Spin

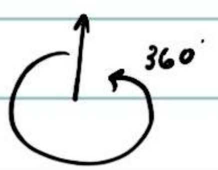
(assume const. field wlog).

- Fields that transform as scalars under the 3-dimensional rotation group
  $\equiv$ spin 0 fields

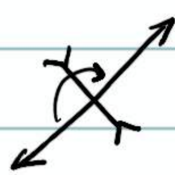
- Fields that transform as 3-vectors under the 3-dimensional rotation group.
  $\equiv$ spin 1 fields.

- other spins (later).

(Heuristic)



$s = \frac{360°}{\theta} = 1$. vector field.



$s = \frac{360}{\theta} = 2$. g-waves

"Spin - 1" Vector Fields

$$\Phi_\alpha \xmapsto{\Lambda} [M(\Lambda)]_\alpha{}^\beta \Phi_\beta(\Lambda^{-1}x)$$

$$[M(\Lambda)]_\alpha{}^\beta = \delta_\alpha{}^\beta - \frac{i\varepsilon}{2} a^{\mu\nu} (\Sigma_{\mu\nu})_\alpha{}^\beta$$

$$(\Sigma_{\mu\nu})_\alpha{}^\beta = i (g_{\mu\alpha} \delta_\nu^\beta - g_{\nu\alpha} \delta_\mu^\beta)$$

(The above Σ satisfies $[\Sigma, \Sigma] = $ "Σ" i.e. the Lorentz Algebra)

This is a "vector" representation of the Lorentz group.

In anticipation that we will eventually get to photons, let $\Phi^\nu(x) \equiv A^\nu(x)$

{ Note : $A_\alpha(x) \xmapsto{\Lambda} [M(\Lambda)]_\alpha{}^\beta A_\beta(\Lambda^{-1}x)$

for $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon a^\mu{}_\nu$

$$= \left[\delta_\alpha^\beta + \frac{1}{2}\varepsilon a^{\mu\nu}\left(g_{\mu\alpha}\delta_\nu^\beta - g_{\nu\alpha}\delta_\mu^\beta\right)\right] A_\beta(\Lambda^{-1}x)$$

$$= \delta_\alpha^\beta + \frac{1}{2}\varepsilon\left(a_\alpha{}^\beta - a^\beta{}_\alpha\right) A_\beta(\Lambda^{-1}x)$$

$$= \left(\delta_\alpha^\beta - \varepsilon\, a^\beta{}_\alpha\right) A_\beta(\Lambda^{-1}x)$$

$$\approx \Lambda_\alpha{}^\beta A_\beta(\Lambda^{-1}x) \quad \text{(as expected)}$$ }

See A. Zee, pg 38 [Ignore this pg if you want].
(eye towards photons)

We are interested in spin-1 vector fields.
What should the Lagrangian be?

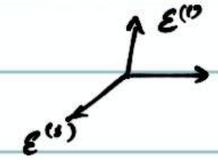
Well we should describe it with a four vector $A_\mu(x)$. [components].

To remain close to our treatment of spin-0 (scalar) fields, we would like

$$(\partial_\mu \partial^\mu + m^2) A^\nu(x) = 0 \qquad \text{massive fields}$$

$$[k^2 - m^2] A^\nu(k) = 0. \qquad \text{(Note, for photons } m \to 0\text{)}.$$

But a spin-1 particle (at least when at rest) has 3-degrees of freedom



corresponding to three spin orientations. So the field must be restricted. A way to do this in a Lorentz friendly way is $\partial_\mu A^\mu = 0$
(or in Fourier space $k_\mu A^\mu(k) = 0$.

It is possible to combine $(\partial^2 + m^2) A^\nu = 0$
& $\partial_\mu A^\mu = 0$
into one eq: $(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^\alpha + m^2 A^\alpha = 0$.

↓ [Ignore if you want]

It is possible to combine $(\partial^2 + m^2)A^\nu = 0$

& $\partial_\mu A^\mu = 0$

into one eq : $(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu + m^2 A_\mu = 0$.

Note $\partial^\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu + m^2\,\partial^\mu A_\mu = 0$

$\Rightarrow \quad (\partial_\nu\partial^2 - \partial^2\partial_\nu)A^\nu + m^2\partial^\mu A_\mu = 0$

$\Rightarrow \quad m^2\partial^\mu A_\mu = 0$

$\Rightarrow \quad \partial^\mu A_\mu = 0$ for $m \neq 0$. ✓

∴ Using $\partial^\mu A_\mu = 0$ in $(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu + m^2 A_\mu = 0$

$\Rightarrow (\partial^2 + m^2)A_\mu = 0$. ✓

What is the Lagrangian density corresponding to $(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu + m^2 A_\mu = 0$?

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu = 0.$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Lagrangian for massive vector fields (Proca)

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$.

This is like E&M, but with a mass term.
This mass term simplifies things (avoids complication from Gauge invariance).

Equations of Motion:

check this!

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \qquad \frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu$$

$\therefore$ EOM $\Rightarrow$ $$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0}$$ Proca Eq. of motion.

Note that $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ $\qquad \because F^{\mu\nu} = -F^{\nu\mu}$

$$\Rightarrow \underline{\underline{\partial_\nu A^\nu = 0}}$$

Furthermore using $\partial_\mu A^\mu = 0$ in the equations of motion,

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

$$\Rightarrow \quad \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) + m^2 A^\nu = 0$$

$$\Rightarrow \quad \left(\partial^2 + m^2 \right) A^\nu - \partial_\nu \cancel{\left(\partial_\mu A^\mu \right)} = 0$$

uses $\partial_\mu A^\mu = 0$

$$\Rightarrow \quad \left(\partial^2 + m^2 \right) A^\nu = 0$$

ie. $A^\nu(x)$ satisfies the Klein Gordon equation.

Recall: This is the same eq. satisfied by massive scalar fields. What is different here is that along with $(\partial^2 + m^2) A^\nu = 0$, A^ν must also satisfy the additional constraint $\partial_\mu A^\mu = 0$.

Lecture 17

Spin-1 vector field : $(\partial^2+m^2)A_\alpha = 0 \qquad \partial^\alpha A_\alpha = 0$

The plane wave solutions.

$$A^\mu(x) = \xi^\mu(p^\nu)e^{-ip\cdot x}$$ satisfy the eq. $(\partial^2+m^2)A^\mu = 0$

as long as $p^2 = m^2$, ie $\omega_p = p^0 = \pm\sqrt{|\vec{p}|^2+m^2}$

{As we had with scalar fields, there are +ve & -ve frequency solutions.

$$A^\mu(x) \sim \xi^\mu(\vec{p})e^{-ip\cdot x}, \quad \tilde{\xi}^\mu(\vec{p})e^{ip\cdot x}$$ }.

There is the additional constraint that

$\partial_\mu A^\mu = 0 \Rightarrow p_\mu \xi^\mu = 0$ ie A^μ is "transverse".

Let us move to the rest frame. (consider +ve frequency).

$$p_\mu = (m, \vec{0}) \Rightarrow p_\mu \xi^\mu(0) = 0 \Rightarrow \xi^\mu(0) = \alpha\underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\xi^{\mu(\lambda)}(0)} + \beta\underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}} + \gamma\underbrace{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}$$

Boost in (for example the z direction) ie, $p_\mu = \{\omega_p, 0, 0, |\vec{p}|\}$

$$\xi^{\mu(1)}(\vec{p}) = \{0,1,0,0\} \quad \xi^{\mu(2)}(\vec{p}) = \{0,0,1,0\} \quad \xi^{\mu(3)}(\vec{p}) = \frac{1}{m}\{|\vec{p}|, 0, 0, \omega_p\}$$

Conditions satisfied by the $\xi^{(\lambda)}(\vec{p})$ vectors.

Orthonormality: $g_{\mu\nu}\, \xi^{\mu(\lambda)} \xi^{\nu(\lambda')}(\vec{p}) = -\delta^{\lambda\lambda'}$

$\because$ ξ's are spacelike for time like p^μ.

Completeness: $\sum_\lambda \xi^{(\lambda)\mu}(\vec{p})\, \xi^{(\lambda)\nu}(\vec{p})$
(outer product)

$$= \underbrace{-\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}\right)}_{\mathcal{P}^{\mu\nu}} \leftarrow \text{projection operator}$$

(projects $\perp$ to momentum)

You can of course orient your axes along & $\perp$ to p^μ, which would lead to somewhat more different expressions for $\xi^{\mu(\lambda)}(\vec{p})$.

More generally, the $\xi^{\mu(\lambda)}(\vec{p})$ must (i) transform as Lorentz vectors (ii) be $\perp$ to p_μ (iii) (for convenience) be orthonormal, and span the space of solutions.

Having explored the plane wave solutions, let us move to quantization & mode expansions.

Returning back to the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu .$$

$$\Pi^{0\nu} = \frac{\partial \mathcal{L}}{(\partial_0 A_\nu)} = -F^{0\nu}$$

Note $\Pi^{00} = 0$! ie A^0 is not dynamical. (reflection of the constraint that $\partial_\mu A^\mu = 0$. The Hamiltonian is less democratic).

Hence can drop A^0 from our considerations, ie. A^0 is not dynamical.

Quantization:

$$\left[A_i(x) , \Pi^{0j}(y) \right] = i\, \delta(\bar{x} - \bar{y})\, \delta_i^j \qquad x^0 = y^0 .$$

All other commutators are zero.

The mode expansion

$$A^\mu(x) = \int (dp) \sum_\lambda \left(\xi^{\mu(\lambda)}(\vec{p})\, a(\vec{p},\lambda) e^{-ip\cdot x} + \left(\xi^{\mu(\lambda)}(\vec{p})\right)^* a^\dagger(\vec{p},\lambda) e^{ip\cdot x} \right)$$

where $\left[a(\vec{p},\lambda), a^\dagger(\vec{p}',\lambda')\right] = 2\omega_p\, \delta(\vec{p}-\vec{p}')\, \delta_{\lambda\lambda'}$

(all other commutators vanish).

Propagator : (directly in momentum space).

$$i\Delta_{\mu\nu}(k) = \frac{-i}{k^2-m^2+i\varepsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right) =: \frac{i\,\mathcal{P}_{\mu\nu}(k)}{k^2-m^2+i\varepsilon}.$$

To see this, verify that

$$i\Delta^{\mu\nu}(x-y) \equiv \langle 0|T\left(A^\mu(x)A^\nu(y)\right)|0\rangle$$

from outer product!

$$= \int (dk)\left(\Theta(x^0-y^0)e^{-ik\cdot(x-y)} + \Theta(y^0-x^0)e^{ik\cdot(x-y)}\right)\mathcal{P}^{\mu\nu}(k)$$

Review!

$$= \int d^4k\, \frac{i e^{-ik\cdot(x-y)}\,\mathcal{P}^{\mu\nu}(k)}{k^2-m^2+i\varepsilon}.$$

Example: Lagrangian with interactions.

$$\mathcal{L} = \underbrace{-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu}_{\mathcal{L}(A_\mu)} + \underbrace{e A_\mu J^\mu(\psi)}_{\text{interaction}} + \mathcal{L}(\psi)$$

complex scalar spin 0

$$\left(A_\mu \left(\psi^\dagger \partial^\mu \psi - \psi \partial^\mu \psi^* \right) + |\partial\psi|^2 - M^2|\psi|^2 \right)$$

$$\therefore \quad \partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu$$

If $\partial_\nu J^\nu = 0$ (conserved current, eg

then $\partial_\nu A^\nu = 0$ (hence the structure is still preserved).

Feynman Rules (for A_ν part).

internal lines :  $\quad \dfrac{-i\left(g_{\mu\nu} - \dfrac{k_\mu k_\nu}{m^2}\right)}{k^2 - m^2}$

external lines :

incoming 

$\xi^{\mu(\lambda)}(\vec{k})$

outgoing 

$\left(\xi^{\mu(\lambda)}\right)^*(\vec{k})$.

Vertex:  $\rightarrow \quad ie\,(p_\mu + p_\mu')$

Massless Spin 1:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \cancel{+\frac{1}{2} m^2 A^\nu A_\nu}$$

$m=0$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

making contact with usual formulation.

This is the Lagrangian for (Sourceless) E & M.

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$E_i = F_{0i} = -\frac{\partial \phi}{\partial t} - \frac{\partial \vec{A}}{\partial t}$$

$$B_k = \frac{1}{2} \varepsilon_{ijk} F_{ij} = \nabla \times \vec{A}$$

where $A^\mu = (\phi, \vec{A})$

The eq. of motion $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \boxed{\nabla \cdot \vec{E} = 0 \quad \& \quad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}}$

in absence of sources

Note that $F_{\mu\nu}$ satisfies the Bianchi identity.

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$$

$$\Rightarrow \boxed{\nabla \cdot \vec{B} = 0 \quad \& \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}} \leftarrow \text{always true.}$$

Lecture 18

Massless Spin 1: (Review).

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

components of $\vec{E}$-field.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad E_i = F_{0i} \;, \quad B_i = \frac{1}{2} \varepsilon_{ijk} F^{jk}$$

components of $\vec{B}$ field.

$$\left.\begin{aligned} &\text{EOM} \Rightarrow \partial_\mu F^{\mu\nu} = 0 \\ &\text{Bianchi} \Rightarrow \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \end{aligned}\right\} \text{Source free Maxwell eqns.}$$

Degrees of freedom

Note that previously (when $m \neq 0$)

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \Rightarrow \partial_\mu A^\mu = 0 \qquad (\text{we used } \partial_\mu \partial_\nu F^{\mu\nu} = 0)$$

(equivalent to $A^0 = -\frac{\partial_i F^{0i}}{m^2}$)

However for $m = 0$, no such constraint is available.

So it seems like we have 4 real degrees of freedom in the massless case (compared to 3 in the massive case)

But this seems all wrong. We expected our "photon" to have two degrees of freedom. (2 polarizations).

* Recall that for a massive A_μ, we could go to the rest frame $p^\mu = (m, \vec{0})$ and define 3 unique polarizations. This was associated with the $SO(3)$ symmetry. For a massless particle, no such rest frame exists. Nevertheless a unique $\hat{p}$ direction exists and there is the $SO(2)$ symmetry of the plane perpendicular to $\hat{p}$. This

corresponds to 2 polarizations.

Thus we need to get from 4 DOF $\to$ 2 DOF for the massless case. How do we get there?

Recall that $\Pi^0 = 0$, ie A^0 is non dynamical.
You can see this explicitly from $\nabla \cdot \vec{E} = 0$.

$$\Rightarrow \nabla^2 A_0 + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0 \Rightarrow A_0(\vec{x}) = \int d^3x' \frac{\nabla \cdot \left(\frac{\partial \vec{A}}{\partial t}\right)}{4\pi |\vec{x} - \vec{x}'|}.$$

Thus specifying $\vec{A}$ & $\dot{\vec{A}}$ is sufficient to get $A_0(\vec{x})$.
Thus we have only 3 dof. But we need to get to two!

Important "symmetry": Gauge Invariance.
Notice that

$$A^\mu(x) \longmapsto A^\mu(x) + \partial^\mu f(x)$$

$$\Rightarrow F_{\mu\nu}(x) \longmapsto F_{\mu\nu}(x) \,! \qquad \text{for arbitrary } f(x).$$

& hence $\partial_\mu F^{\mu\nu} = 0 \mapsto \partial_\mu F^{\mu\nu} = 0$

ie the equations of motion (& $F_{\mu\nu}$) do not change under $A_\mu \to A_\mu + \partial_\mu f$ for arbitrary $f(x)$.

This is called Gauge Invariance:

This is a pretty large symmetry of our theory. You are allowed to add the 4-gradient of an arbitrary function of spacetime!

How should we interpret this?

- We should view this symmetry as a redundancy in our description of the physical state of our system. That is, $A_\mu(x)$ & $A_\mu(x) + \partial_\mu f(x)$ are actually the same physical state.

- You might be tempted to revert to $\vec{E}$ & $\vec{B}$ instead of $A_\mu(x)$ which do not have this redundancy. But quantum phenomenon like the Ahranov Bohm effect require $\vec{A}(\vec{x},t)$. So it seems necessary. Also charged fields (like dirac-fermions -electrons) require $A_\mu(t,\vec{x})$.

- We can turn the fact that $A_\mu(x) + \partial_\mu f(x)$ represents the same physical state for arbitrary $f(x)$ to our advantage.

By an astute choice of $f(x)$ (ie choice of gauge) we can simplify our problems significantly. It also helps in removing the extra degree of freedom we were worried about.

Examples of Gauge Choices:

1. Coulomb or Radiation Gauge:
Suppose a solution $A^\mu(t,\vec{x})$ is known. Now Consider the following choice of $f(t,\vec{x})$

$$f(x) = -\int d^3x' \frac{\partial_{i'} A^i(t,\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} + \lambda(\vec{x})$$

will be specified later

Then
$$\tilde{A}^\mu(x) = A^\mu(x) + \partial^\mu f(x)$$

$$\mu = 0 \Rightarrow \quad \tilde{A}^0(x) = A^0(x) - \int d^3x' \, \partial^0 \left(\frac{\partial_i A^i(t,\vec{x})}{4\pi|\vec{x}-\vec{x}'|}\right) \qquad \partial^0 = \partial_t$$

$$= 0$$

why: because $A^0(x) = \int d^3x' \, \partial_0 \frac{(\partial_i A^i)}{4\pi|\vec{x}-\vec{x}'|}$ is a soln. to $\nabla\cdot\vec{E} = 0$

Thus $\tilde{A}^0(x) = 0$ in this Gauge.

For $\mu = i$

$$\tilde{A}^i(x) = A^i(x) - \partial^i \int d^3x' \frac{\partial_{i'} A^i(t,\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} + \partial^i \lambda(\vec{x}).$$

Recall that $\nabla^2 \tilde{A}^0 + \frac{\partial}{\partial t}(\partial_i \tilde{A}^i) = 0$ (from $\nabla \cdot \vec{E} = 0$)

But since $\tilde{A}^0 = 0$, we have

$$\frac{\partial}{\partial t}(\partial_i \tilde{A}^i) = 0 \quad \Rightarrow \quad \partial_i \tilde{A}^i(t,\vec{x}) = f(\vec{x}).$$

We can now use the additional freedom in $\lambda(\vec{x})$ to set $\partial_i \tilde{A}^i(t,\vec{x}) = 0$.

To see this explicitly, consider specifying $f(t,\vec{x})$ in two steps.

First $f_{(1)}(t,x) = -\int d^3x' \frac{\partial_{i'} A^i(t,\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$

This yields. $A^\mu_{(1)}(x) = A^\mu(x) + \partial^\mu f_{(1)}(t,\vec{x})$

$$\Rightarrow A^0_{(1)}(x) = 0 \quad \& \quad \frac{\partial}{\partial t}\left(A^i_{(1)}(t,\vec{x})\right) = 0$$

Now, consider:

$$f_{(2)}(\vec{x}) = -\int d^3x' \, \frac{f(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$$

$$\therefore \quad A^{\mu}_{(2)}(x) = A^{\mu}_{(1)}(x) + \partial^{\mu} f_{(2)}(\vec{x}).$$

does not affect the time component.

$$A^{0}_{(2)}(x) = A^{0}_{(1)}(x) + 0 = 0$$

$$A^{i}_{(2)}(x) = A^{i}_{(1)}(x) + \partial^{i} \int d^3x' \, \frac{f(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \quad \partial_i A^{i}_{(2)}(x) = f(\vec{x}) + \nabla^2 \int d^3x' \, \frac{f(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$$

$$= f(\vec{x}) - f(\vec{x})$$

$$= 0$$

Thus, $\partial_i A^{i}_{(2)}(x) = 0$ & $A^{0}_{(2)}(x) = 0$ ← Coulomb/Radiation Gauge.

These constraints on A^{μ}, now correctly yield 2 dof as we needed. Also $\partial_i A^i = 0$ $\Rightarrow$ the fields are transverse in 3D sense., Note : $\pi^0 = 0$ only reduces 1 dof. Gauge freedom removed another.

Explicit in Coulomb/Radiation Gauge : $A^0 = 0$

$$\partial_i A^i = 0 .$$

But this Gauge condition cannot be written in a nice co-variant form.

Other choices : Lorentz Gauge : $\partial_\mu A^\mu = 0$
(Recall this is not imposed by EOM for $m \neq 0$).

How do we choose $f(t, \vec{x})$ to impose this?
Again start with a $A^\mu(t, \vec{x})$ that is known.

$$\tilde{A}^\mu(x) = A^\mu(x) + \partial^\mu f(x)$$

$$\partial_\mu \tilde{A}^\mu = 0 \Rightarrow \quad \partial^2 f(x) = -\partial_\mu A^\mu(x) .$$

The solution to this equation yields the requisite $f(x)$. Note that for any solution we can always add $\lambda(x)$ such that $\partial^2 \lambda(x) = 0$
Then $\tilde{f}(t, \vec{x}) = f(x) + \lambda(x)$ would still yield the Lorentz condition.

When we move towards Quantization, we have the choice of Gauge to work in. Each comes with its own problems.

Let us first consider the Coulomb Gauge. $\begin{pmatrix} A^0 = 0 \\ \nabla \cdot \vec{A} = 0 \end{pmatrix}$

Since $A^0 = 0$, we need to only think about A^i & $\pi^i = -F^{0i} = E^i$. We impose the canonical commutation relationships (at least try).

$$[A_i(t, \vec{x}), \pi^j(t, \vec{y})] = i\delta(\vec{x} - \vec{y})\delta_i^j$$

$E^i = -F^{0i} = \pi^i$

$$\Rightarrow [A_i(t, \vec{x}), E^j(t, \vec{y})] = -i\delta(\vec{x} - \vec{y})\delta_i^j$$

(and all others being zero)

But this can't be correct. The quantization condition is inconsistent with the constraints $\nabla \cdot \vec{A} = \nabla \cdot \vec{E} = 0$ | $0 = \partial^i \partial_j [A_i, E^j] \neq i\nabla^2 \delta(\vec{x} - \vec{y})$

The correct structure should be (Quantization with constraints).

$$[A_i(t, \vec{x}), E_j(t, \vec{y})] = i\left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right)\delta(\vec{x} - \vec{y}).$$

To write down the mode expansion, we need the plane wave solns. (with $p^2 = 0$)

$\vec{A}(t,x) = \vec{\xi}(\vec{p})e^{-ip\cdot x}$, But $\nabla\cdot\vec{A} = 0 \Rightarrow \vec{\xi}(\vec{p})\cdot\vec{p} = 0$

$\Rightarrow \vec{\xi}(\vec{p})$ is spanned by a two dimensional vector space $\perp$ to $\vec{p}$. Hence the mode expansion.

$$\vec{A}(t,\vec{x}) = \int (dp) \sum_{\lambda=1}^{2} \left(\vec{\xi}^{(\lambda)}(\vec{p})\, a(\vec{p},\lambda) e^{-ip\cdot x} + \vec{\xi}^{*(\lambda)}(\vec{p})\, a^{\dagger}(\vec{p},\lambda)\, e^{ip\cdot x} \right)$$

where $\vec{p}\cdot\vec{\xi}^{(\lambda)}(\vec{p}) = 0$ & $\vec{\xi}^{(\lambda)}(\vec{p}) \cdot \vec{\xi}^{(\lambda')}(\vec{p}) = \delta_{\lambda\lambda'}$ inner

outer

$$\sum_{\lambda} \xi^{i(\lambda)}(\vec{p})\, \xi^{j(\lambda)}(\vec{p}) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}$$

Our quantization is consistent with

$$[a(\vec{p},\lambda), a^{\dagger}(\vec{p}',\lambda')] = 2\omega_p \delta(\vec{p}-\vec{p}')\, \delta_{\lambda\lambda'}$$

(all others zero)

The normal ordered Hamiltonian.

$$H = \int (dp)\ |\vec{p}|^2 \sum_{\lambda=1}^{2} a^\dagger(\vec{p},\lambda)\, a(\vec{p},\lambda).$$

Propagator?

transverse.

$$\mathcal{D}^T_{ij}(p) = \frac{i}{p^2}\left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right)$$

[Not so great because we have lost nice Lorentz structure But otherwise fine for the moment]

Now let us consider the <u>Lorentz Gauge</u>.

$\partial_\mu A^\mu = 0$.

Instead of finding a $f(t, \vec{x})$ that imposes this condition, we can equivalently write a Lagrangian of the form.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$$

The EOM $\Rightarrow \partial_\mu F^{\mu\nu} + \partial^\nu (\partial_\alpha A^\alpha) = 0$

$\Rightarrow \partial^2 A^\nu = 0$

(which is identical to the EOM for $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ with $\partial_\mu A^\mu = 0$)

We haven't yet used the constraint; we have just constructed a theory where $A^\nu(x)$ satisfies the same eq. of motion as the constrained one. We will do that (and hence deal with the residual d.o.f later).

First note that in this new theory,

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = -\partial_\mu A^\mu \neq 0$$ and for the

moment all 4 d.o.f are dynamical

Now we quantize all 4 d.o.f., and finally impose the condition $\partial_\mu A^\mu = 0$ in a somewhat non-trivial way.

$$[A_\mu, \Pi_\nu] = i g_{\mu\nu}\, \delta(\vec{x}-\vec{y})$$

$$A_\mu(\vec{x}) = \int (dp) \sum_{\lambda=0}^{3} \left(\xi^{(\lambda)\mu}(\vec{p})\, a(\vec{p},\lambda) e^{-ip\cdot x} + \xi^{(\lambda)\mu}(\vec{p})\, a^\dagger(\vec{p},\lambda) e^{ip\cdot x} \right)$$

This unfortunately leads to -ve normed states for $\lambda = 0$. The constraints $\partial_\mu A^\mu = 0$ should help somehow, but how?

Turns out that we need to impose $\langle\psi|\partial_\mu A^\mu|\psi\rangle = 0$

rather than $\partial_\mu A^\mu = 0$. (← this becomes incompatible with commutation relations).

Feynman Rules (for $\mathcal{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$

$$-i\,\frac{g_{\mu\nu}}{k^2}$$

$$\xi^{\mu(\lambda)}(k).$$

$$\left(\xi^{\mu(\lambda)}\right)^*(k)$$

Lot more here, see D. Tong's notes as well as P. Stevenson's notes for details.

Spin 1/2 fields

Recall : $\Phi_\alpha(x) \xmapsto{\Lambda} \delta_\alpha^\beta + \frac{i\varepsilon}{2} a^{\mu\nu} (\Sigma_{\mu\nu})_\alpha^\beta \Phi_\beta(\Lambda^{-1}x)$

For spin 1/2 fields.

$\Sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ where $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are 4×4 matrices

satisfying $\{\gamma^\mu, \gamma^\nu\} = 4g^{\mu\nu} \mathbb{1}$

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 4g^{\mu\nu}\mathbb{1}$

Note $\gamma_\mu = g_{\mu\nu}\gamma^\nu$; that is $\gamma^0 = \gamma_0$

$\gamma^i = -\gamma_i$

Note that $\Sigma_{\mu\nu}$ satisfies the Lorentz algebra.

$[\text{"}\Sigma\text{"}, \text{"}\Sigma\text{"}] = \text{"}\Sigma\text{"}$ (Check in HW).

Mathematical Preliminaries

Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k .$$

$\{\sigma_i, \mathbb{1}\}$ form a basis for 2×2 Hermitian Matrices.

A explicit form of the γ^μ matrices.

$$\gamma^0 = \left(\begin{array}{c|c} 1 & 0 \\ \hline & -1 \end{array}\right) \qquad \gamma^i = \left(\begin{array}{c|c} & \sigma_i \\ \hline -\sigma_i & \end{array}\right)$$

(this is the Dirac-Pauli Form, There are others.)

Note: $(\gamma^0)^\dagger = \gamma^0$ $\qquad (\gamma^i)^\dagger = -\gamma^i$ $\leftarrow$ Not hermitian.

$(\gamma^0)^2 = \mathbb{1}$ $\qquad (\gamma^i)^2 = -1$

But $\qquad \overline{\gamma^\mu} \equiv \gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$

$\Rightarrow \qquad \overline{\Sigma}^{\mu\nu} \equiv \gamma^0 (\Sigma^{\mu\nu})^\dagger \gamma^0 = \Sigma^{\mu\nu}$

} will need this soon.

Lecture 19

Spin ½ fields

Recall : $\Phi_\alpha(x) \overset{\Lambda}{\longmapsto} \delta_\alpha^\beta - \frac{i\varepsilon}{2} a^{\mu\nu} (\Sigma_{\mu\nu})_\alpha^\beta \Phi_\beta(\Lambda^{-1}x)$

For spin ½ fields.

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$ where $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are 4×4 matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}$

Note $\gamma_\mu = g_{\mu\nu}\gamma^\nu$; that is $\gamma^0 = \gamma_0$

$\gamma^i = -\gamma_i$

Note that $\Sigma_{\mu\nu}$ satisfies the Lorentz algebra.
$[$ "Σ", "Σ" $]$ = "Σ" (Check in HW).

Mathematical Preliminaries

Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k .$$

$\{\sigma_i , \mathbb{1}\}$ form a basis for 2×2 Hermitian Matrices.

A explicit form of the γ^μ matrices.

$$\gamma^0 = \left(\begin{array}{c|c} 1 & 0 \\ \hline & -1 \end{array}\right) \qquad \gamma^i = \left(\begin{array}{c|c} & \sigma_i \\ \hline -\sigma_i & \end{array}\right)$$

(this is the Dirac-Pauli Form, there are others.)

Note: $(\gamma^0)^\dagger = \gamma^0$ $\qquad (\gamma^i)^\dagger = -\gamma^i$ $\leftarrow$ Not hermitian.

$(\gamma^0)^2 = \mathbb{1}$ $\qquad (\gamma^i)^2 = -1$

But $\overline{\gamma^\mu} \equiv \gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$

$\Rightarrow \quad \overline{\Sigma^{\mu\nu}} \equiv \gamma^0 (\Sigma^{\mu\nu})^\dagger \gamma^0 = \Sigma^{\mu\nu}$

will need these soon.

Recall that $\psi(x) \xmapsto{\Lambda} M(\Lambda)\psi(\Lambda^{-1}x)$.

For $F(x)$ to be a Lorentz scalar, $F(x) \xrightarrow{\Lambda} F(\Lambda^{-1}x)$.

Is $\psi^{\dagger}(x)\psi(x)$ a Lorentz scalar? That is

$$\psi^{\dagger}(x)\psi(x) \xmapsto{\Lambda} \psi^{\dagger}(\Lambda^{-1}x)\psi(\Lambda^{-1}x) \; ? \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\psi^{\dagger} = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$$

$$\psi^{\dagger}(x)\psi(x) \xmapsto{\Lambda} \psi^{\dagger}(\Lambda^{-1}x)M^{\dagger}(\Lambda)\, M(\Lambda)\psi(\Lambda^{-1}x).$$

So we need $M^{\dagger}(\Lambda)M(\Lambda) = \mathbb{1}$

ie $M(\Lambda)$ is unitary. Lets check if this is true

We will use $\quad M(\Lambda) = \mathbb{1} - \frac{i\varepsilon}{2} a^{\mu\nu} \Sigma_{\mu\nu} \qquad \varepsilon \ll 1.$

$$M^\dagger(\Lambda) = \mathbb{1} + \frac{i\varepsilon}{2} a^{\mu\nu} \Sigma^\dagger_{\mu\nu}$$

$$\therefore \quad M^\dagger(\Lambda)\, M(\Lambda) = \mathbb{1} + \frac{i\varepsilon}{2} a^{\mu\nu}\left(\Sigma^\dagger_{\mu\nu} - \Sigma_{\mu\nu}\right)$$

But $\Sigma^\dagger_{\mu\nu} \neq \Sigma_{\mu\nu}$ so this is not going to work!

$\therefore \quad M^\dagger(\Lambda) M(\Lambda) \neq \mathbb{1}$. (check in HW).

But we previously checked that

$$\bar{\Sigma}_{\mu\nu} = \gamma^0 \Sigma^\dagger_{\mu\nu} \gamma^0 = \Sigma_{\mu\nu} \Rightarrow \quad M^\dagger(\Lambda)\gamma^0 = \gamma^0 + \frac{i\varepsilon}{2} a^{\mu\nu} \Sigma^\dagger_{\mu\nu}\gamma^0$$

$$\Rightarrow \quad M^\dagger(\Lambda)\gamma^0 M(\Lambda) = \gamma^0 \qquad \text{(check in HW)}$$

Motivated by this, let try $\psi^\dagger(x)\gamma^0\psi(x)$.

$$\psi^\dagger(x)\gamma^0\psi(x) \overset{\Lambda}{\longmapsto} \psi^\dagger(\Lambda^{-1}x) \underbrace{M^\dagger(\Lambda)\gamma^0 M(\Lambda)}_{\gamma^0} \psi(\Lambda^{-1}x)$$

$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x) \qquad \checkmark$$

Lorentz Scalar

Define $\quad \bar{\psi}(x) = \psi^\dagger(x)\gamma^0$

So $\bar{\psi}(x)\psi(x)$ is a Lorentz scalar.

Similarly, we can show that (in HW).

$\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector:

ie $\bar{\psi}(x)\gamma^\mu\psi(x) \xrightarrow{\Lambda} \Lambda^\mu{}_\nu\,\bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x)$

Using $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ (along with a derivative) we now construct a Lagrangian (which due to the above properties will be a Lorentz scalar)

<u>Dirac Lagrangian</u>!

$$\mathcal{L} = \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) \qquad \text{scalar}$$

to make the Lagrangian real. $\bar{\psi} = \psi^\dagger\gamma^0$

Note: Only ∂_μ (first order appears).

Note $[\psi] = M^{3/2}$ so as to make $S = \int d^4x\,\mathcal{L}$

mass dimension. have no mass dimension.

As usual we can find the equations of motion using the Euler Lagrange equations. Treating ψ, $\bar{\psi}$ as independent variables.

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu$$

$$\Rightarrow \quad \partial_\mu (i\bar{\psi}\gamma^\mu) + m\bar{\psi} = 0$$

— ①

Equivalently.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\gamma^\mu \partial_\mu - m)\psi \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

$$\therefore \quad (i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad — ②$$

① & ② are equivalent. Let us focus on ②. New "slash notation".

$$a^\mu \gamma_\mu = \not{a} \qquad \gamma^\mu \partial_\mu = \not{\partial}$$

$\therefore$ Dirac Equation.

$$(i\not{\partial} - m)\psi = 0$$

Next, We will discuss solutions of the Dirac equation. Before we do this, note the following "wonderful" property of $\psi(x)$.

Consider a rotation by an angle θ, around the z axis.

ie $\quad \Lambda = \exp\left[i\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \qquad (\Lambda = \mathbb{1} + \underset{\substack{\uparrow \\ \text{read off } a^{\mu\nu}.}}{a} + \ldots)$

The spinor will transform as (assuming it is const. in space)

$$\psi \overset{\Lambda}{\longmapsto} e^{-\frac{i}{2} a^{\mu\nu} \Sigma_{\mu\nu}} \psi$$

where $a^{12} = -\theta$ $\;a^{21} = \theta$, and all other $a^{\mu\nu} = 0$

$$\therefore \psi \xrightarrow{\Lambda} e^{-\frac{i}{2}\theta(\Sigma_{21} - \Sigma_{12})}\psi = e^{-i\theta\Sigma_{12}}\psi \qquad (\because \Sigma_{12} = -\Sigma_{21})$$

But $\Sigma_{12} = \frac{i}{4}[\gamma_1, \gamma_2]$

Using the explicit forms $\gamma_i = \left(\begin{array}{c|c} 0 & -\sigma_i \\ \hline \sigma_i & 0 \end{array}\right)$

$$\Rightarrow \quad \Sigma_{12} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\therefore \quad \psi \overset{\wedge}{\longmapsto} \exp\left[-i\frac{\theta}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\right]\psi$$

What if the rotation is $\theta \to 2\pi$?

$$\exp\left[+i\pi\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\therefore \quad \psi \overset{\wedge}{\xrightarrow{\theta \to \theta + 2\pi}} -\psi$$

So a 2π rotation does not bring a spinor back to itself!

[From here on, I recommend reading P. Stevenson's notes "QFTlec14.pdf". It has more details (nothing new below)]

With this "oddity" out of the way, let us now look for solutions of

$$(i\not{\partial} - m)\psi(x) = 0$$

First, note that $\underset{\text{act from left.}}{(i\not{\partial} + m)}(i\not{\partial} - m)\psi(x) = 0$

$$\Rightarrow \quad (\partial_\mu \partial^\mu + m^2)\psi = 0$$

ie each component now satisfies $(\underbrace{\partial_\mu \partial^\mu}_{\partial^2} + m^2)\psi_\alpha(x) = 0$

[Note: $(i\not{\partial} - m)\psi = 0$ is a stronger constraint. ie
$$(i\not{\partial} - m)\psi = 0 \underset{\not\Leftarrow}{\Rightarrow} (\partial^2 + m^2)\psi_\alpha = 0$$]

Plane wave solutions:

$$\psi_\alpha(x) = W(p^\mu)e^{-ip\cdot x} \qquad [p\cdot x = p_\mu x^\mu = p^0 x^0 - \vec{p}\cdot\vec{x}]$$

satisfied $(\partial^2 + m^2)\psi_\alpha(x) = 0$ provided $p^2 = p^\mu p_\mu = m^2$

For $(i\partial\!\!\!/ - m)\psi = 0$, we also need

$$(i\partial\!\!\!/ - m) W e^{-ip\cdot x} = 0$$

$$\Rightarrow (p\!\!\!/ - m) W(p^\mu) = 0.$$

Note that $(p\!\!\!/ + m)(p\!\!\!/ - m) W = 0$

$$\Rightarrow (p^2 - m^2) W = 0$$

$$\Rightarrow p^2 = m^2 \qquad \text{if } W \neq 0.$$

Thus, a plane wave solution is

$$\psi_\alpha(x) = W_\alpha(p^\mu) e^{-ip\cdot x} \qquad \text{where } p^2 = m^2$$

$$p^0 = \pm\sqrt{|\vec{p}|^2 + m^2}$$

Let us first consider $p^0 = +\sqrt{|\vec{p}|^2 + m^2}$

; define $u(\vec{p}) = W(p^\mu)\Big|_{p^0 = \sqrt{|\vec{p}|^2 + m^2}}$ "+ve energy soln" then,

$$(\not{p} - m)\, u(\vec{p}) = 0$$

Let us go to the rest frame $\vec{p} = (m, \vec{0})$

Then $\not{p} = p^0 \gamma_0 = m\gamma^0$

$$\therefore \quad (\not{p} - m)\, u(\vec{p}) = 0 \qquad\qquad \gamma^0 = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$$

$$\Rightarrow \quad m(\gamma^0 - \mathbb{1})\chi = 0 \qquad\qquad u(\vec{p} = 0) = \chi$$

$$\Rightarrow \quad \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -2 \end{array}\right) \begin{pmatrix} \\ \chi \\ \\ \end{pmatrix} = 0$$

$$\Rightarrow \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix} = \chi_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\equiv \chi^{1/2}} + \chi_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\equiv \chi^{-1/2}}$$

What is the solution in a boosted frame?

Ans: $$u^{(\pm 1/2)}(\vec{p}) = \frac{(\not{p} + m)}{\sqrt{E+m}} \chi^{(\pm \frac{1}{2})}$$

$\sqrt{E+m}$ ← normalization.

check: $$(\not{p} - m)\, u^{\pm\frac{1}{2}}(\vec{p}) = \frac{(\not{p} - m)(\not{p} + m)}{\sqrt{E+m}} \chi^{\pm\frac{1}{2}}$$

$$= 0$$

Why the labels? $\pm\frac{1}{2}$. Note that (from QM).
$\{\frac{1}{2}, \frac{-1}{2}\}$ are the eigenvalues of $J_z = \frac{1}{2}\sigma_3$ and
$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ are its eigenvectors. Thus, the

spinors are labelled by the "spin" along the z axis.
* But there is nothing special about the z-axis.
Convenient to label instead by spin along the $\vec{p}$ direction; using eigenvalues(λ) of the "Helicity" operator. $\frac{1}{2}\vec{\sigma}\cdot\hat{p}$ $\quad \vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$

(Note Helicity commutes with every momentum; is a good quantum # to label states)

So $\chi^{(\lambda)}(\hat{p}) = \underset{\uparrow}{R(\hat{p})}\, \chi^{\pm\frac{1}{2}}$

appropriate rep. for Rotation of spinors.

After rotation.
still labelled by $\pm\frac{1}{2}$ (sorry for the confusing notation!)

$$\chi^{\frac{1}{2}}(\hat{p}) = \begin{pmatrix} \cos(\theta/2) \\ \sin\left(\frac{1}{2}\theta\, e^{i\phi}\right) \\ 0 \\ 0 \end{pmatrix} ; \quad \chi^{-\frac{1}{2}}(\hat{p}) = \begin{pmatrix} -\sin\theta/2\, e^{-i\phi} \\ \cos\frac{1}{2}\theta \\ 0 \\ 0 \end{pmatrix}$$

where $\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

From now on.

$$u^{(\lambda)}(\vec{p}) = \frac{\not{p} + m}{\sqrt{E+m}} \, \chi^{(\lambda)}(\hat{p})$$

Rest frame spinor

$\lambda =$ spin in $\hat{p}$ direction.

$\Rightarrow \lambda = \frac{1}{2}$ $\xrightarrow{\hat{p}}$ $\qquad$ $\Leftarrow \lambda = -\frac{1}{2}$ $\xrightarrow{\hat{p}}$

A different & equivalent useful form of spinors:

$$u^{(\lambda)}(\vec{p}) = \left(\sqrt{E+m} + 2\lambda \sqrt{E-m} \, \gamma_5 \right) \chi^{(\lambda)}(\hat{p})$$

$$\left\{ \text{where} \quad \gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5 \quad ; \quad \{\gamma^\mu, \gamma^5\} = 0 \right.$$

$$(\gamma_5)^2 = \mathbb{1}$$

$$\gamma_5^\dagger = \gamma_5$$

$$\left. \gamma_5 = \left(\begin{array}{c|c} & 1 \\ \hline 1 & \end{array} \right) \right\}$$

Adjoint spinors:

$$\bar{u}(\vec{p}) = u^{\dagger}(\vec{p})\gamma^{0}$$

$$= \chi^{\dagger(\lambda)}\left[\sqrt{E+m} - 2\lambda\sqrt{E-m}\,\gamma_5\right] \qquad \text{(check!)}$$

note sign.

Two useful identities: (can be proved using above form & properties of γ matrices).

* $\bar{u}^{(\lambda')}(\vec{p})\, u^{(\lambda)}(\vec{p}) = 2m\,\delta_{\lambda\lambda'}$ ← Orthonormality.

* $\sum_{\lambda=\frac{1}{2},-\frac{1}{2}} u^{(\lambda)}(\vec{p})\,\bar{u}^{(\lambda)}(\vec{p}) = \not{p} + m$ ← Semi completeness (outer product).

[For proofs see P. Stevenson's notes].

Hint: often useful to pick $\hat{p} = \hat{z}$.

Lecture 20

Recall that

$$(i\not{\partial} - m)\psi = 0. \qquad\qquad \text{Dirac Equation.}$$

Plane wave solutions.

$$\psi(x) = W(p^\mu)\, e^{-ip\cdot x}$$

with $\quad p^2 = m^2 \quad \& \quad (\not{p} - m)W = 0.$

For $\quad p^0 = +\sqrt{|\vec{p}|^2 + m^2} \quad , \quad W \equiv u.$

ie $\quad \psi(x) = u(\vec{p})\, e^{-ip\cdot x} \qquad p^0 = \sqrt{|\vec{p}|^2 + m^2}$

$$u^{(\lambda)}(\vec{p}) = \left(\sqrt{E+m} + 2\lambda\sqrt{E-m}\,\gamma_5\right)\chi^{(\lambda)}(\hat{p})$$

rest frame spinor

$\lambda =$ spin in $\hat{p}$ direction.
$= \pm 1/2.$

Orthonormality : $\quad \bar{u}^{(\lambda)}(\vec{p})\, u^{(\lambda)}(\vec{p}) = 2m\,\delta_{\lambda\lambda'}$

Completeness : $\quad \sum_\lambda u^{(\lambda)}(\vec{p})\, \bar{u}^{(\lambda)}(\vec{p}) = \not{p} + m.$

"Anti-particle" spinors

$$\psi(x) = v(\vec{p})\, e^{ip\cdot x} \qquad p^0 = \sqrt{|\vec{p}|^2 + m^2}.$$

$$(\not{p} + m)\, v(\vec{p}) = 0$$

flip in helicity.

$$v^{(\lambda)}(\vec{p}) = (-1)^{\frac{1}{2}-\lambda}\, \gamma_5\, u^{(-\lambda)}(\vec{p})$$

phase factor, usually not important

Canonical Quantization

Recall that the Lagrangian density for the free Dirac Field was

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

The conjugate momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i\psi^\dagger$$

The Hamiltonian density

skip for now.

$$\begin{aligned}
\mathcal{H} &= \Pi\dot{\psi} - \mathcal{L} \\
&= i\psi^\dagger\dot{\psi} - \bar{\psi}(i\partial\!\!\!/ - m)\psi \\
&= (-\bar{\psi} i\gamma^i\partial_i\psi + m\bar{\psi}\psi) \\
&= -i\psi^\dagger\gamma^0\gamma^i\partial_i\psi + m\bar{\psi}\psi \qquad \vec{\gamma}\cdot\nabla = \gamma^i\partial_i \\
&= \psi^\dagger(-i\gamma^0\vec{\gamma}\cdot\nabla\psi + m\gamma^0)\psi
\end{aligned}$$

Following our "nose" we might want to impose the canonical commutation relationships

$$[\psi_\alpha(\vec{x}), \Pi_\beta(\vec{y})] = i\,\delta(\vec{x}-\vec{y})\,\delta_{\alpha\beta} \qquad \text{components win}$$

X ie $$[\psi_\alpha(\vec{x}), i\psi^\dagger_\beta(\vec{y})] = i\,\delta(\vec{x}-\vec{y})\,\delta_{\alpha\beta}$$

But as we will find out, imposing these conditions leads to unsurmountable difficulties.

We will find that we need to specify

$$\{\psi_\alpha(\vec{x}), \underset{\substack{\uparrow \\ i\psi^\dagger_\beta(\vec{y})}}{\Pi_\beta(\vec{y})}\} = i\delta(\vec{x}-\vec{y})\,\delta_{\alpha\beta}. \checkmark$$

anti-commutation relation.

To see the problem and its resolution, we need to write down the mode expansion (in terms of the plane wave solns) and creation and annihilation operators.

$$\psi_\alpha(x) = \int (dp) \sum_\lambda \left(u_\alpha^{(\lambda)}(\vec{p})\, b(\vec{p},\lambda)\, e^{-ip\cdot x} + v_\alpha^{(\lambda)}(p)\, d^\dagger(\vec{p},\lambda)\, e^{ip\cdot x} \right)$$

$b(\vec{p},\lambda)$ annihilates a particle with helicity λ & momentum $\vec{p}$

$d^\dagger(\vec{p},\lambda)$ creates an anti-particle with helicity λ & momentum $\vec{p}$

[Recall that $u^{(\lambda)}(\vec{p})e^{-ip\cdot x}$ and $v^{(\lambda)}(\vec{p})e^{ip\cdot x}$ both satisfy the dirac eq.

$$(\not{p}-m)u(\vec{p})=0 \qquad (\not{p}+m)v(\vec{p})=0]$$

Plugging this in the commutation relationships $[\psi_\alpha(\vec{x}), i\psi^\dagger_\beta(\vec{y})] = i\delta(\vec{x}-\vec{y})\delta_{\alpha\beta}$.

along with our usual $[b(\vec{k},\lambda), b(\vec{p},\lambda')] = 0$

$[b(\vec{k},\lambda), b^\dagger(\vec{p},\lambda')] = 2\omega_p\delta(\vec{k}-\vec{p})\delta_{\lambda\lambda'}$

$[d(\vec{k},\lambda), d(\vec{p},\lambda')] = 0$

$[d(\vec{k},\lambda), d^\dagger(\vec{p},\lambda')] = 2\omega_p\delta(\vec{k}-\vec{p})\delta_{\lambda\lambda'}$

we find

$[\psi_\alpha(\vec{x}), i\psi^\dagger_\beta(\vec{y})]$

$$i\int(dp)(dp')\sum_\lambda\sum_{\lambda'}\Big[u^{(\lambda)}_\alpha(\vec{p})b(\vec{p},\lambda)e^{-ip\cdot x} + v^{(\lambda)}_\alpha(\vec{p})d^\dagger(\vec{p},\lambda)e^{ip\cdot x},$$
$$v^{(\lambda')\dagger}_\beta(\vec{p}')d(\vec{p}',\lambda')e^{-ip'\cdot y} + u^{(\lambda')\dagger}_\beta(\vec{p}')b^\dagger(\vec{p}',\lambda')e^{ip'\cdot y}\Big]$$

$$= i\int(dp)(dp')\sum_{\lambda\lambda'}\Big(u^{(\lambda)}_\alpha(\vec{p})\,u^{\dagger(\lambda')}_\beta(\vec{p}')\,2\omega_p\,\delta(\vec{p}-\vec{p}')\delta_{\lambda\lambda'}e^{ip'\cdot y}e^{-ip\cdot x}$$

because $-[d,d^\dagger] = [d^\dagger,d]$

$$- v^{(\lambda)}_\alpha(\vec{p})\,v^{(\lambda')\dagger}(\vec{p}')\,2\omega_p\,\delta(\vec{p}-\vec{p}')\delta_{\lambda\lambda'}e^{ip\cdot x}e^{-ip'\cdot y}\Big)$$

$$= i\int(dp)\sum_\lambda\Big(u^{(\lambda)}_\alpha(\vec{p})\,u^{\dagger(\lambda)}_\beta e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - v^{(\lambda)}_\alpha(\vec{p})\,v^{\dagger(\lambda)}_\beta(\vec{p})e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big)$$

$\underline{\underline{x^0 = y^0}}$

(1)

In your homework, you will prove the following useful identities

$$\sum_\lambda u_\alpha^{(\lambda)}(\vec{p})\, u_\beta^{\dagger(\lambda)}(\vec{p}) = \left[(\not{p}+m)\gamma^0\right]_{\alpha\beta} = \omega_p - \vec{\gamma}\cdot\vec{p} + m\gamma^0$$

$$\sum_\lambda v_\alpha^{(\lambda)}(\vec{p})\, v_\beta^{\dagger(\lambda)}(\vec{p}) = \left[(\not{p}-m)\gamma^0\right]_{\alpha\beta} = \omega_p - \vec{\gamma}\cdot\vec{p} - m\gamma^0$$

Recall that $\int(dp) = \int\frac{d^3p}{(2\pi)^3}$ so we are free to flip the sign of $\vec{p}$ in the second term on the last line of the previous page. Then we have.

from sign flip

$$[\psi_\alpha(\vec{x}), i\psi_\beta(\vec{y})] = i\int(dp)\left(\left(\omega_p - \vec{\gamma}\cdot\vec{p}\gamma^0 + \gamma^0 m\right)_{\alpha\beta} - \left(\omega_p + \vec{\gamma}\cdot\vec{p} - \gamma^0 m\right)_{\alpha\beta}\right)e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$= 2i\int(dp)\,\left(m\gamma^0 - \vec{\gamma}\cdot\vec{p}\gamma^0\right)_{\alpha\beta} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$\neq\quad i\,\delta(\vec{x}-\vec{y})\,\delta_{\alpha\beta}.$$

So at the very minimum we need to change $[b,b^\dagger]$ or $[d,d^\dagger]$ relations.

Looking back through the calculation, you will notice that the culprit is the -ve sign between the $uu^\dagger$ & $vv^\dagger$ terms. This arose because of $[d, d^\dagger] = 2\omega\delta(\ )\delta_{\lambda\lambda'}$. If we flip the sign of this commutation relation for $[d, d^\dagger]$ ie,

$$[d(\vec{p},\lambda), d^\dagger(\vec{p}',\lambda')] = \overset{\text{extra -ve sign}}{\overset{\downarrow}{-}} 2\omega_p\, \delta(\vec{p}-\vec{p}')\,\delta_{\lambda\lambda'} \quad \text{then}$$

we would get $[\psi_\alpha(\vec{x}), i\psi_\beta(\vec{y})] = i\delta(\vec{x}-\vec{y})\delta_{\alpha\beta}$.

That was easy, right? Unfortunately, that -ve sign has disastrous consequences.

It leads to states with -ve norms, which in turn leads to violation of Unitarity (conservation of probability).

This is bad!

After many different attemps at fixing this, you might be lead to despair. (Try this! Useful exercise)

It turns out that the way make everything work is to impose anti-commutation relations

$$\{\psi_\alpha(\vec{x}), i\psi_\beta^\dagger(\vec{y})\} = i\delta(\vec{x}-\vec{y})\,\delta_{\alpha\beta}$$

& $$\{b(\vec{p},\lambda), b^\dagger(\vec{p}',\lambda')\} = 2\omega_p\, \delta^3(\vec{p}-\vec{p}')\,\delta_{\lambda\lambda'}$$

$$\{d(\vec{p},\lambda), d^\dagger(\vec{p}',\lambda')\} = 2\omega_p\, \delta^3(\vec{p}-\vec{p}')\,\delta_{\lambda\lambda'}$$

with $\{d, d\} = \{b, b\} = \{\psi, \psi\} = 0$
$\{d^\dagger, d^\dagger\} = \{b^\dagger, b^\dagger\} = \{\psi^\dagger, \psi^\dagger\} = 0$

These anti-commutation relations are the source of Fermi-dirac statistics & the Pauli exclusion principle. (fermions!)

Single fermion state: $|\vec{p},\lambda\rangle = b^\dagger(\vec{p},\lambda)|0\rangle$.

$\Downarrow$

Two fermions cannot occupy same state!

$$b^\dagger(\vec{p},\lambda)|\vec{p},\lambda\rangle = b^\dagger(\vec{p},\lambda)b^\dagger(\vec{p},\lambda)|0\rangle$$
$$= 0$$
$$\because \{b^\dagger, b^\dagger\} = 2b^\dagger b^\dagger = 0 \,!$$

(From P. Stevenson)

So consistency within QFT $\Rightarrow$ Pauli exclusion.

$\Downarrow$

Periodic Table!

Extra: Look up spin-statistics theorem

integer spin $\leftrightarrow$ commutation relation $\longrightarrow$ BE statistics.

half integer spin $\leftrightarrow$ anti-commutation relation. $\rightarrow$ FD statistic.

As we did with scalars, we want to calculate scattering amplitudes $\langle f|S|i\rangle$. requires evaluation of the S matrix

$$S = T\exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(x)\, d^4x\right)$$

$\mathcal{H}_I$ is the interaction hamiltonian.
(We will write $\mathcal{H}_I$ down explicity for fermions & scalars or fermions & gauge fields later)

Recall that we went through the process of defining normal ordering, time ordering, wick contractions, propagators, Wick's theorem, Feynman rules etc. for scalar fields

Now let us do it for fermions. It is rather similar, apart from the changes due to $\{,\}$ instead of $[,]$. and the u, v's.

Normal Ordering

Fermions

$$:b^\dagger b: \;=\; b^\dagger b$$

$$:bb^\dagger: \;=\; -\,b^\dagger b$$

notice -ve sign!

Bosons

$$:a^\dagger a: \;=\; a^\dagger a$$

$$:aa^\dagger: \;=\; a^\dagger a$$

+ve & -ve frequency parts & contractions

Fermions

(the index for components ψ_α is suppressed)

$$\psi_\alpha(x) = \psi_{+\alpha} + \psi_{-\alpha}$$

$$\psi_{+\alpha} = \int(dp)\sum_\lambda u_\alpha^{(\lambda)}(\vec{p})\,b(\vec{p},\lambda)\,e^{-ip\cdot x}$$

$$\psi_{-\alpha} = \int(dp)\sum_\lambda v_\alpha^{(\lambda)}(\vec{p})\,d^\dagger(\vec{p},\lambda)\,e^{ip\cdot x}$$

$$\bar{\psi}_\beta(y) = \bar{\psi}_{+\beta} + \bar{\psi}_{-\beta}$$

$$\bar{\psi}_{+\beta} = \bar{v}_\beta^{(\lambda)}(\vec{p})\,d(\vec{p},\lambda)\,e^{-ip\cdot x}$$

$$\bar{\psi}_{-\beta} = \bar{u}_\beta^{(\lambda)}(\vec{p})\,b^\dagger(\vec{p},\lambda)\,e^{ip\cdot x}$$

Bosons (complex)

$$\varphi(x) = \varphi_+ + \varphi_-$$

$$\varphi_+ = \int(dp)\,b(\vec{p})\,e^{-ip\cdot x}$$

$$\varphi_- = \int(dp)\,d^\dagger(\vec{p})\,e^{ip\cdot x}$$

$$\varphi^\dagger(y) = \varphi_+^\dagger(y) + \varphi_-^\dagger(y)$$

$$\varphi_+^\dagger(y) = \int(dp)\,d(\vec{p})\,e^{-ip\cdot x}\,;\quad \varphi_-^\dagger(y) = \int(dp)\,b^\dagger(\vec{p})\,e^{ip\cdot x}$$

$$\psi_\alpha(x)\bar{\psi}_\beta(y) = :\psi_\alpha(x)\bar{\psi}_\beta(y): + \{\psi_{+\alpha}(x),\bar{\psi}_{-\beta}(y)\}$$

$$\{\psi_{+\alpha}(x),\bar{\psi}_{-\beta}(y)\} = \underbrace{\psi_\alpha(x),\bar{\psi}_\beta}(y)$$

$$= (i\partial\!\!\!/ + m)_{\alpha\beta}\,\Delta_+(x-y)$$

$$\varphi(x)\varphi^\dagger(y) = :\varphi(x)\varphi^\dagger(y): + [\varphi_+(x),\varphi_-^\dagger(y)]$$

$$[\varphi_+(x),\varphi_-^\dagger(y)] = \underbrace{\varphi(x)\varphi^\dagger}(y)$$

$$= \Delta_+(x-y)$$

$$\Delta_+(x-y) = \int(dp)\,e^{-ip\cdot(x-y)}$$

Let us quickly review where we are at the present.

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\Pi = i\psi^{\dagger}$$

$$\psi_{\alpha}(x) = \int (dk) \sum_{\lambda} \left(u_{\alpha}^{(\lambda)}(\vec{k}) b(\vec{k},\lambda) e^{-ik\cdot x} + v_{\alpha}^{(\lambda)} d^{\dagger}(\vec{k},\lambda) e^{ik\cdot x} \right)$$

$$\left\{ \psi_{\alpha}(x), i\psi_{\beta}^{\dagger}(y) \right\} = i\delta(\vec{x}-\vec{y})\delta_{\alpha\beta} . \qquad x^{0} = y^{0}$$

$$\left\{ b(\vec{p},\lambda), b^{\dagger}(\vec{p}\,',\lambda') \right\} = 2\omega_{p}\,\delta(\vec{p}-\vec{p}\,')\,\delta_{\lambda\lambda'}$$

$$\left\{ d(\vec{p},\lambda), d^{\dagger}(\vec{p}\,',\lambda') \right\} = 2\omega_{p}\,\delta(\vec{p}-\vec{p}\,')\,\delta_{\lambda\lambda'}$$

$$\{b,b\} = \{d,d\} = \{d^{\dagger},d^{\dagger}\} = \{b^{\dagger},b^{\dagger}\} = \{\psi,\psi\} = 0$$

Normal ordering (fermionic).

$$:b^{\dagger}b: \; = \; b^{\dagger}b$$

$$:bb^{\dagger}: \; = \; -b^{\dagger}b$$

$$\psi_{\alpha}(x)\bar{\psi}_{\beta}(y) = \; :\psi_{\alpha}(x)\bar{\psi}_{\beta}(y): + (i\partial\!\!\!/ + m)\Delta_{+}(x-y)$$

Fermions | Bosons

$$\psi_\alpha(x)\bar{\psi}_\beta(y) = :\psi_\alpha(x)\bar{\psi}_\beta(y): + \{\psi_{+\alpha}(x), \bar{\psi}_{-\beta}(y)\}$$

$$\varphi(x)\varphi^\dagger(y) = :\varphi(x)\varphi^\dagger(y): + [\varphi_+(x), \varphi_-^\dagger(y)]$$

$$\{\psi_{+\alpha}(x), \bar{\psi}_{-\beta}(y)\} = \underbrace{\psi_\alpha(x), \bar{\psi}_\beta(y)}$$
$$= (i\partial\!\!\!/ + m)_{\alpha\beta}\Delta_+(x-y)$$

$$[\varphi_+(x), \varphi_-^\dagger(y)] = \underbrace{\varphi(x)\varphi^\dagger(y)}$$
$$= \Delta_+(x-y)$$

$$\Delta_+(x-y) = \int (dp)\, e^{-ip\cdot(x-y)}$$

Time Ordering

Fermions

$$T(\psi_\alpha(x)\psi_\beta(y)) = \begin{cases} \psi(x)\psi(y) & x^0 > y^0 \\ -\psi(y)\psi(x) & x^0 < y^0 \end{cases}$$

-ve sign!

Bosons

$$T(\phi(x)\phi(y)) = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & x^0 < y^0 \end{cases}$$

Propagator

Fermions

$$\overbrace{\psi_\alpha(x)\bar{\psi}_\beta(y)} \equiv T(\psi_\alpha(x)\bar{\psi}_\beta(y)) - :\psi_\alpha(x)\bar{\psi}_\beta(y):$$
$$= \Theta(x^0-y^0)(i\partial\!\!\!/+m)_{\alpha\beta}\Delta_+(x-y) + \Theta(y^0-x^0)(i\partial\!\!\!/+m)_{\alpha\beta}\Delta_-(x-y)$$
$$= (i\partial\!\!\!/+m)\, i\Delta_F(x-y)$$

Bosons

$$\overbrace{\varphi(x)\varphi^\dagger(y)} \equiv T(\varphi(x)\varphi(y)) - :\varphi(x)\varphi^\dagger(y):$$
$$= \Theta(x^0-y^0)\Delta_+(x-y) + \Theta(y^0-x^0)\Delta_-(x-y)$$
$$\equiv i\Delta_F(x-y)$$

Fermions	Bosons
Fourier transform.	Fourier trans:
$(i\cancel{\partial}+m)_{\alpha\beta}\, i\Delta_F(x-y)$	$i\Delta_F(x-y) \longrightarrow i\Delta_F(p) = \dfrac{i}{p^2-m^2}$
$\rightarrow i\dfrac{(\cancel{p}+m)_{\alpha\beta}}{p^2-m^2}$	

In summary

$$iS_F(x-y) = (i\cancel{\partial}+m)\, i\Delta_F(x-y)$$

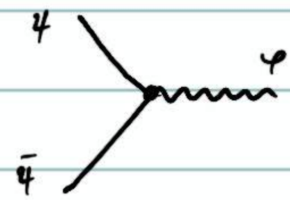
$$S_F(p) = \frac{\cancel{p}+m}{p^2-m^2} \sim \text{"} \frac{1}{\cancel{p}-m} \text{"}$$

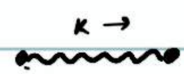
Dirac Fields Feynman Propagator.

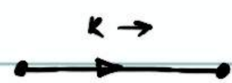
Consider the following Lagrangian for a real scalar field φ an fermions ψ (and $\bar\psi$).

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\mu^2\varphi^2}_{\text{Scalar}} + \underbrace{\bar\psi(i\not{\partial} - m)\psi}_{\text{Dirac.}} - \underbrace{\lambda\varphi\bar\psi\psi}_{\text{interaction.}}$$

Feynman Rules? (~~~ φ ; —— $\psi, \bar\psi$)

vertex :  $= (-i\lambda)\,\delta^4(\Sigma\,\text{momenta})$

internal lines :  $= i\int d^4k\,\dfrac{1}{k^2-\mu^2}$

 $= i\int d^4k\,\dfrac{(\not{k}+m)}{k^2-m^2}$

external lines : 

$= 1$ incoming φ

$= 1$ outgoing φ.

$= u^{(\lambda)}(\vec{p})$ incoming particle . (f)

$= \bar{u}^{(\lambda)}(\vec{p})$ outgoing particle (f)

$= \bar{v}^{(\lambda)}(\vec{p})$ incoming anti-particle ($\bar{f}$)

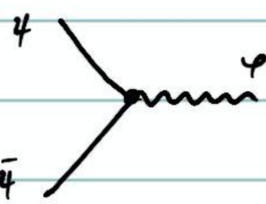
$= v^{(\lambda)}(\vec{p})$ outgoing anti-particle ($\bar{f}$)

Lecture 21

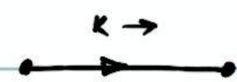
Consider the following Lagrangian for a real scalar field φ an fermions ψ (and $\bar{\psi}$).

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\mu^2\varphi^2}_{\text{Scalar}} + \underbrace{\bar{\psi}(i\not{\partial} - m)\psi}_{\text{Dirac}} - \underbrace{\lambda\varphi\bar{\psi}\psi}_{\text{interaction}}$$

Feynman Rules? ( φ ; $\psi, \bar{\psi}$)

vertex :  $= (-i\lambda)\,\delta^4(\Sigma\,\text{momenta})$

internal lines :  $= i\int d^4k\,\dfrac{1}{k^2-\mu^2}$

 $= i\int d^4k\,\dfrac{(\not{k}+m)}{k^2-m^2}$

external lines : 

$= 1$ incoming φ

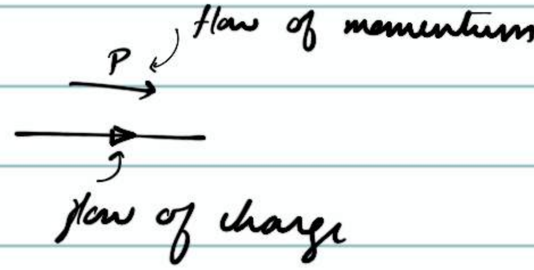
$= 1$ outgoing φ.

$= u^{(\lambda)}(\vec{p})$ incoming particle (f)

$= \bar{u}^{(\lambda)}(\vec{p})$ outgoing particle (f)

$= \bar{v}^{(\lambda)}(\vec{p})$ incoming anti-particle ($\bar{f}$)

$= v^{(\lambda)}(\vec{p})$ outgoing anti-particle ($\bar{f}$)

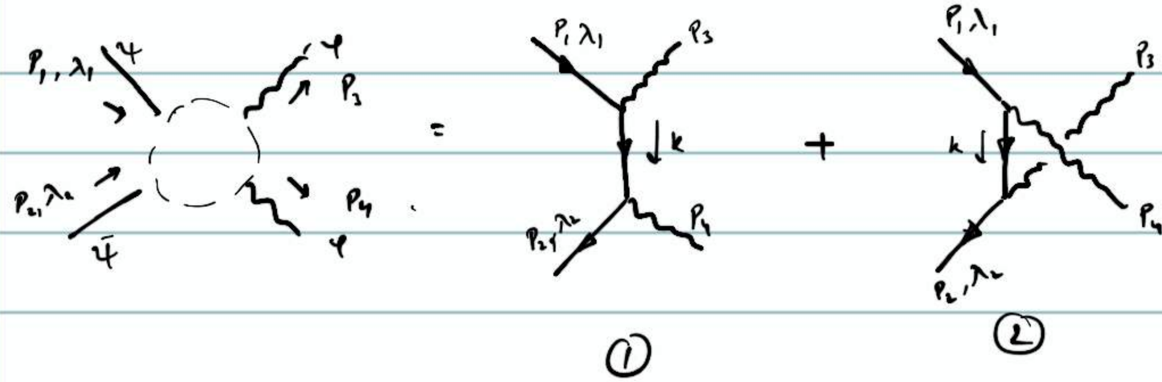
Note ①: 

② Note that $\bar{v}$ is incoming
$\bar{u}$ is outgoing

③ internal line $\xrightarrow{k}$ = same direction of k & →

Sample Calculations:

Example: $\psi\bar{\psi} \to \varphi\varphi$



$$① = (-i\lambda)^2 \delta^{(4)}(p_1+p_2-p_3-p_4)\int d^4k\, \delta^4(p_1-p_3-k)\, \bar{v}^{(\lambda_2)}(\vec{p}_2)\cdot \frac{i(\not{k}+m)}{k^2-m^2}\cdot u^{(\lambda_1)}(\vec{p}_1)$$

$$= -i\lambda^2 \delta^{(4)}(p_1+p_2-p_3-p_4)\left[\bar{v}^{(\lambda_2)}(\vec{p}_2)\cdot \frac{\not{p}_1-\not{p}_3+m}{(p_1-p_3)^2-m^2}\, u^{(\lambda_1)}(\vec{p}_1)\right]$$

$$② = -i\lambda^2 \delta^{(4)}(p_1+p_2-p_3-p_4)\left[\bar{v}^{(\lambda_2)}(\vec{p}_2)\cdot \frac{\not{p}_1-\not{p}_4+m}{(p_1-p_4)^2-m^2}\cdot u^{(\lambda_1)}(\vec{p}_1)\right]$$

$$\therefore \langle f | S-1 | i \rangle_{(2)} = -i\lambda^2 \delta^{(4)}(\Sigma p_i) \left\{ \left[\bar{v}^{(\lambda_2)}(\vec{p}_2) \cdot \frac{\not{p}_1 - \not{p}_3 + m}{(p_1-p_3)^2 - m^2} \, u^{(\lambda_1)}(\vec{p}_1) \right] + \left[\bar{v}^{(\lambda_2)}(\vec{p}_2) \cdot \frac{\not{p}_1 - \not{p}_4 + m}{(p_1-p_4)^2 - m^2} \cdot u^{(\lambda_1)}(\vec{p}_1) \right] \right\}$$

Note : $\dfrac{\not{p}_i - \not{p}_j + m}{(p_i - p_j)^2 - m^2}$ is a 4×4 matrix.

It is sandwiched between $\bar{v}$ & u to yield a scalar.

A couple of extra points

 $\longrightarrow$ leads has an extra -ve sign.

(If confused go back to the Wick expansion)

Spin Sums & Cross sections

cross sections $\sim$ $|\mathcal{M}_{\lambda\lambda'}|^2 \times$ kinematic factors

In many cases (when spins are not measured in the final state or when the initial state is unpolarized) we need quantities like

$$\overline{\sum_{\lambda}} \sum_{\lambda'} \left| \bar{u}^{(\lambda')}(\vec{p}\,') M u^{(\lambda)}(\vec{p}) \right|^2$$

($\overline{\sum_{\lambda}}$ avg over initial spins
$\sum_{\lambda'}$ sum over final spins)

Now $(\bar{u} M u)^\dagger = u^\dagger M^\dagger \bar{u}^\dagger$

$$= u^\dagger M^\dagger \gamma_0 u$$

$$= \bar{u} \bar{M} u$$

$\therefore \overline{\sum_{\lambda}}\sum_{\lambda'} |\quad|^2$

$$= \overline{\sum_{\lambda}} \sum_{\lambda'} \bar{u}^{(\lambda')}(\vec{p}\,') M u^{(\lambda)}(\vec{p}) \, \bar{u}^{(\lambda)}(\vec{p}) \bar{M} u^{(\lambda')}(\vec{p}\,')$$ $\leftarrow$ scalar = Tr

permute cyclically inside trace.

$$= \overline{\sum_{\lambda}} \sum_{\lambda'} \mathrm{Tr}\Big[\underbrace{u^{(\lambda')}(\vec{p}\,') \bar{u}^{(\lambda')}(\vec{p}\,')}_{\not{p}' + m} M \underbrace{u^{(\lambda)}(\vec{p}) \bar{u}^{(\lambda)}(\vec{p})}_{\not{p} + m} \bar{M} \Big]$$

$$= \frac{1}{2} \mathrm{Tr}\left[(\not{p}' + m) M (\not{p} + m) \bar{M} \right]$$

$\frac{1}{2}$ from avg. $\overline{\Sigma}$

<u>Lesson</u>: Instead of working out $\bar{u} M u$, wait till you need to evaluate $\sum\sum |\bar{u} M u|^2$ when calculating observables. Then it becomes a matter of evaluating some traces.

Useful Trace Theorems

$\mathrm{Tr}[\mathbb{1}] = 4$

$\mathrm{Tr}[\text{odd \# of } \gamma^\mu \text{ matrices}] = 0$

$\mathrm{Tr}[\not{a}\not{b}] = 4\, a\cdot b$

$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$

See more in P. Stevenson's notes.

Having discussed spin-½ fields and (earlier) massless spin-1 fields, we are now in a position to write down one of the most exquisitely tested theories of physics. "Quantum Electrodynamics"

Lagrangian for Quantum Electrodynamics.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - \underbrace{e A_\mu \bar{\psi} \gamma^\mu \psi}_{\text{interaction}}$$

eg photons   eg e^+, e^-
μ^+, μ^-

Note that $J^\mu \equiv \bar{\psi}\gamma^\mu\psi$ is a conserved current. $\partial_\mu J^\mu = 0$. [How did we find it?

Note that $\mathcal{L}_\psi = \bar{\psi}(i\not{\partial} - m)\psi$ has a continuous symmetry $\psi \to e^{i\alpha}\psi = (1 + i\alpha)\psi = \psi + \delta\psi$ means there is a Noether current (internal symmetry)
$J^\mu \equiv \pi^\mu \delta\psi = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi \propto \bar{\psi}\gamma^\mu\psi$]

As you can check $\partial_\mu J^\mu = 0$ on the eq. of motion.

This is nice, but this Lagrangian has another remarkable invariance.

If we make the transformations

$$\psi \longrightarrow e^{i\Lambda(x)}\psi \qquad \leftarrow \text{Note } \Lambda(x) \text{ is not a const!}$$

$$A_\mu \longrightarrow A_\mu - \frac{1}{e}\partial_\mu \Lambda(x) \qquad \text{for arbitrary } \Lambda(x)$$

the Lagrangian does not change.

Let us check: $\bar{\psi}\psi \longmapsto \bar{\psi}\psi$

However $\bar{\psi} i \not{\partial} \psi \longmapsto -\bar{\psi}\gamma^\mu \psi \partial_\mu \Lambda + \bar{\psi} i \not{\partial}\psi$

& $-eA_\mu \bar{\psi}\gamma^\mu\psi \longmapsto \bar{\psi}\gamma^\mu\psi\partial_\mu\Lambda - eA_\mu\bar{\psi}\gamma^\mu\psi$

Hence the extra terms from the transformation of $\bar{\psi} i\not{\partial}\psi$ and $-eA_\mu\bar{\psi}\gamma^\mu\psi$ precisely cancel each other, leaving an unchanged Lagrangian. (Note $F_{\mu\nu}$ is unchanged because of Gauge invariance)

It is convenient (and conceptually useful) to rewrite the above QED Lagrangian using a "covariant derivative"

$$\mathcal{D}_\mu \equiv \partial_\mu + ie A_\mu \, , \text{ so that}$$

$$\not{\mathcal{D}} = \not{\partial} + ie\gamma^\mu A_\mu .$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\mathcal{D}} - m)\psi \qquad \text{(check!)}.$$

— x —

Furthermore $F_{\mu\nu} = \frac{-i}{e} [\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu .$

check this too!

Insisting on "local" gauge invariance would have naturally led up to Quantum Electrodynamics as well. ie

* Start with $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$ which is invariant under $\psi \to e^{i\theta_0}\psi$ $\theta_0 = \text{const}$. (global U(1) symmetry)
* Now make this global U(1) symmetry $\to$ local

ie $\theta_0 \to \theta(x)$.

and insist on the Lagrangian still remaining invariant under $\psi \to e^{i\theta(x)}\psi$.

* A way to ensure this is to replace.

$\partial_\mu \longrightarrow D_\mu = \partial_\mu + ieA_\mu$

where $A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu \theta(x)$.

* With this "minimal substitution", the new

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \qquad \text{is locally gauge invariant!}$$

* But A_μ needs some dynamics, so give it a kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ constructed out of $[D_\mu, D_\nu]$.

* Thus we naturally arrive at the QED Lagrangian.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

Note the interaction term is sitting inside $\bar{\psi}\not{D}\psi$.

This idea works beyond QED. You can get the "Yang-Mills Theories" that describe the Electroweak theory & Quantum Chromodynamics using a similar procedure.

We will briefly outline this in the next class.

Lecture 22

Review

$$\mathcal{L}_0 = \bar{\psi}\,(i\not{\partial} - m)\,\psi$$

Dirac Fermions

$\psi \xrightarrow{\alpha} e^{i\alpha}\psi$ - global $U(1)$ transformation

$\Rightarrow \mathcal{L}_0 \xrightarrow{\alpha} \mathcal{L}_0$ - invariance under global $U(1)$ transformation

Consider

$\psi \xmapsto{\alpha(x)} e^{i\alpha(x)}\psi$ - "local" $U(1)$ trans.

$\mathcal{L}_0 \xmapsto{\alpha(x)} \mathcal{L}_0 - \bar{\psi}\gamma^\mu \partial_\mu \alpha(x)\psi$ - not invariant!

Fix by $\partial_\mu \longrightarrow \mathcal{D}_\mu = \partial_\mu + ieA_\mu(x)$

with $A_\mu(x) \xrightarrow{\alpha(x)} = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$

$\mathcal{L}_1 = \bar{\psi}\,(i\not{\mathcal{D}} - m)\,\psi \xmapsto{\alpha(x)} \bar{\psi}(i\not{\mathcal{D}} - m)\psi$

↑ invariant under "local" $U(1)$ transformations.

Dynamics for A_μ? : $F_{\mu\nu} \equiv \frac{-i}{e}[\mathcal{D}_\mu, \mathcal{D}_\nu]$

$= \partial_\mu A_\nu - \partial_\nu A_\mu.$

$$\mathcal{L}_2 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\,(i\not{\mathcal{D}} - m)\,\psi$$

$$\mathcal{L}_2 = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi \quad \leftarrow \{\underset{\sim}{\widetilde{QED}}\}$$

- it is invariant under local $U(1)$
- includes interaction term $A_\mu \bar{\psi}\gamma^\mu\psi$.
- A_μ is massless (photons)

* Generalize : $U(1) \longmapsto SU(N)$
  yields "Yang Mills" theories.

$SU(N)$: $N \times N$ unitary matrices $U^\dagger U = 1$ with $\det U = 1$. $\Big\}$ Fundamental Rep.

: $N^2 - 1$ generators

Important : $U(1) = SU(1)$ is an "Abelian" group.

$g_1 \cdot g_2 = g_2 \cdot g_1$ for all $g \in SU(1)$

$SU(N)$ is a non-Abelian group.

$g_1 \cdot g_2 \neq g_2 \cdot g_1$ for at least some $g \in SU(N)$.

Example of non-abelian group : $SO(3) \simeq SU(2)$.

For $SU(2)$: $U = e^{i\alpha_A T_A}$ $\quad A = 1, 2, 3$

$f_{ABC} \propto \varepsilon_{ABC}$; $[T_A, T_B] = i f_{ABC} T_C$

Consider a Lagrangian which is invariant under global $SU(N)$: $\mathcal{U} = e^{i\alpha_A T_A}$ $\quad A = 1, \ldots, \underbrace{N^2-1}$

($T_A = N \times N$ matrix)

$$\mathcal{L}_0 = \bar{\psi}_a (i\not{\partial} - m)\psi_a \qquad a = 1, \ldots N$$

$$\psi_a \xrightarrow{\alpha_A} e^{i\alpha_A (T_A)_{ab}} \psi_b$$

T_A are traceless & Hermitian

$$\mathcal{L}_0 \xrightarrow{\alpha_A} \mathcal{L}_0$$

Consider.

$$\alpha_A \longrightarrow \alpha_A(x) \qquad A = 1 \ldots N^2-1$$

$$\mathcal{L}_0 \rightarrow \mathcal{L}_0 - \bar{\psi}_a \gamma^\mu \partial_\mu \alpha_A(x) (T_A)_{ab} \psi_b$$

To fix this, let $\partial_\mu \longrightarrow \mathcal{D}_\mu = \partial_\mu + ig(T_A) W_\mu^A(x)$

"charge"

$W_\mu^A(x) = N^2-1$ vector fields.

with

$$W_\mu^A(x) \xrightarrow{\alpha_A(x)} W_\mu^A(x) - \frac{1}{g}\partial_\mu \alpha^A(x) - \underbrace{f_{ABC}\,\alpha^B W_\mu^C(x)}$$

needed because $T^A T^B \neq T^B T^A$.

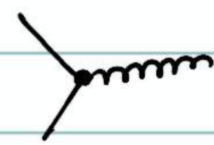
Thus $\mathcal{L}_1 = \bar{\psi}_a (i \not{D} - m) \psi_a \xrightarrow{\alpha_A(x)} \mathcal{L}_1$

What about the Kinetic term for W_μ^A ?

$$[D_\mu, D_\nu] = ig T_c \underbrace{\left[\partial_\mu W_\nu^c - \partial_\nu W_\mu^c - g f_{cAB} W_\mu^A W_\nu^B \right]}_{\equiv F_{\mu\nu}^c}$$

$$\therefore \mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu}^c F_{\mu\nu c} + \bar{\psi}_a (i \not{D} - m) \psi_a .$$

- invariant under local $SU(N)$
- $A = 1, 2 \ldots N^2-1$ massless vector fields.
- a) interaction terms. $g(\bar{\psi} \gamma^\mu T_A \psi) W^A{}_\mu(x)$.



- b) $g^4 (WWWW)$ 

$g^2 (\partial W) W^2$ 

} new! – self-interaction of gauge fields!

↓

leads to asymptotic freedom in QCD.

Electromagnetic : $U(1)$

Strong Interactions : $SU(3)$ ← symmetry group
(QCD)

Electroweak : $U(1) \otimes SU(2)$.

STANDARD MODEL : $SU(3) \otimes SU(2) \otimes U(1)$.

Broken Symmetries:

Symmetry of the Hamiltonian / Lagrangian is "broken" by the ground state of the system.

Important for QFT, even more so when combining it with gauge theories like the ones we just discussed.

1. Connects breaking symmetries to appearance of massless / gapless excitations
2. Allows us to understand the connection between the Higgs field & the mass of the W & Z bosons.
3. Important for phase transitions, as well as the appearance of "defects" in condensed matter systems as well as cosmology
4. Goldstone (gapless) excitation do not get mass due to quantum corrections.

5. Superconductivity - (Landau Ginzburg Theory)

6. Origin of anisotropies in the Cosmic Microwave Background! - (EFT of Inflation).

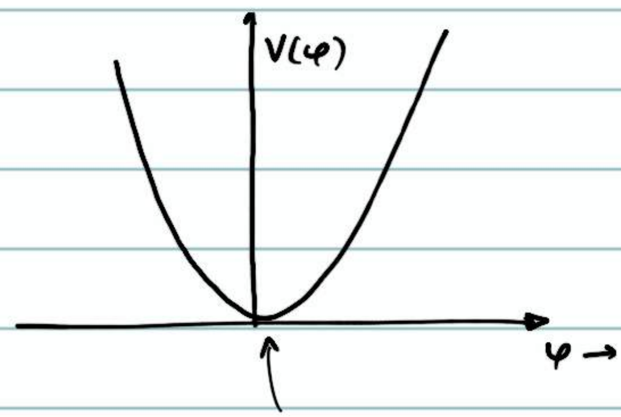
$\vdots$

Consider:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi).$$

where $V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4$.

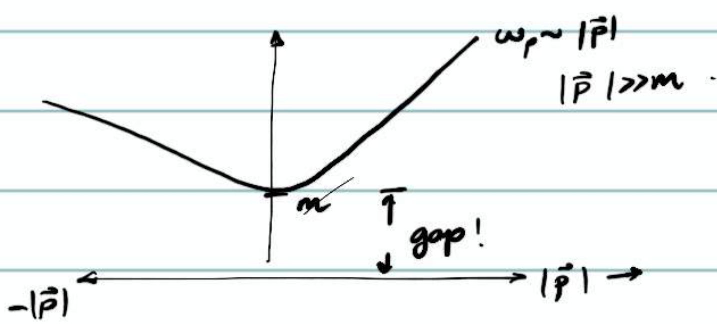
Note $\varphi \to -\varphi$ is a symmetry of the Lagrangian.



Ground state: $\langle 0|\varphi(x)|0\rangle = 0$.

Excitation around the ground state (ie particles) have energy

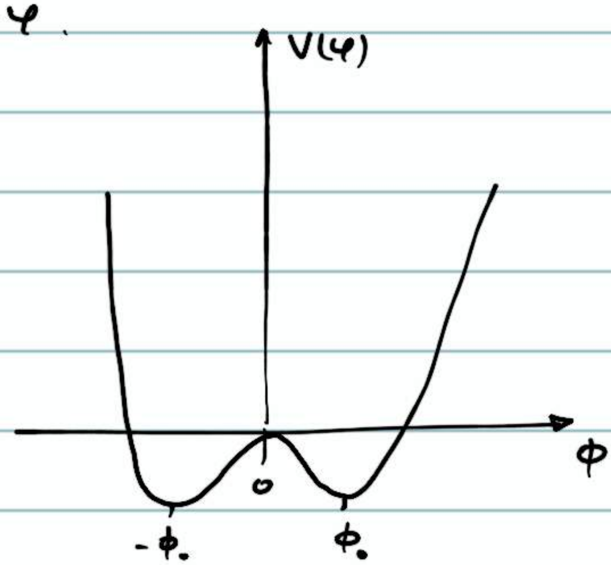
$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$



$m^2 = V''(\varphi=0)$ determines the gap. All particles cost at least m in energy, more for $\vec{p} \neq 0$.

Now consider: $V(\varphi) = -\frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4$.

Lagrangian still possesses the symmetry $\varphi \to -\varphi$.



$\phi_0 = \sqrt{\frac{6m^2}{\lambda}}$.

Ground state: $\langle 0 | \varphi(x) | 0 \rangle = \begin{cases} \sqrt{\frac{6m^2}{\lambda}} = \phi_0 \\ -\sqrt{\frac{6m^2}{\lambda}} = -\phi_0 \end{cases}$

Picking the ground state, obviously breaks the symmetry.

Excitations about $\phi_0 = \sqrt{\frac{6m^2}{\lambda}}$?

Expanding $V(\phi)$ near $\phi = \phi_0$

because at min

$$\Rightarrow \quad V(\phi) = V(\phi_0) + \underbrace{V'(\phi_0)}_{0}(\phi - \phi_0) + \frac{V''(\phi_0)}{2!}(\phi-\phi_0)^2$$

$$= V(\phi_0) + m^2 u^2 \qquad \text{where } u = \phi - \phi_0$$

Similarly, expanding the kinetic term & putting it all together

$$\mathcal{L} = \text{const} + \frac{1}{2}\partial_\mu u \, \partial^\mu u - \frac{\mu^2 u^2}{2} \qquad \mu = \sqrt{2}m .$$

So the excitations around $\phi = \phi_0$, have a mass $\sqrt{2}\, m$ (different from m).

Quantization would then proceed as usual.

Thus : a) The ground state breaks the symmetry of the Lagrangian ($\phi \to -\phi$)

b) new type of excitations (eg. different mass appeared).

Lecture 23

The ground state would lie anywhere around the circle of radius $\sqrt{\phi_1^2+\phi_2^2} = \sqrt{\frac{6m^2}{\lambda}}$.

Let us pick one : $\phi_1 = \sqrt{\frac{6m^2}{\lambda}}$, $\phi_2 = 0$. (which of course breaks the symmetry)

and expand the field around this ground state value.

Let $u = \phi_1 - \sqrt{\frac{6m^2}{\lambda}}$

$v = \phi_2 - 0$

$$V(\phi_1, \phi_2) = \text{const.} + m^2 u^2 + O(u^3, \ldots).$$

The entire Lagrangian becomes.

$$\mathcal{L} = \frac{1}{2}\partial_\mu u\,\partial^\mu u + \frac{1}{2}\partial_\mu v\,\partial^\mu v - \mu^2 u^2 + \ldots$$

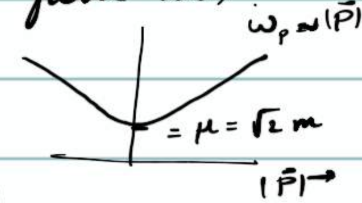
where $\mu = \sqrt{2}m$

The excitations around this ground state have the following features:

a) $u = \phi_1 - \sqrt{\frac{6m^2}{\lambda}}$ (ie along the ϕ_1 direction, near our picked ground state)

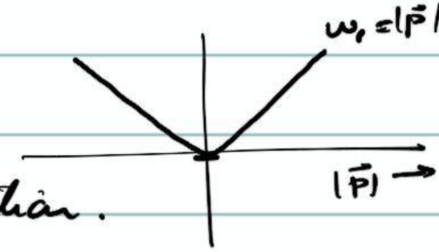
has a mass $\mu = \sqrt{2}\, m$ (different from m)

It is not gapless.



b) $v = \phi_2 - 0$ (ie in the "gutter").

This excitation has no mass term.
This excitation is massless!
There is no "minimal" energy needed. This is a gapless excitation.



The vanishing of the mass (along the gutter) ie the existence of massless excitation is a manifestation of Goldstone's theorem.

It says the following: Breaking of a continuous symmetry results in a massless excitation, known as a Goldstone mode. The massless particle associated with the Goldstone mode is known as a Goldstone boson.

Let us reconsider the ϕ_1, ϕ_2 theory as a theory of a complex scalar field.

$$\mathcal{L} = \partial^\mu \psi^\dagger \partial_\mu \psi + m^2 \psi^\dagger \psi - \lambda (\psi^\dagger \psi)^2$$

This Lagrangian is invariant under $\psi \to e^{i\alpha}\psi$ where $\alpha = \text{const}$.

Let us change co-ordinates : $\psi(x) = \rho(x) e^{i\theta(x)}$

2 real d.o.f   Polar form.
2 real. d.o.f.

The Lagrangian then becomes.

$$\mathcal{L} = \partial_\mu \rho \, \partial^\mu \rho + \rho^2 \partial_\mu \theta \partial^\mu \theta + m^2 \rho^2 - \lambda \rho^4$$

The continuous symmetry $\psi \to e^{i\alpha}\psi$, now becomes $\theta \to \theta + \alpha$.
$\rho \to \rho$.

We break the symmetry by picking the ground state $\rho = \rho_0 = \sqrt{\frac{m^2}{2\lambda}}$ and $\theta = \theta_0 = 0$.

What do the excitations around this ground state look like?

Let $u = \rho - \rho_0$
$v = \theta - \theta_0$

The Lagrangian at leading order (ignoring additive) constants becomes

$$\mathcal{L} = \left(\frac{m^2}{2\lambda}\right) \partial_\mu v \, \partial^\mu v$$

$$+ (\partial_\mu u)(\partial^\mu u) - 2m^2 u^2 - 4\left(\frac{m^2\lambda}{2}\right)^{1/2} u^3 - \lambda u^4$$

$$+ \left(u^2 + \left(\frac{2m^2}{\lambda}\right)^{1/2} u\right)(\partial_\mu v)(\partial^\mu v) + \dots$$

Note : $v = \rho - \rho_0$ has a mass $= \sqrt{2}\, m$
$u = \theta - \theta_0$ (excitation along the gutter)
is massless as expected

Generalize

* Let us consider. $\vec{\varphi} = (\varphi_1, \ldots \varphi_N)$ φ_a are real.

$$\mathcal{L} = \frac{1}{2}\partial^\mu \vec{\varphi}\cdot\partial_\mu \vec{\varphi} - \underbrace{\left(-\frac{m^2}{2}\vec{\varphi}\cdot\vec{\varphi} + \frac{\lambda}{4!}(\vec{\varphi}\cdot\vec{\varphi})^2\right)}_{V(\vec{\varphi}\cdot\vec{\varphi})}$$

$$= \frac{1}{2}\partial^\mu \vec{\varphi}\cdot\partial_\mu \vec{\varphi} - \frac{\lambda}{4!}\left(\vec{\varphi}\cdot\vec{\varphi} - v^2\right)^2 \qquad v^2 = \frac{6m^2}{\lambda}.$$

$\mathcal{L}$ is invariant under global: $SO(N)$.

# of Goldstone bosons : $N-1$
(massless excitations).

* Let us consider $\vec{\varphi} = (\varphi_1, \varphi_2 \ldots \varphi_N)$ φ_a are complex.

$$\mathcal{L} = \partial_\mu \vec{\varphi}\cdot\partial^\mu \vec{\varphi}^\dagger - \frac{\lambda}{4!}\left(\vec{\varphi}\cdot\vec{\varphi}^\dagger - v^2\right)^2$$

$\mathcal{L}$ is invariant under global $SU(N)$.

# of Goldstone bosons : $2N-1$.

* For such cases, dimensionality of the vacuum manifold is the # of Goldstone bosons.
(See for example. chapter 11, Maggiore)

Here is another general way of stating Goldstone's theorem:

continuous symmetry

Suppose $\mathcal{L} \xrightarrow{G} \mathcal{L}$ for $\vec{\varphi} \rightarrow 1 - i\alpha_A T_A \vec{\varphi}$

$\mathcal{N}$ generators of G

Let $V(\vec{\varphi} \neq 0) = V_{min}$ define the vacuum manifold

and $\vec{v}_0 \neq 0$ be an arbitrarily chosen vacuum (space time independent)

Then $\vec{v}_0 \rightarrow \vec{v}_0 - i\alpha_A T_A \vec{v}_0$.

If $T_B \vec{v}_0 = 0$ for $B = 1, \ldots, \mathcal{M}$

then the # of Goldstone modes is $\underline{\underline{\mathcal{N} - \mathcal{M}}}$

[Note * $SO(N)$: $\mathcal{N} = \frac{N(N-1)}{2}$, $\mathcal{M} = \frac{(N-1)(N-2)}{2}$.

$\mathcal{N} - \mathcal{M} = N - 1$.

= dim of vacuum manifold.

* $SU(N)$: $\mathcal{N} = N^2 - 1$ $\quad \mathcal{M} = (N-1)^2 - 1$

$\mathcal{N} - \mathcal{M} = 2N - 1$

= dim. of vacuum manifold]

(Proof in Zee or Peskin & Schroeder).

$\overbrace{\text{SSB.}}$

Spontaneous Symmetry Breaking in a Gauge Theory

Let us consider the following lagrangian:

$$\mathcal{L} = (\mathcal{D}_\mu \psi)^\dagger (\mathcal{D}^\mu \psi) + m^2 \psi^\dagger \psi - \lambda (\psi^\dagger \psi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$\mathcal{D}_\mu = \partial_\mu + i q A_\mu .$$

The Lagrangian is invariant under the following transformations:

$$\psi \longmapsto e^{i\alpha(x)} \psi$$

$$A_\mu \longrightarrow A_\mu - \frac{1}{q} \partial_\mu \alpha(x)$$

The Lagrangian includes:

1) a massless spin 1 field $(\omega_p = |\vec{p}|)$

2) a massive particles & anti particles (charged).

As before let us move to polar co-ordinates

$$\psi(x) = \rho(x)\, e^{i\theta(x)}$$

Let us pick a ground state at

$$\rho(x) = \rho_0 = \sqrt{\frac{m^2}{2\lambda}} \quad \& \quad \theta(x) = \theta_0$$

This state breaks global and local symmetry.

What about excitations around this state? To understand the excitations, let us rewrite the Lagrangian in polar co-ordinates

First $\quad D_\mu \psi = (\partial_\mu \rho) e^{i\theta} + i\left(\partial_\mu \theta + q A_\mu\right) \rho e^{i\theta}$.

Define the locally gauge invariant combination

$$C_\mu = A_\mu + \frac{1}{q}\partial_\mu \theta .$$

[Check: $A_\mu \to A_\mu - \frac{1}{q}\partial_\mu \alpha(x)$

$\frac{1}{q}\partial_\mu \theta \to \frac{1}{q}\partial_\mu \theta + \frac{1}{q}\partial_\mu \alpha$.]

$$\therefore \quad \mathcal{D}_\mu \psi = (\partial_\mu \rho) e^{i\theta} + i q C_\mu \rho e^{i\theta}$$

$$\Rightarrow (\mathcal{D}_\mu \psi)^\dagger (\mathcal{D}^\mu \psi) = (\partial_\mu \rho)(\partial^\mu \rho) + \rho^2 q^2 C_\mu C^\mu$$

Further note that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu C_\nu - \partial_\nu C_\mu$

$$\therefore \quad \mathcal{L} = (\partial_\mu \rho)(\partial^\mu \rho) + \rho^2 q^2 C_\mu C^\mu + m^2 \rho^2 - \lambda \rho^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

which is completely in terms of ρ & C_μ.

Let us now consider the Lagrangian for

$\frac{u}{\sqrt{2}} = \rho - \rho_0 = \rho - \sqrt{\frac{m^2}{2\lambda}}$ ($\sqrt{2}$ makes things easier).

$v = \theta - \theta_0 = \theta$ $\qquad \theta_0 = 0$.

$$\mathcal{L} = \frac{1}{2} \partial_\mu u \, \partial^\mu u - m^2 u^2 - \sqrt{\lambda}\, m\, u^3 - \frac{\lambda}{4} u^4$$

$$- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{M^2}{2} C_\mu C^\mu + q^2 \left(\frac{m^2}{\lambda}\right)^{1/2} u\, C_\mu C^\mu + \frac{1}{2} q^2 u^2 C^\mu C_\mu.$$

$+ \dots$ (ignoring const. terms).

where $M = q\sqrt{\frac{m^2}{\lambda}}$.

Let us pause to analyze the behavior of u & v excitations around the ground state.

u (radial), v (angular)

* The radial u excitations are massive, with a mass $= \sqrt{2}\, m$. v on the other hand does not appear at all!

Moreover,

** the field C_μ now has a mass

$$M = q\sqrt{\frac{m^2}{\lambda}} .$$

We started with a massless gauge field, now we have a massive one! Its excitations

$$\omega_p = \sqrt{|\vec{p}|^2 + \left(\frac{q^2 m^2}{\lambda}\right)}$$

are not gapless anymore.

*** Naively, this seems to be bad, we have gained a d.o.f by making the gauge field massive. But this is as it should be, since the angular d.o.f (the goldstone excitation) is no where visible in this formulation.

This is often referred to as follows:

"The massless gauge field eats the goldstone and becomes massive!"

Mathematically, we removed the Goldstone when we defined $C_\mu = A_\mu + \frac{1}{q}\partial_\mu \theta$. (this is a gauge transformation, so goldstones were just gauge modes!)

If we think about the ψ field as the Higgs, then this is how the W and Z bosons get their mass based on the vev of the Higgs!

<u>Accounting</u>:

$$\begin{array}{lcl} 2 \text{ massive scalars} & & 1 \text{ massive scalar} \\ 2 \text{ massless photons} & \longrightarrow & 3 \text{ massive vectors} \\ \overline{4} & & \overline{4} \end{array}$$

Lecture 24

Superconductors and Spontaneous Symmetry Breaking.

Experimentally, it is known that

1) Certain Materials become superconductors below some critical temperature $T < T_c$. (conductivity $\sigma \to \infty$).
2) Static magnetic and electric fields do penetrate the conductor (actually, they penetrate, but only a little bit).

Let us first consider some simple "classical" arguments

1) $$\underset{\text{current}}{\vec{J}} = \overset{\text{conductivity}}{\sigma} \underset{\vec{E}\text{ field}}{\vec{E}}$$

Superconductor $\Rightarrow \sigma \to \infty$. Let $\vec{J} = -n e \vec{v}$

n: # density of e's is finite

$\Rightarrow \vec{J}$ is finite

$\Rightarrow \vec{E} \to 0$

inside superconductors.

From $\dot{\vec{B}} = -\nabla \times \vec{E}$

$\to 0$ inside Superconductor (sc)

ie $\vec{B}(0) = 0 \Rightarrow \vec{B}(t) = 0$, ie even if we apply external fields, the sc expels them.

2). Is there a field theoretical description of this?

Landau Ginzburg Theory

At temperature $T < T_c$, electrons pair up into Cooper pairs (charge $2e$, spin 0). The collection of cooper pairs can be described by an effective complex scalar field $\varphi(\vec{x})$. This field can be coupled to a gauge field $A_i(\vec{x})$, via

$$\partial_i \varphi \rightarrow \mathcal{D}_i \varphi = \partial_i \varphi + 2ie A_i$$

(we ignore time derivatives to focus on static configurations).

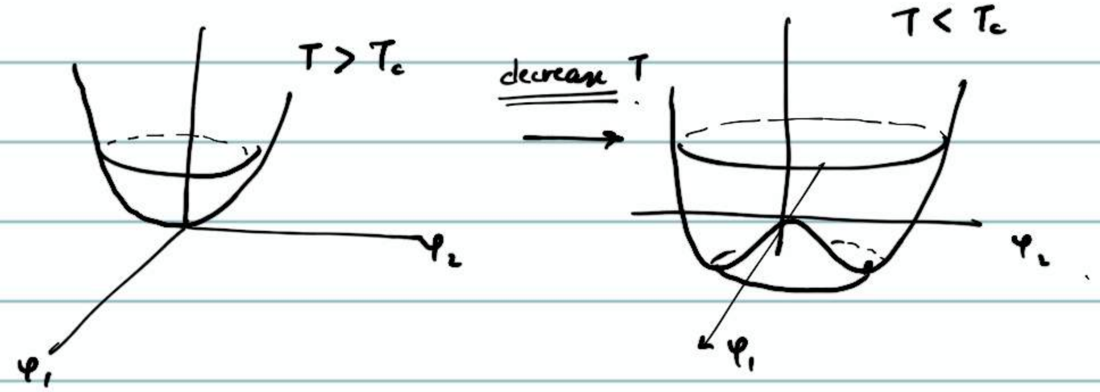
$$\mathcal{L} = \frac{1}{4} F_{ij} F^{ij} + |\mathcal{D}_i \varphi|^2 + \overbrace{a(T)|\varphi|^2 + b(T)|\varphi|^4 + \cdots}^{V(|\varphi|)}$$

where the temperature dependent co-efficients $a(T)$ and $b(T)$ are such that

for $T \simeq T_c$.

$a(T) \sim a_1 (T - T_c)$

$b(T) > 0$



Thus the effective theory $\varphi(\vec{x})$ looks like the Higgs field coupled to a gauge field — and the theory has spontaneous symmetry breaking.

For $T > T_c$, the photon is massless

For $T < T_c$, in SC phase the photon gains a mass M via SSB. $\left(M \sim e\sqrt{\frac{b}{-a}}\right)$

So what? Well, the wave equations for $\vec{E}$ and $\vec{B}$ now become.

$$(\nabla^2 + M^2)\vec{E} = 0 \quad \& \quad (\nabla^2 + M^2)\vec{B} = 0$$

with solutions $|\vec{B}| \sim \underset{\uparrow \text{ surface}}{B_s} e^{-Mx}$ (similarly $\vec{E}$).

That is, in the broken phase $\vec{B}$ and $\vec{E}$ only penetrate a distance $\sim \frac{1}{M}$ inside the superconductor!

Thus the mass gained by the photon via SSB. explain why there are no $\vec{E}$ & $\vec{B}$ fields inside superconductors.

__Note:__

* Typically $M^{-1} \sim 50 - 500$ nm
* The cooper pair formation allows electron pairs to condense & behave like bosons. A mass gap at low temp. prevents collisional interaction $\Rightarrow \sigma \to \infty$.

"Defects" :

When symmetry breaking leads to different vacua being realized in different regions of space, this can lead to the formation of defects.

Examples of such defects include:

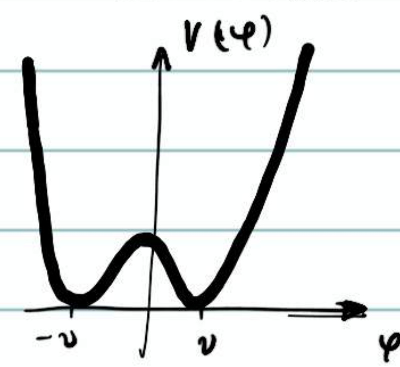
1) Domain walls in a ferro magnet
2) vortices.
3) domain walls, strings & textures that could have formed in the early universe

A illustrative example: Kinks.

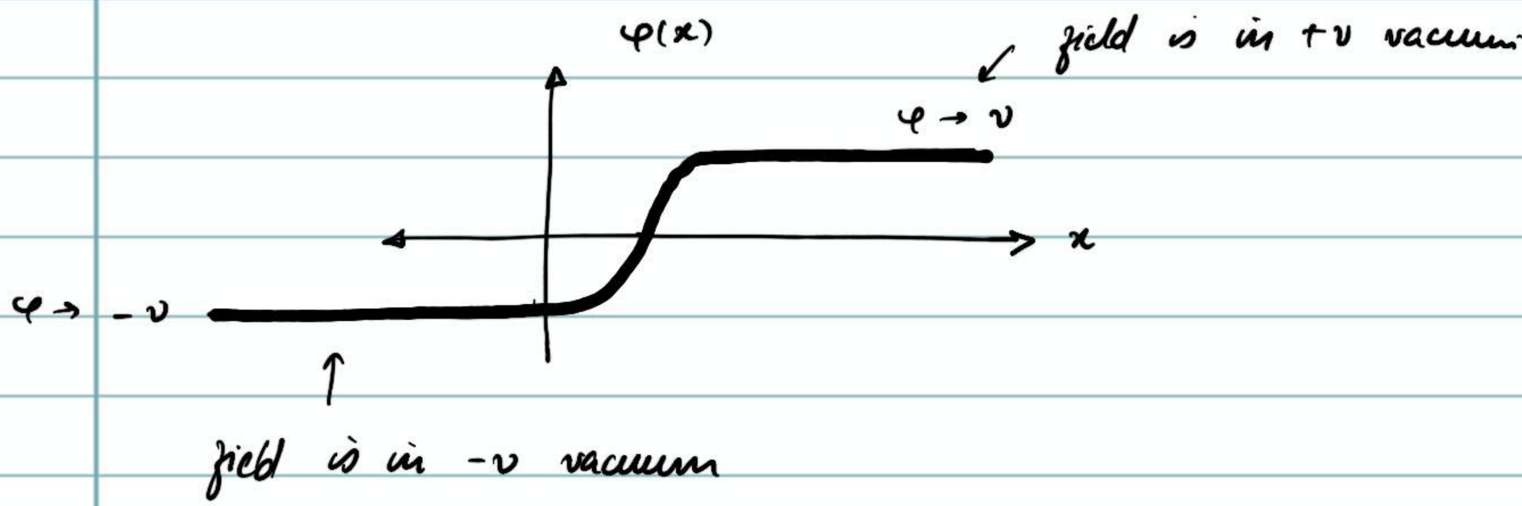
Consider: $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{\lambda}{4}(\varphi^2 - v^2)^2$

$v^2 = \frac{m^2}{\lambda}$

in 1+1 spacetime dimensions:



The vacuum configuration can be $\phi = v$ or $\phi = -v$. Let us consider a situation where at $x \to +\infty$ $\phi(x) \to v$. and as $x \to -\infty$ $\phi(x) \to -v$



Is such a static configuration possible? (ie does it have a finite amount of energy?

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_x \varphi)^2 + V(\varphi)\right]$$

check.
$\because \partial_x^2 \varphi = V'(\varphi)$
$\Rightarrow \frac{1}{2}(\partial_x \varphi)^2 = V(\varphi)$.

$$= \int_{-\infty}^{\infty} dx \; 2V(\varphi)$$

$$= \int_{-v}^{v} d\varphi \, \frac{dx}{d\varphi} \, 2V(\varphi) = \int_{-v}^{v} d\phi \left[2V(\varphi)\right]^{1/2}$$

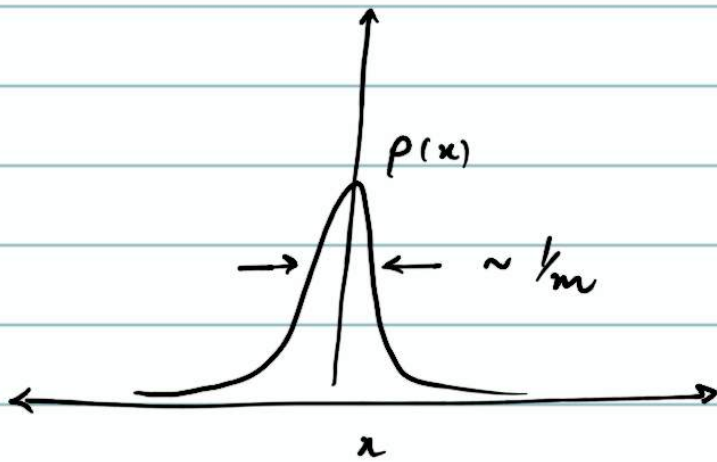
$$= \frac{1}{\sqrt{2}} \, \frac{4}{3} \frac{m^3}{\lambda}$$

$v^2 = \frac{m^2}{\lambda}$.

Finite!

Note the non-perturbative nature as $\lambda \to 0$.

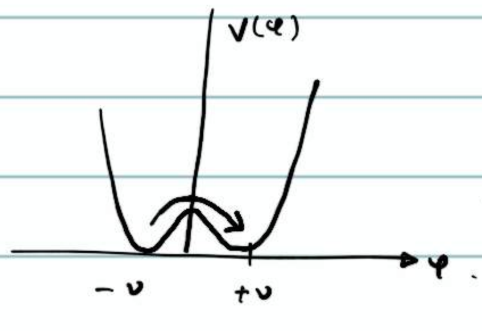
The energy density?

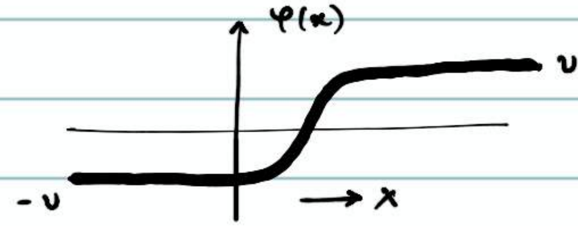


* No way to "unwind" for finite amount of energy.
* Conceptually similar to domain walls in magnets.

* Note that $\varphi(x)$ is a solution $\Rightarrow$
  $\varphi\left(\gamma(x - vt)\right)$ is also a solution (Lorentz invariance)
  - Kinks in motion. Behave like particles.

Let us now think about this soliton from a Topological perspective.





Q = 1

Kink

Let us define a current $J^\mu \equiv \frac{1}{2v} \varepsilon^{\mu\nu} \partial_\nu \varphi$.

where $\varepsilon^{01} = 1 \quad \varepsilon^{10} = -1 \quad \varepsilon^{11} = \varepsilon^{00} = 0$

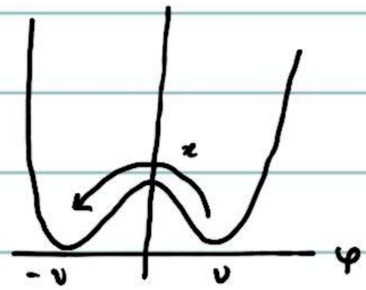
Note the $\partial_\mu J^\mu = 0$

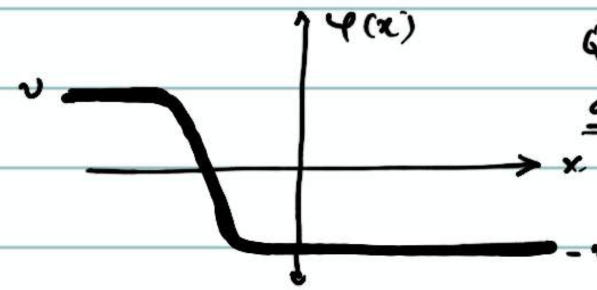
The charge associated with this conserved current

$$Q = \int dx \, J^0(x) = \frac{1}{2v} \left[\varphi(\infty) - \varphi(-\infty) \right] = 1 .$$

This current and charge have nothing to do with Noether, and unlike Noether, we did not rely on the equations of motion for $\partial_\mu J^\mu = 0$.

The current and charge are "topological" and care about the structure of the vacuum manifold $= Z_2$ with $\varphi(+\infty) = -\varphi(-\infty) = v$.

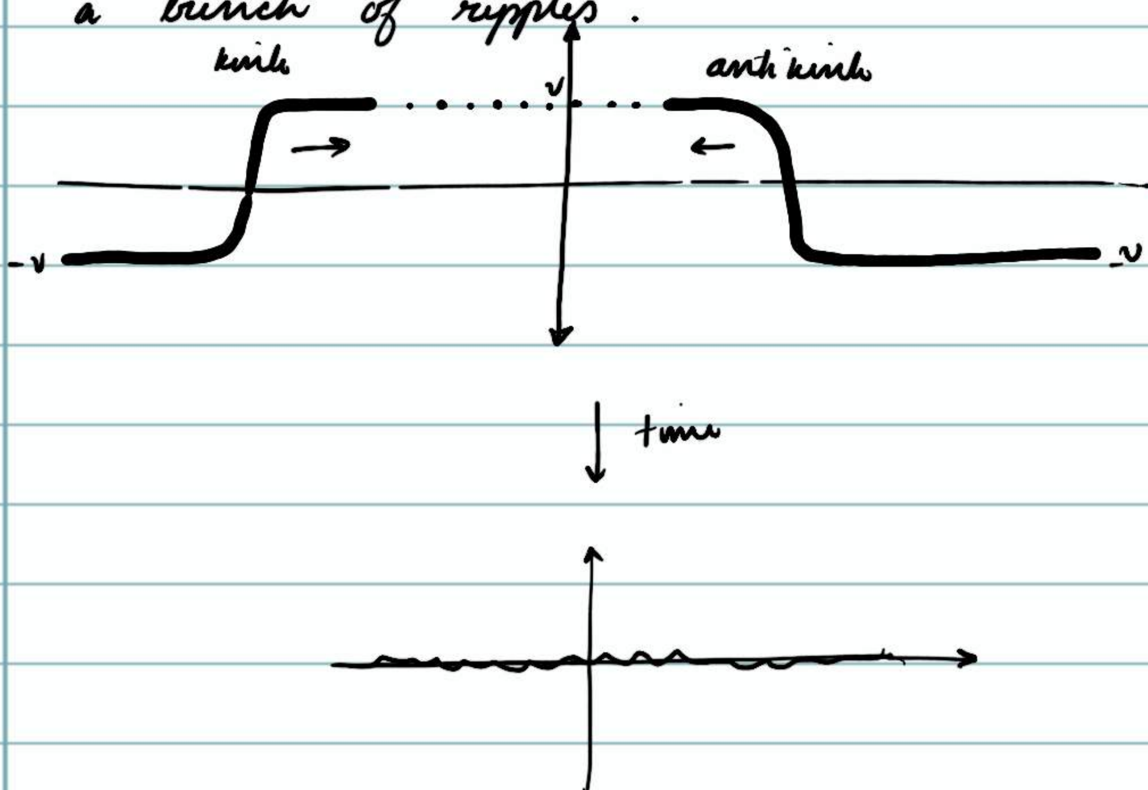




$Q = -1$

<u>anti kink</u>.

Note that a kink - antikink pair can have $Q = 0$ and can "annihilate" each other into a bunch of ripples.



$Q = 0$

$Q = 0$

Note that a kink or an anti-kink alone cannot decay into ∼∼∼ because Q would not be conserved! Kinks and antikinks are (seperately) protected by topology!

We have restricted ourselves to 1 dim of space and to one (real) scalar field.

Let us now consider the real scalar field in D dimensions. Do <u>static</u>, <u>finite energy</u> solutions exist in D-dimensions?

$$S[\varphi] = -\int d^D x \left[\frac{1}{2}(\partial_i \varphi)^2 + V(\varphi)\right] \qquad \partial_t \varphi = 0$$

$$= -E[\varphi] \qquad = \text{energy.}$$

Since the solutions extremize the action, they also extremize the energy.

Let $\varphi(\vec{x})$ be a solution, then for $\varphi(\lambda \vec{x})$

$$E_\lambda[\varphi] = \int d^D x \left[\frac{1}{2}\left(\partial_i \varphi(\lambda \vec{x})\right)^2 + V(\varphi(\lambda \vec{x}))\right]$$

$$= \lambda^{2-D} \int d^D x' \frac{1}{2}\left(\partial_{i'} \varphi(\vec{x}')\right)^2 + \lambda^{-D}\int d^D y\, V(\varphi(\vec{y}))$$

$$= \lambda^{2-D} \underbrace{I_G[\varphi]}_{\geq 0} + \lambda^{-D} I_V[\varphi]$$

Since $\varphi(\vec{x})$ is a solution

$$\left.\frac{dE_\lambda[\varphi]}{d\lambda}\right|_{\lambda=1} = 0$$

scaling by λ represents a variation. The action & hence in the static case, the energy must be stationary.

$$\Rightarrow \quad 0 = (2-D)\, I_k[\varphi] - D\, I_V[\varphi]$$

For stability $\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=1} > 0$

$$\Rightarrow \quad (2-D)(1-D)\, I_k[\varphi] + D(D+1)\, I_V[\varphi] > 0$$

$$\Rightarrow \quad \left[(2-D)(1-D) + D(D+1)\frac{(2-D)}{D}\right] \underbrace{I_k[\varphi]}_{>0} > 0 .$$

$$\Rightarrow \quad D-2 > 0 .$$

Thus we have stable, static solutions only in $\underline{D=1}$!

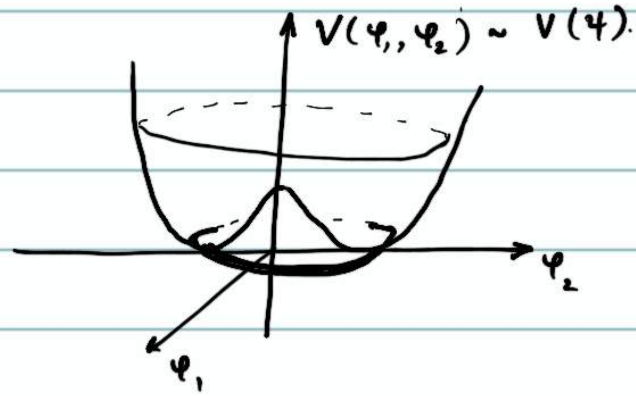
This is a baby version of Derrick's Theorem. It can be generalized to N scalar fields.

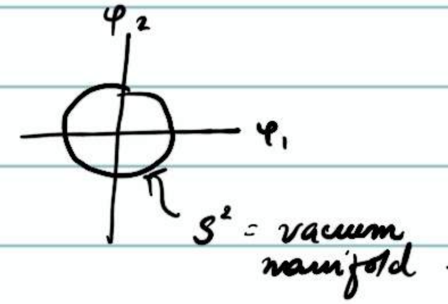
Lecture 25

Refer to QFTGA chapter 29.

Now let us consider a theory with continuous symmetry breaking : $\psi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$.

$$V(\psi) = |\partial_\mu \psi|^2 - \lambda(|\psi|^2 - v^2)^2$$





We will consider the case that this field lives in 2 spatial dimensions $\vec{\varphi}(t, x, y)$
$= (\varphi_1(x,y), \varphi_2(x,y))$

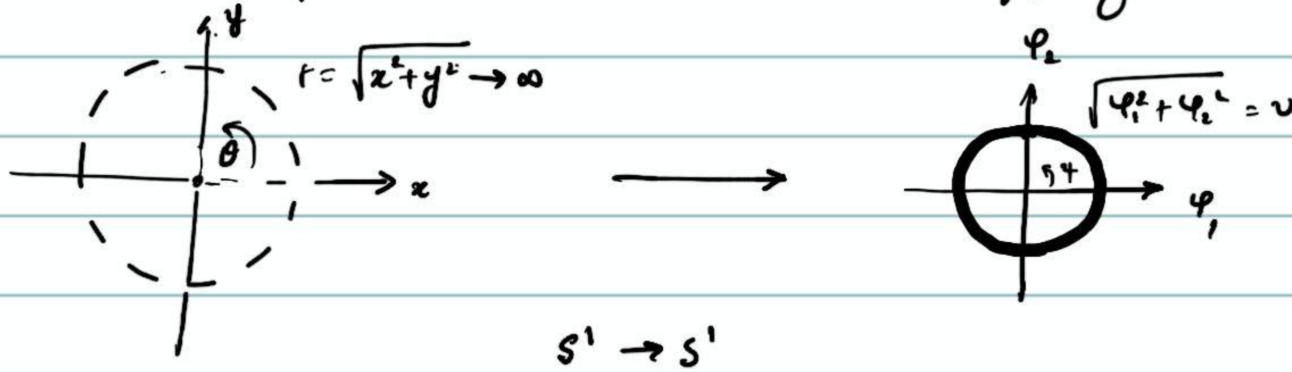
Let us try to construct the analog of a kink in this theory.

In the case of the kink we mapped
$\varphi(x \to \infty) = v \qquad \varphi(x \to -\infty) = -v$

$x = \pm\infty$ spatial boundaries.
$V(\varphi = \pm v) = 0$ vacuum manifold.

(static config)

In the present case consider mapping



$S^1 \to S^1$

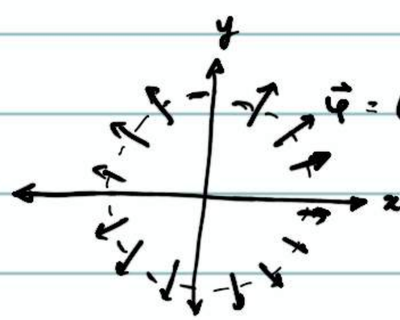
We can imagine a one-to-one mapping

$$(r\cos\theta,\ r\sin\theta) \longmapsto (\rho\cos\psi,\ \rho\sin\psi)$$

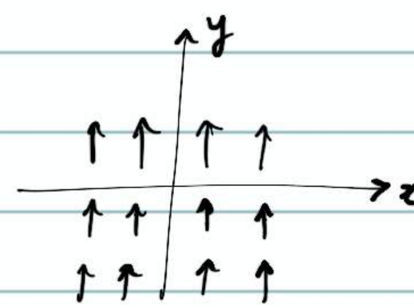
for $r \to \infty$, $\theta \in (0, 2\pi)$ $\qquad \rho \to v$

Possibilities :

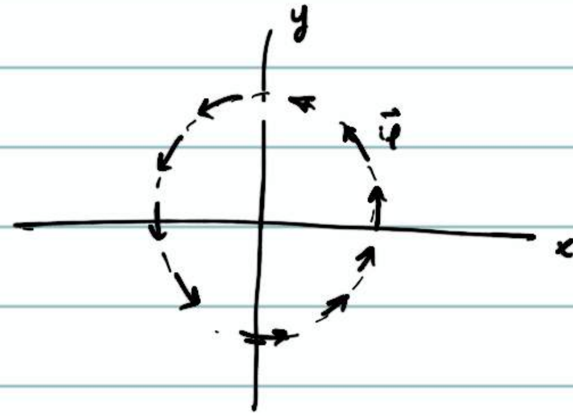
(i) $\psi = \theta$



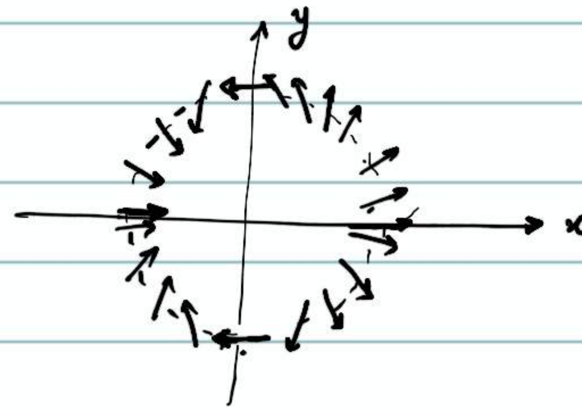
Note cannot be smoothly deformed to a homogeneous vacuum.

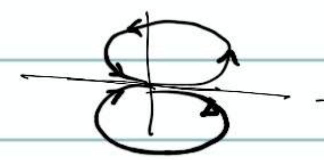


(ii) $\psi = \theta + \frac{\pi}{2}$



(iii) $\psi = 2\theta$.





In general $\psi = n\theta + \alpha$

n: winding number (integer).

α: phase

All of these configurations cannot be deformed to a spatially homogeneous, uni-directional vacuum.

These are "vortices".

Note $\quad \vec{\varphi}(r,\theta) \xrightarrow{r\to\infty} \left(v\cos(n\theta+\alpha), \sin(n\theta+\alpha)\right)$

equivalently $\quad \psi \sim v\, e^{i(n\theta+\alpha)}$

$$\nabla\psi = \cancelto{0}{\partial_r\psi} + \frac{1}{r}\partial_\theta\psi = \frac{in}{r}\psi.$$

However there is trouble here

Note that the energy of these vertices.

$$E[\psi] = \int d^2x \left(|\partial_i \psi|^2 + V(\psi) \right)$$

for $r \to \infty$, $V(\psi) \to 0$ $\because$ ψ settles into the minimum of the potential.

However $|\partial_i \psi|\Big|_{r\to\infty} \sim \frac{v}{r}$

$$\Rightarrow \int d^2x \, \frac{v^2}{r^2} \sim \ln r \Big|_{r\to\infty}$$

So the energy of these vertices is not finite!

[This is as it should be. Derrick's theorem already told us that there are no finite energy, static soln. in greater that one dim for scalar fields]

How can we fix this? Let us introduce a gauge field.

$$\mathcal{L} = |\mathcal{D}_\mu \psi|^2 - \frac{\lambda}{4!}(|\psi|^2 - v^2)$$

$$\mathcal{D}_\mu = \partial_\mu + iqA_\mu.$$

The trouble arose because

$$\nabla \psi = i\nabla(n\theta)\,\psi = i\frac{1}{r}\underset{\uparrow_{\theta\text{ derivative}}}{\partial_\theta}(n\theta)\,\psi = \frac{in}{r}\psi$$

$A_r = 0$

$A_\theta = -\frac{n}{qr}$

as $r \to \infty$.

So we want $\vec{A} = \frac{1}{q}\nabla(n\theta)$ $\left[A^i = \overset{\text{Note}}{\downarrow}\!\!-A_i\right]$

In this case $|\mathcal{D}_i\psi| \to 0$ as $r \to \infty$.
and we can get a finite energy solution.

Should we worry about the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ contribution of the gauge field?

Note that at infinity $\vec{A} \sim \underset{\uparrow}{\nabla}(n\theta) \Rightarrow F_{ij} \to 0$
So no contribution!

Apart from making the energy of the vortex finite, what does the gauge field actually do?

Consider the magnetic flux.

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l} = \oint_{r\to\infty} \frac{\nabla(n\theta)}{q}\cdot r\,d\theta$$

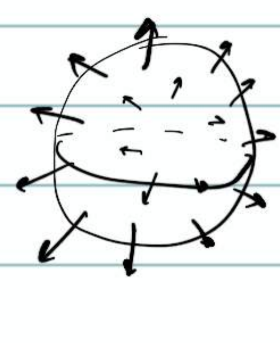
Stoke's theorem.

$$= -\frac{2\pi n}{q} \longleftarrow \int_0^{2\pi} A_\theta \, r\, d\theta$$

Thus, the flux for a vortex is quantized : $\left(\frac{2\pi}{q}\right) n$ ← winding #.

In 3 dimensions, we would have described the object as a monopole!

(SSB with gauge fields has topological solitons!)



Much of the description here can be carried over to superfluids and such.

Let me drive home some general statements about the role of topology here.

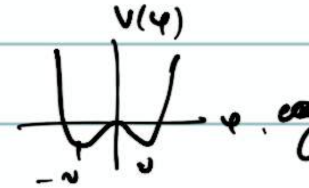
We made maps between spatial infinity in 1, 2, 3 dimensions and the vaccum manifold.

For example in 2 dimensions we asked how to map the circle at infinity to the circle of the (field) vacuum manifold. We mapped circle at ∞ to multiple windings around the vaccuum manifold. None of these maps can be reduced to a point by a deformation. You can think of the class of maps with the same winding # as the element of a group: $\pi_1(S^1) = \mathbb{Z}$ ← integers.

This is a homotopy group.

More generally $\Pi_n(S^n) = Z$

topologically S^n

$\Pi_0(S^0) = Z_2$ (this is the 

e.g.).

These homotopy groups which rely only on topology (not geometry) can give immediate insights into field configurations and their stability.