PHYS 622 (Spring 2017)

Astrophysics II: Galaxies and Cosmology
Tues & Thurs, 2:30-3:45 pm, HBH 453

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office: HBH 364
office hrs: Thurs 4:00-5:00 pm (or by appointment)
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Course Description: This course is an introduction to modern cosmology and aspects of galaxies. On the cosmological front, you will learn about the standard cosmological model: inflation followed by a radiation, a dark-matter and finally a cosmological constant dominated universe. You will learn about (1) the interplay between the contents of the universe and its space-time dynamics, (2) about how different particle species emerge from the thermal soup after the big bang and (3) the evolution of structure in our universe: from tiny quantum fluctuations during our universe's infancy to correlated distribution of galaxies we see in the sky today. In second half of the course we will discuss galaxies in detail including our own host galaxy. We will emphasize the kinematics and dynamics of galaxies, their classifications as well as important scaling relations between observables. Throughout the course, we will emphasize the the interplay between different areas of physics in an astrophysical/cosmological setting and how our current theoretical understanding of galaxies and cosmology is firmly rooted in observations.

Learning Objectives:
By the end of the course, you (the student) should be able to to do the following:

- Describe and calculate the dynamics of the homogeneous universe, understand how the dynamics relates to its contents as well as observational probes of these dynamics.
- Be familiar with the thermal history of our universe and be able to calculate how different species of particles emerge from the thermal soup in the early universe.
- Calculate the evolution of density perturbations in our universe; from initial conditions during inflation to the present day and understand corresponding observations.
- Describe properties of our host galaxy including its morphology and kinematics.
- Understand galaxy classifications and be familiar with scaling relations between different observables of galaxies.
- Make order of magnitude estimates, apply concepts from many different areas of physics and perform detailed calculations of astrophysical/cosmological observables with an understanding of the approximations involved.

Prerequisites: A good base in electromagnetism, classical mechanics, statistical mechanics, special relativity, quantum mechanics, and of course prior exposure to astrophysics in general will be helpful. I will introduce relevant ideas as needed for (aspects of) general relativity, familiarity with GR will make your life easier. I will assume familiarity with systems of ordinary differential equations, multivariable calculus, Fourier analysis and linear algebra. Formally, for undergraduate students at Rice the prerequisites include (ASTR 350 OR ASTR 360) and (PHYS 301 and PHYS 302).

Class Website: All course materials including problem sets, links to relevant websites, supplementary material, class updates and announcements will be posted on the ASTR 452 Canvas page. It is your responsibility to check Canvas regularly for the most recent information concerning the class.

Main Text(s): I will provide hand written class notes after class. For the cosmology section, you should refer to the cosmology notes by D. Baumann which be available on Canvas. For the galaxies part, I recommend the book by Peter Schneider. The book by Schneider is recommended, but not required for you to buy.

- *Class Notes*: My hand written notes for the class will be provided after each class.
- *Cosmology Lecture Notes* by Baumann (U. of Amsterdam): We will mostly follow Baumann's notes for the cosmology part of this course, with some omissions and additions. At the minimum, you should read these notes and class notes.
- *Extragalactic Astronomy and Cosmology* by Peter Schneider: We will use this book for the galaxies part of the course. It has a good balance of relevant observational details along with the theory. It is also a gentler, less mathematical treatment for the cosmology part of the course.

Additional Resources:

- *Cosmology* by Steven Weinberg: If you want rigor, this is the place to go to. The presentation is "clean". It has most of the interesting things you can do "by hand" in cosmology. It might not be easy on the first reading but I recommend referring to it if you are confused elsewhere.
- *Physical Cosmology* by Mukhanov: I like the treatment of hydrodynamical perturbations in this book. It also has some more advanced topics relevant for the very early universe.
- *Spacetime and Geometry: An Introduction to General Relativity* by Carroll. This is an excellent textbook for GR at the graduate level.
- *Physical Cosmology* by Scott Dodelson: This is a detailed, graduate level text. It is particularly good in its detailed treatment of the cosmic microwave background.
- *An Introduction to Modern Astrophysics* by Carroll & Ostlie. Less advanced than the course, but comprehensive.

Caution: Notation varies across texts! Baumann and Mukhanov use the "mostly minus" $+ - --$ metric convention while Weinberg, Dodelson and Carroll use the "mostly plus" $- + ++$ metric convention.

Exams and Problem Sets:
Problem sets every 1 - 2 weeks
2 Oral Exams (a Midterm and a Final)

Grading Policies: Homeworks will account for 60% of the grade and the two exam will account 20% each of the total grade. Late Homeworks are annoying. For unexcused tardiness, there will be a 20% reduction/per day in credit, with no credit 2 days after the homework is due. If illness or other circumstances beyond your control lead to a delay in submission, please contact me as soon as possible (preferably before the deadline). The final grade for the course will include some discretion based on class participation/interactions etc. Attendance is not mandatory, but highly recommended.

Collaboration and Help: For a better understanding of the material you are strongly encouraged to talk to other students, postdocs and faculty (including me!). For the problem sets, you should work on each problem independently first ($\geq$ 1 hr). When needed, get help about the general idea of how to do the problem but not the details of the entire calculation. If you collaborate/get help from others, they should be acknowledged in the problem sets along with details of what you collaborated/got help on. The write-up should always be done independently. Do not look up solutions online or from previous years. The Honor Code applies. You can review Rice's Honor Council documentation online at: honor.rice.edu/index.cfm

Special Needs: If you have a documented disability that requires special consideration for this class then please contact the me as soon as possible to discuss your needs. Students with disabilities should also contact the Disability Support Services Office in the Ley Student Center (dss.rice.edu).

Note: I reserve the right to change the contents and policies in this syllabus during the semester to suit the needs of the class.

Acknowledgements

Special thanks to Daniel Baumann & Anthony Challinor for their excellent cosmology notes. For the cosmology section, I use them extensively. They were made available to the students along with their hand written notes.

I have also used material from Dodelson's "Modern Cosmology" & Peter Schneider's "Extragalactic astrophysics & cosmology". In the Galaxies part, I benefited from discussions with Andrea Isella.

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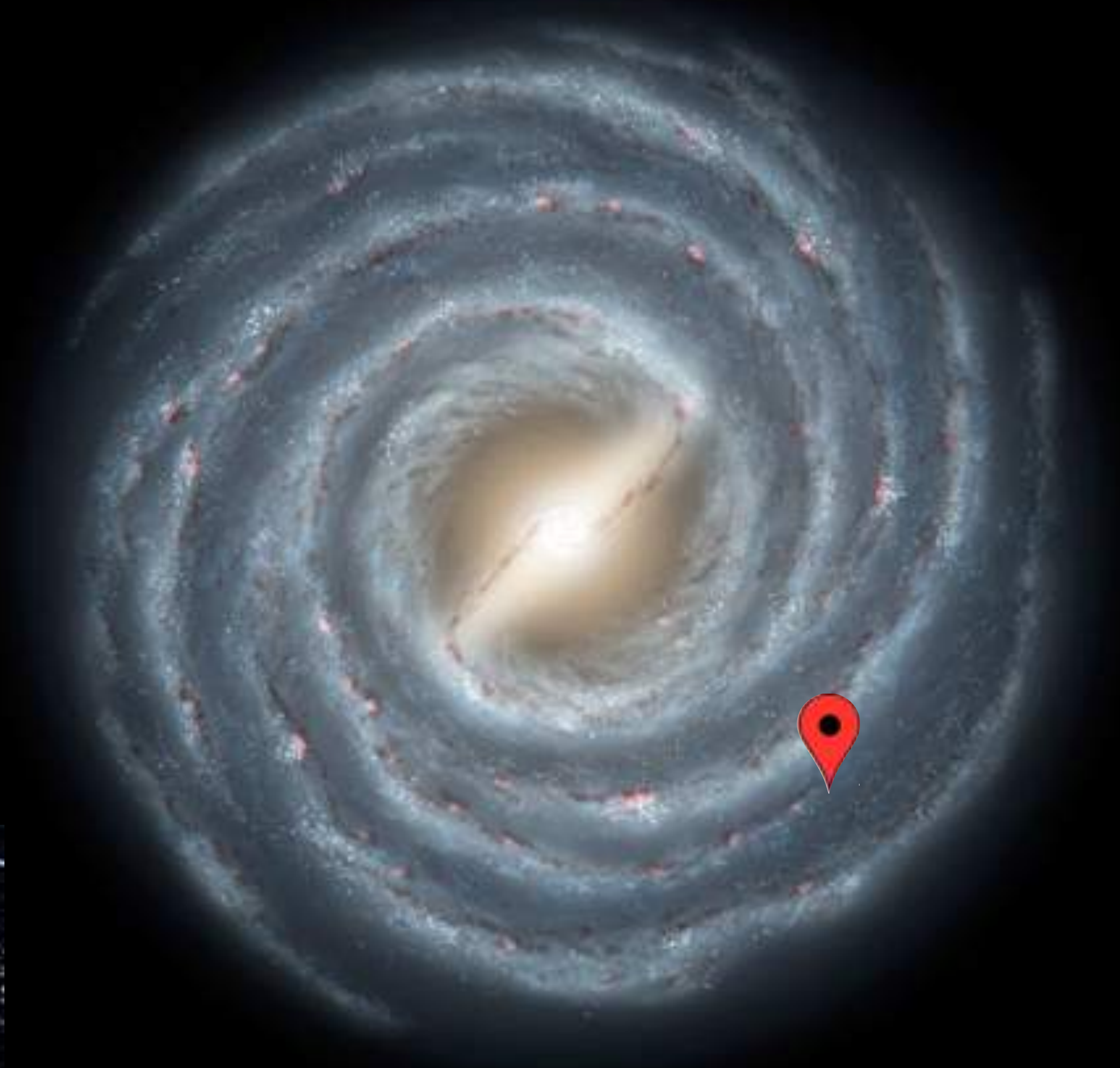
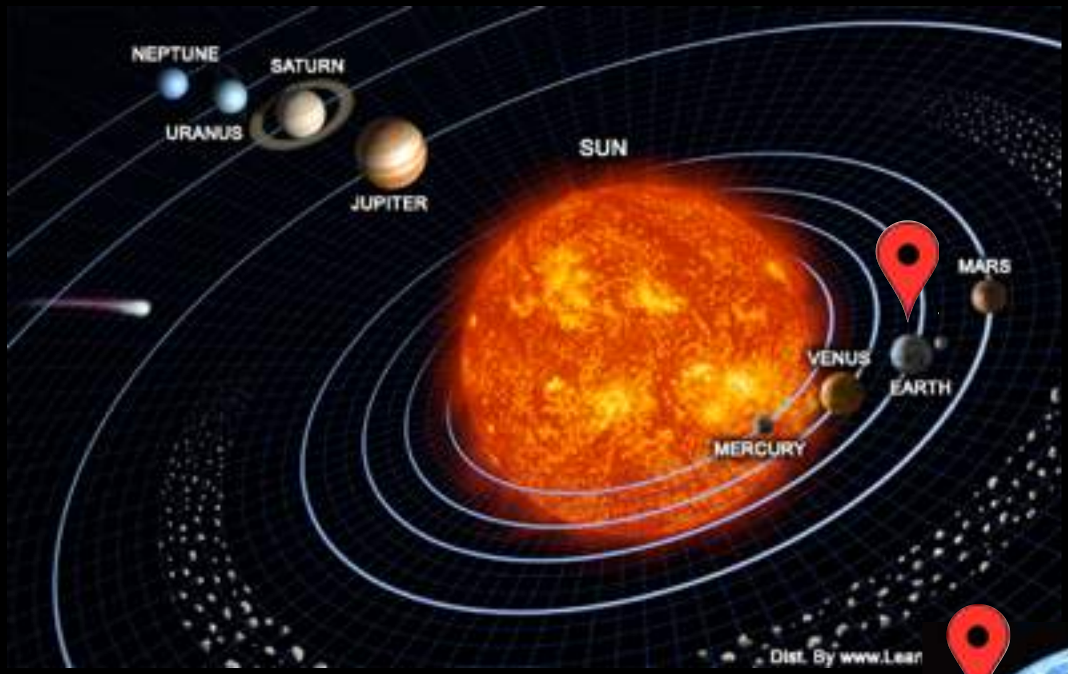
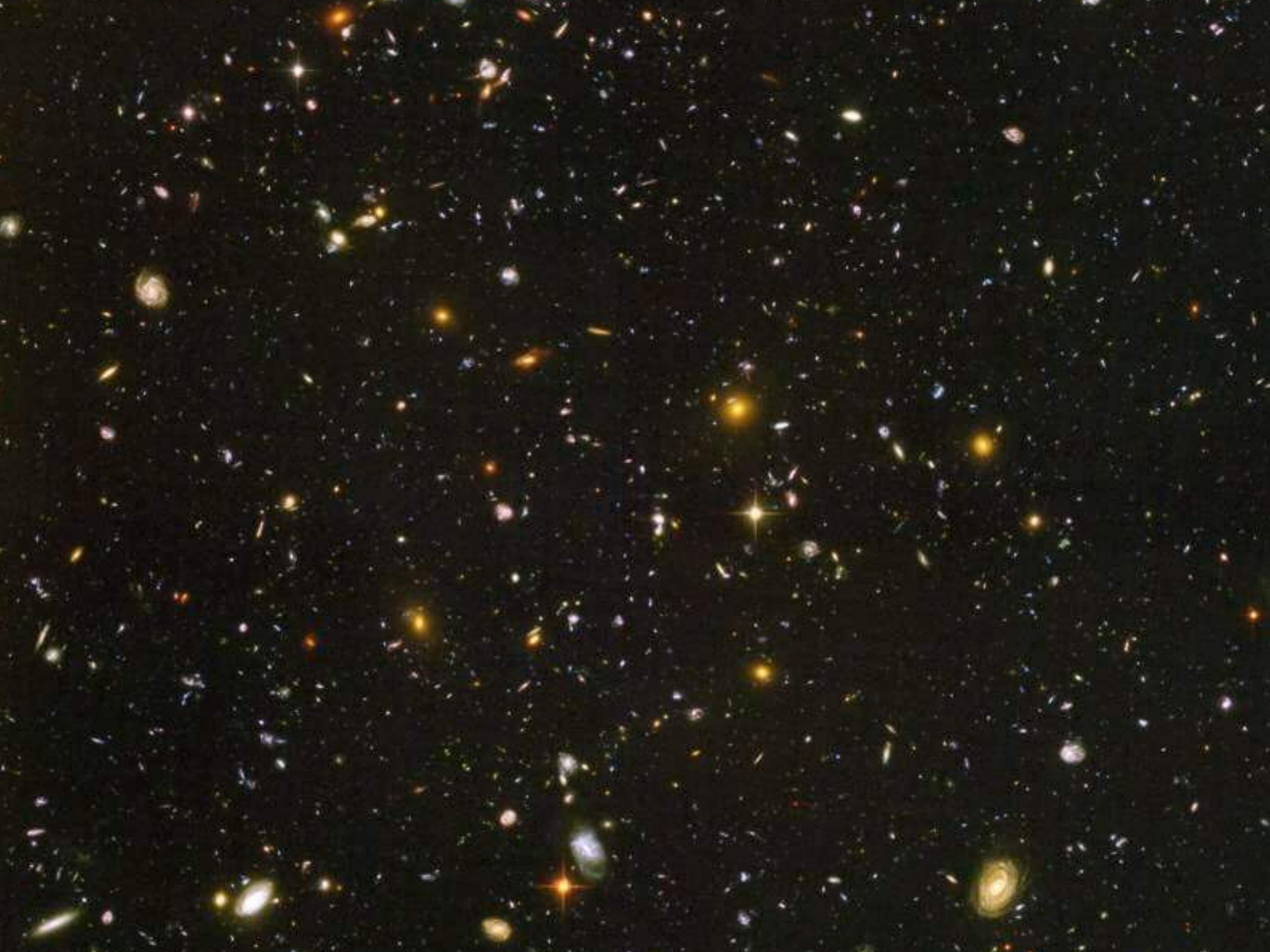
an orientation
and an invitation




NEPTUNE
SATURN
URANUS
SUN
JUPITER
MARS
VENUS
EARTH
MERCURY

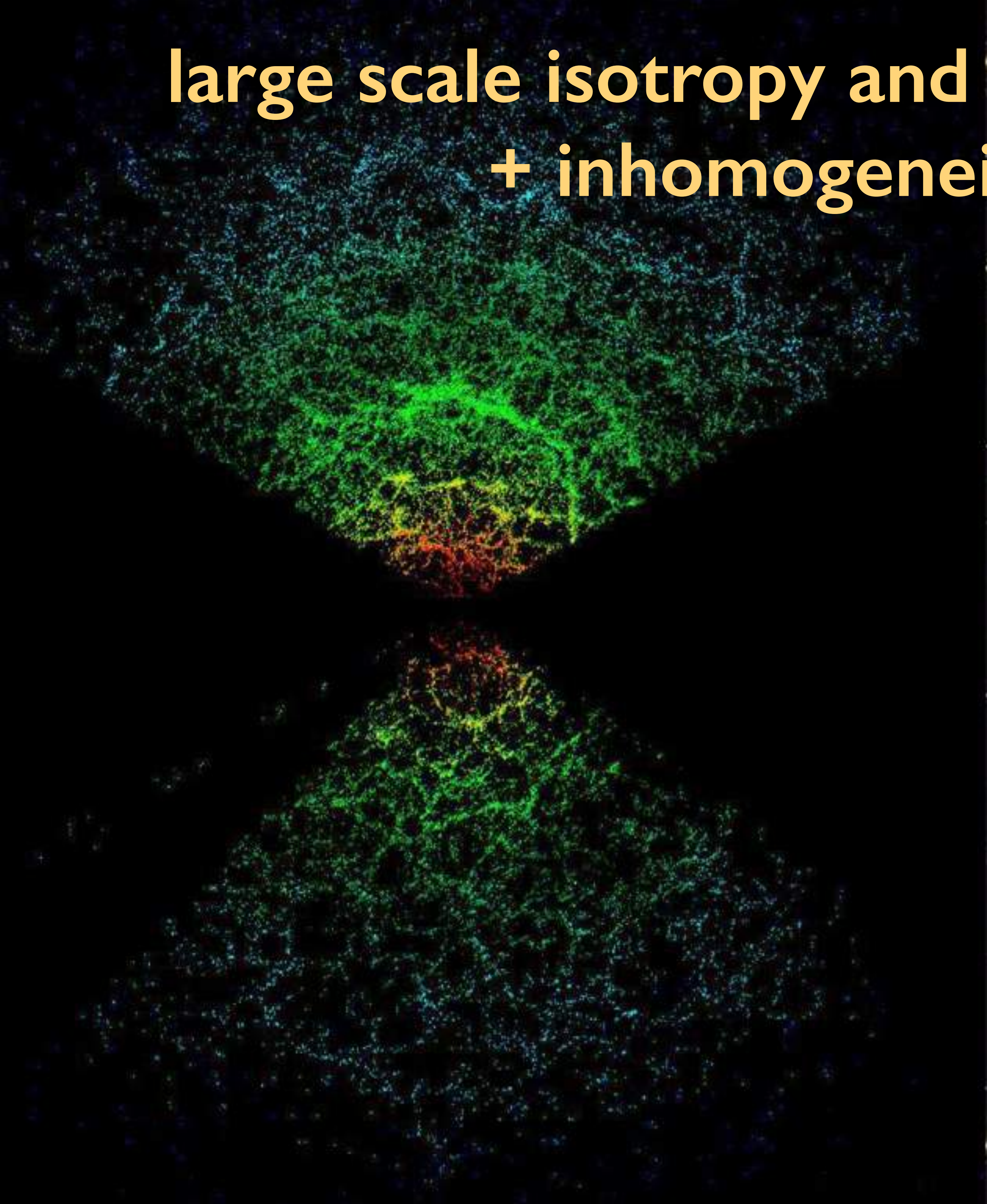
we are here





1930's

large scale isotropy and homogeneity
+ inhomogeneities



Cosmic Microwave Background (CMB)

Penzias and Wilson (1965)

Credit: Roger Ressmeyer

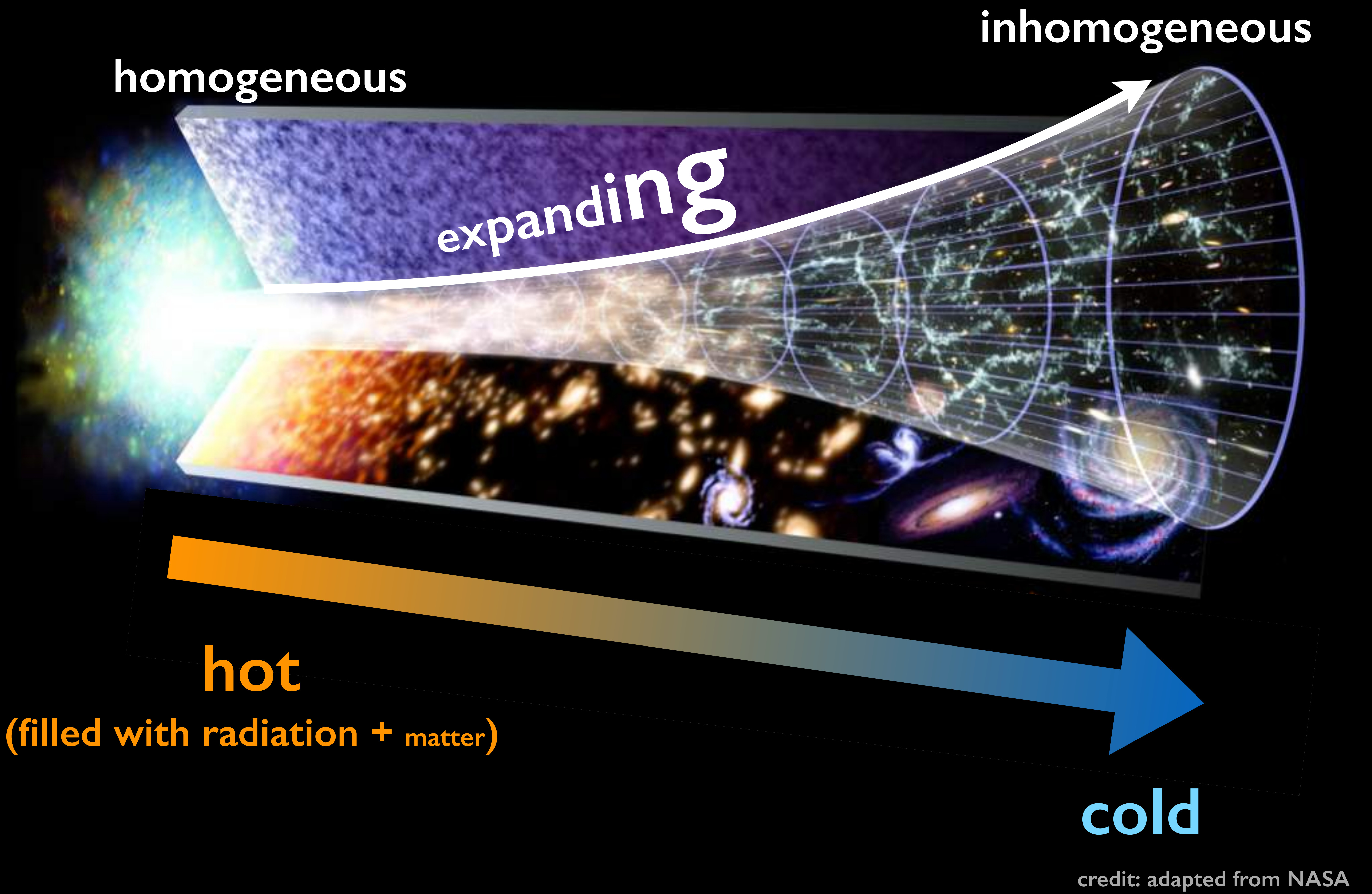
cosmic microwave background

$$T = 2.726 \pm 0.001\,\mathrm{K}$$

COBE/FIRAS
Mather et. al 1994
Fixen et. al 1996

our cosmic story

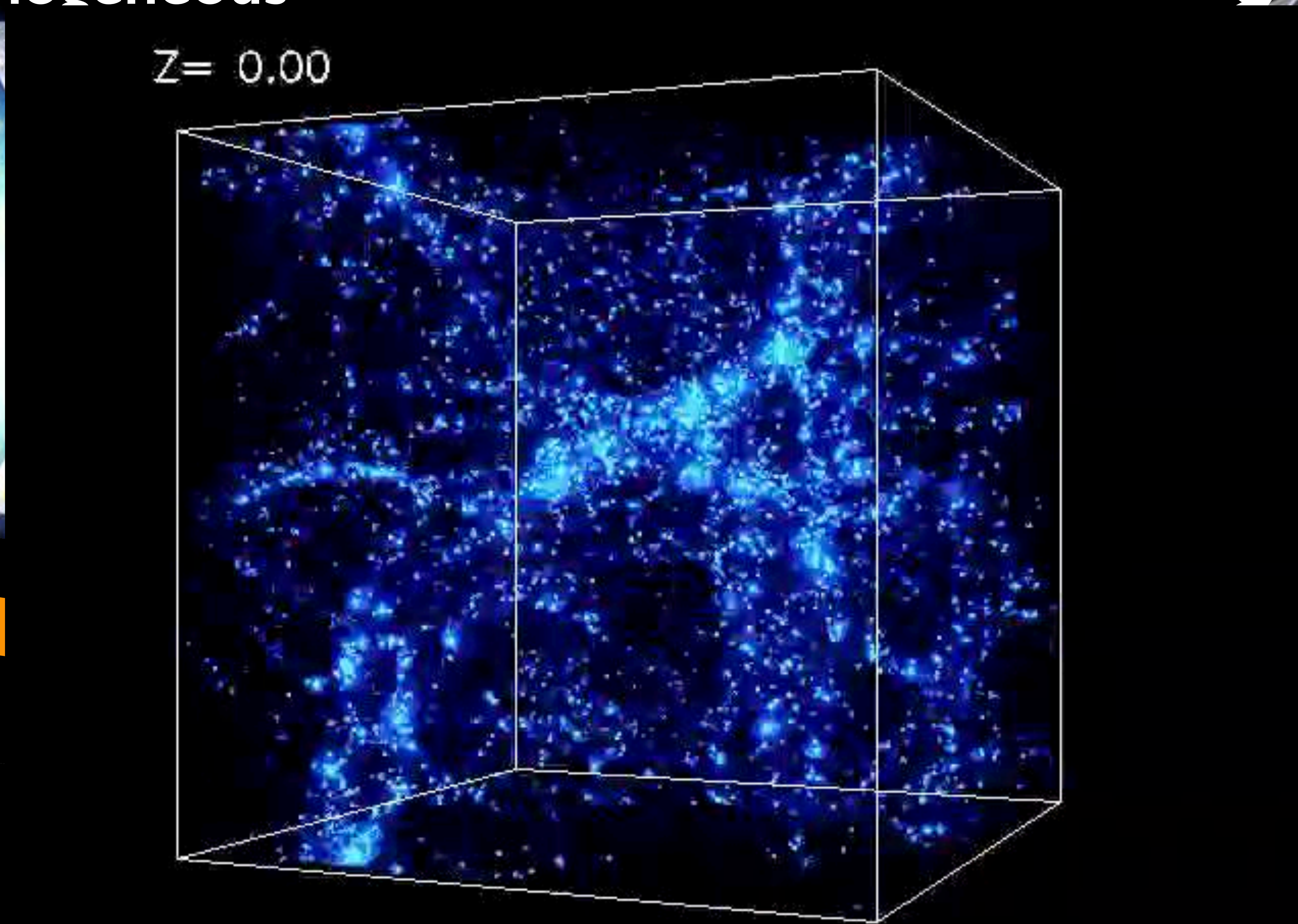
The Standard Cosmology



formation of structure

homogeneous

inhomogeneous



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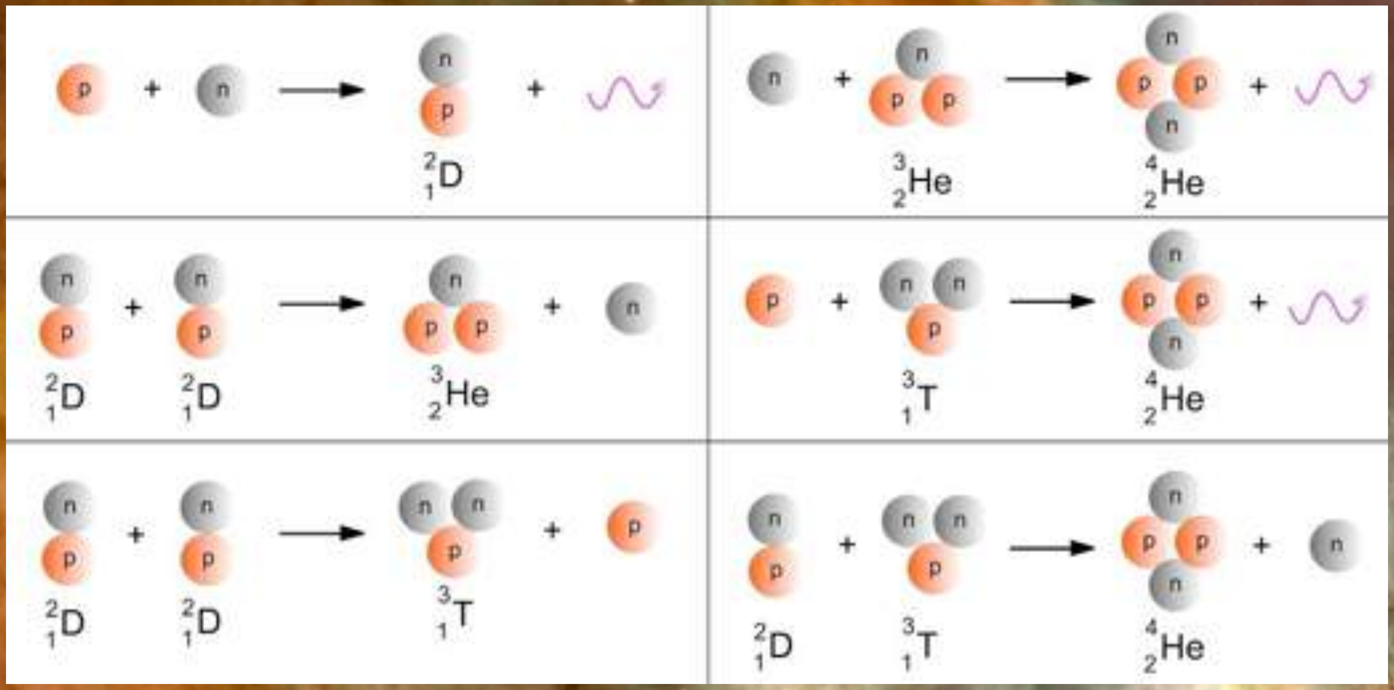
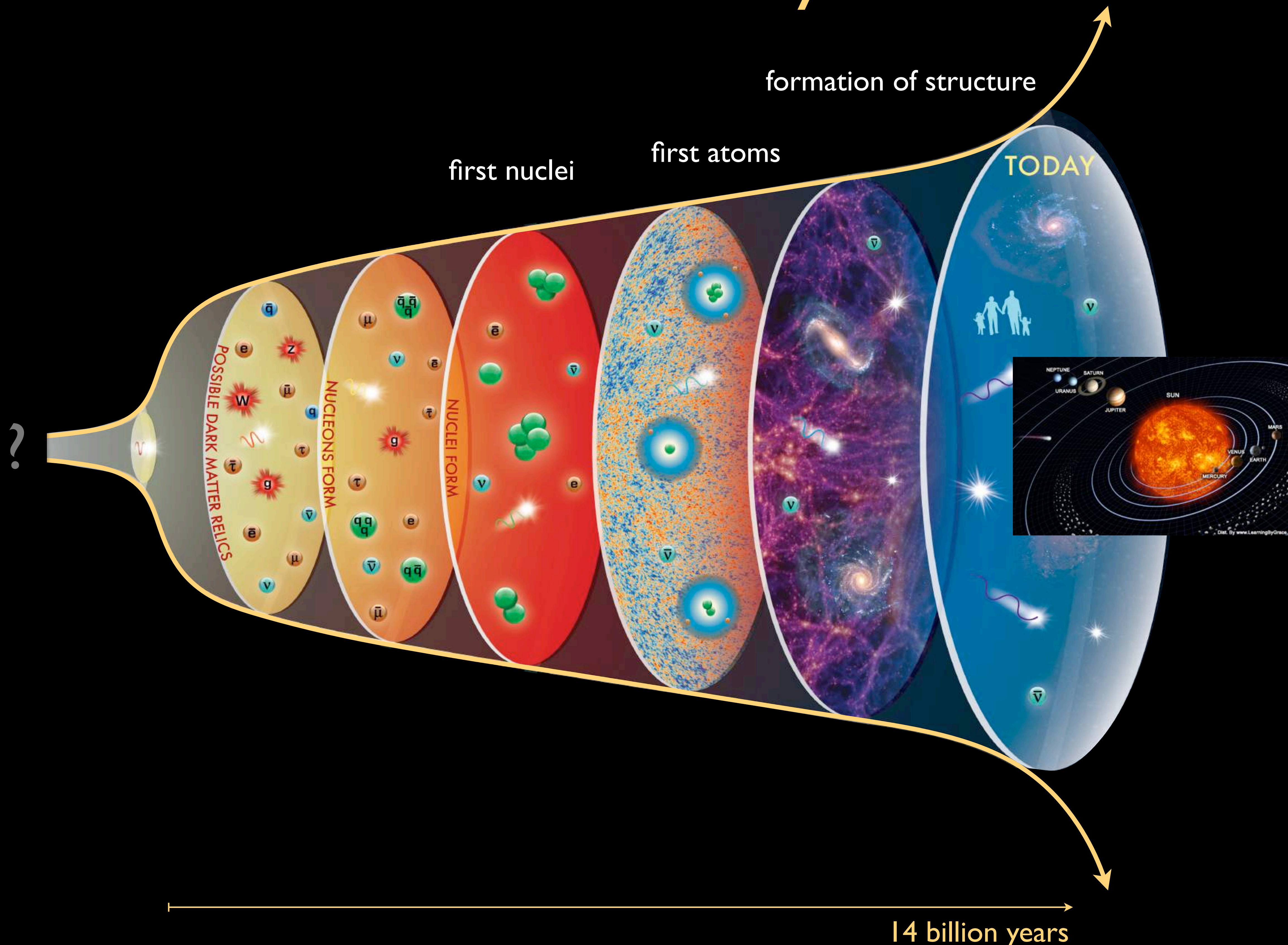
credit: adapted from NASA

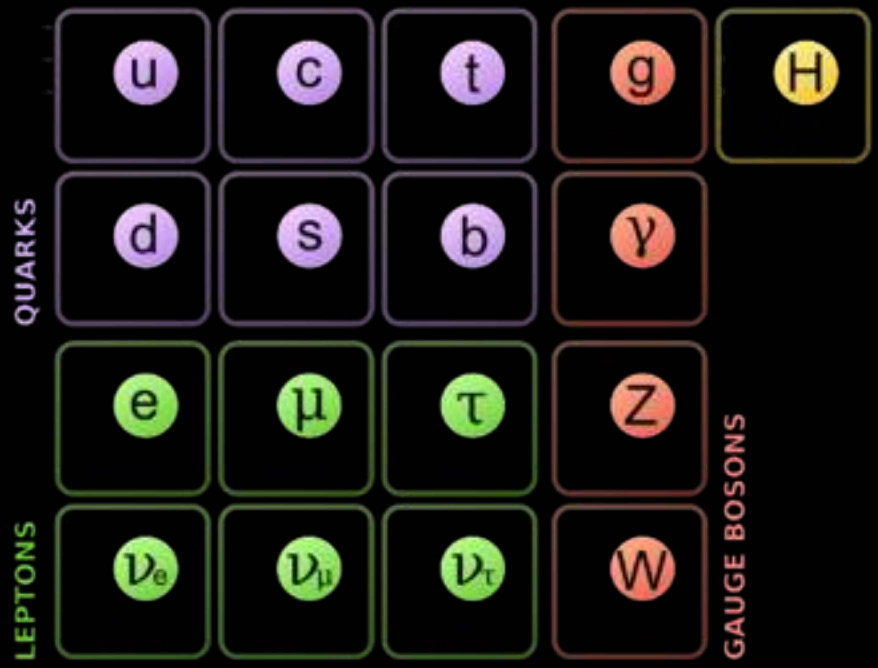


chemical enrichment

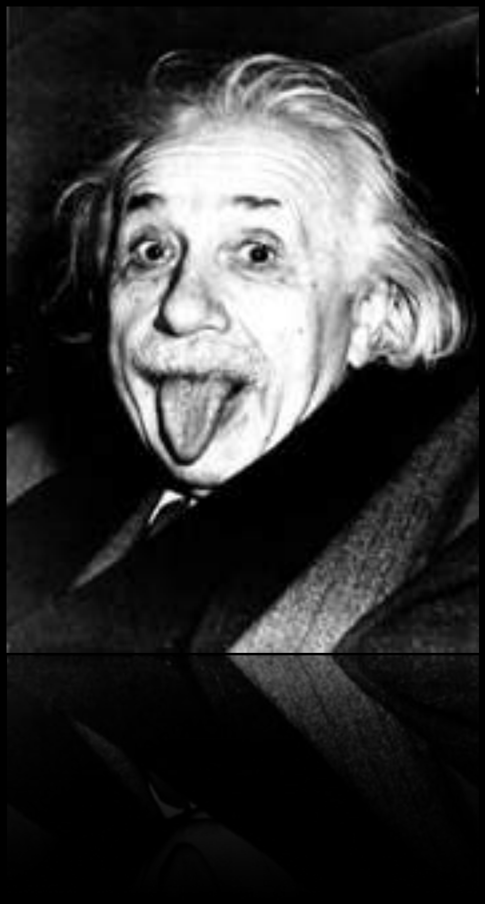
our cosmic story



seems sufficient …

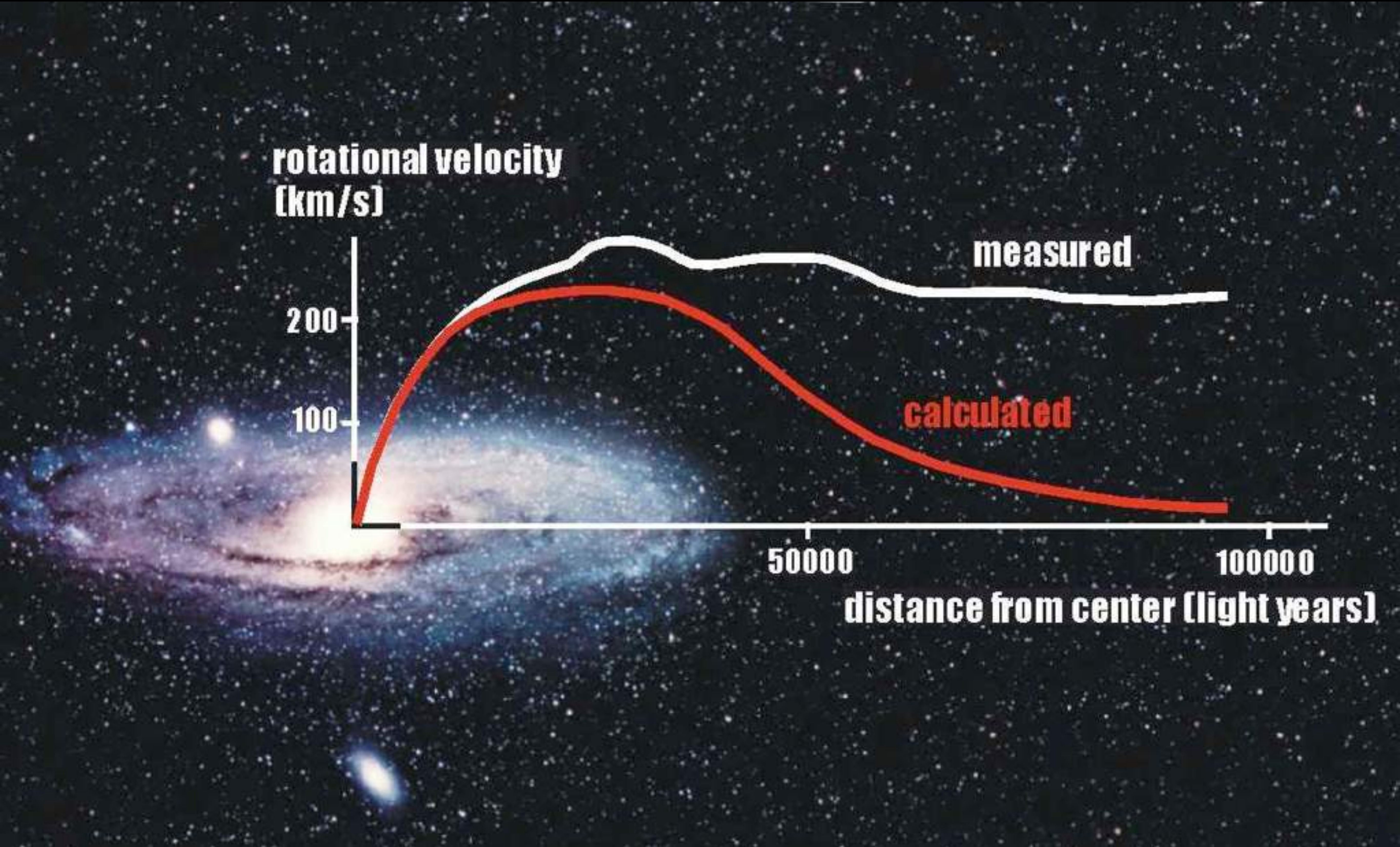


+ QFT) + GR



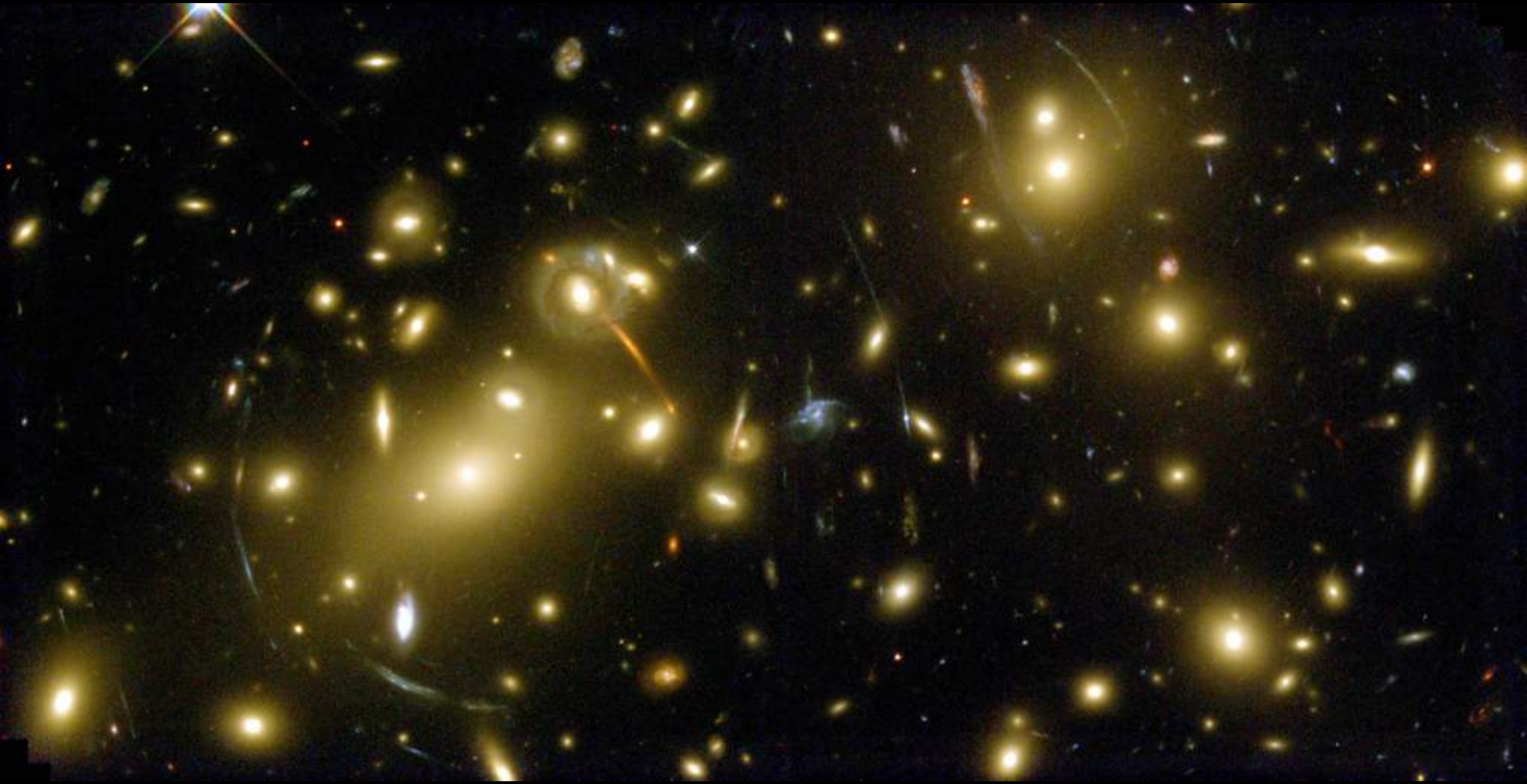
but wait …

galaxies are rotating too fast …

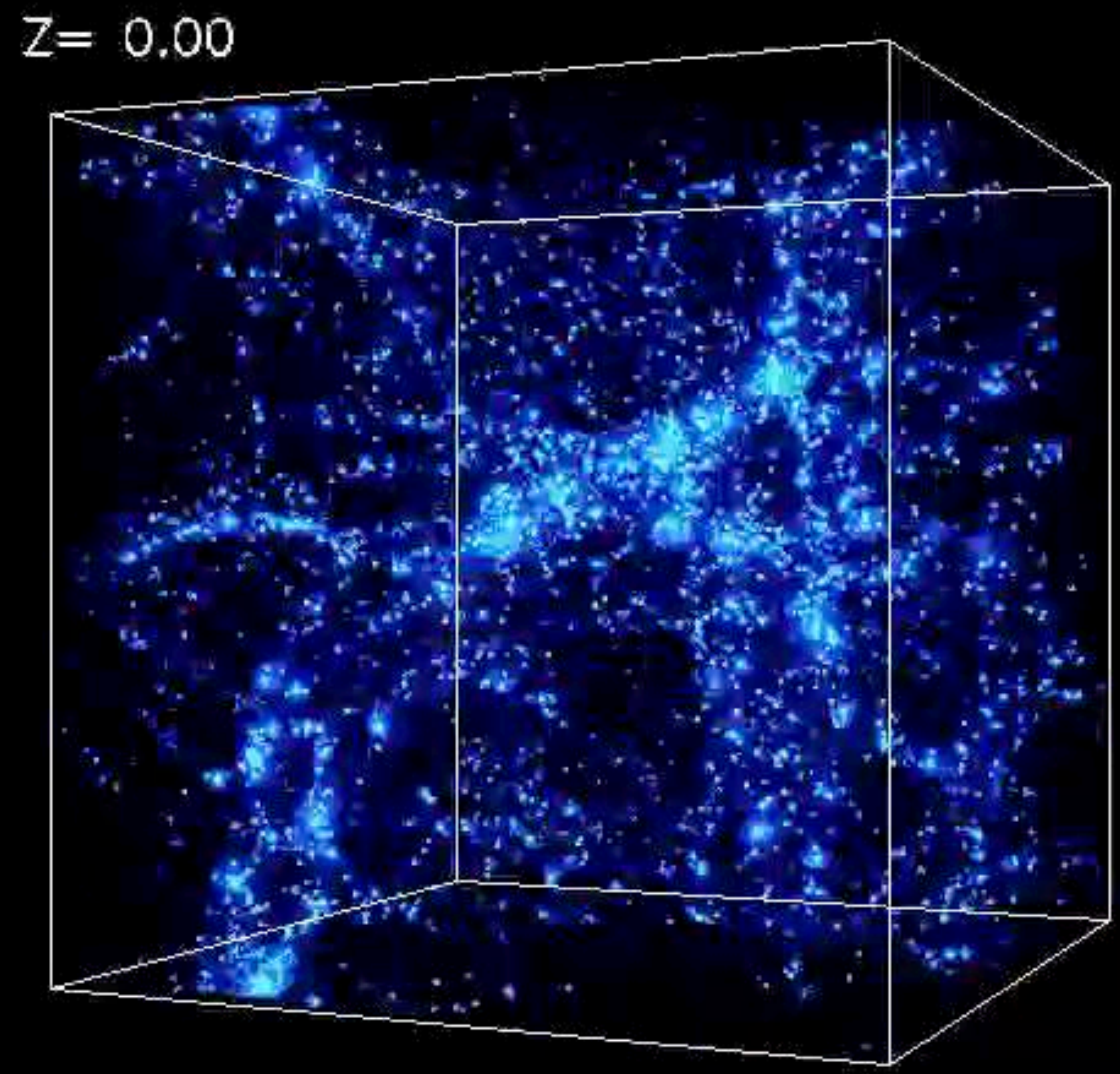


galaxies cluster are way too hot …

too much distortion of images



need "faster" gravitational growth



and then there is …


universe is undergoing
accelerated expansion !

and …

why is this so uniform?

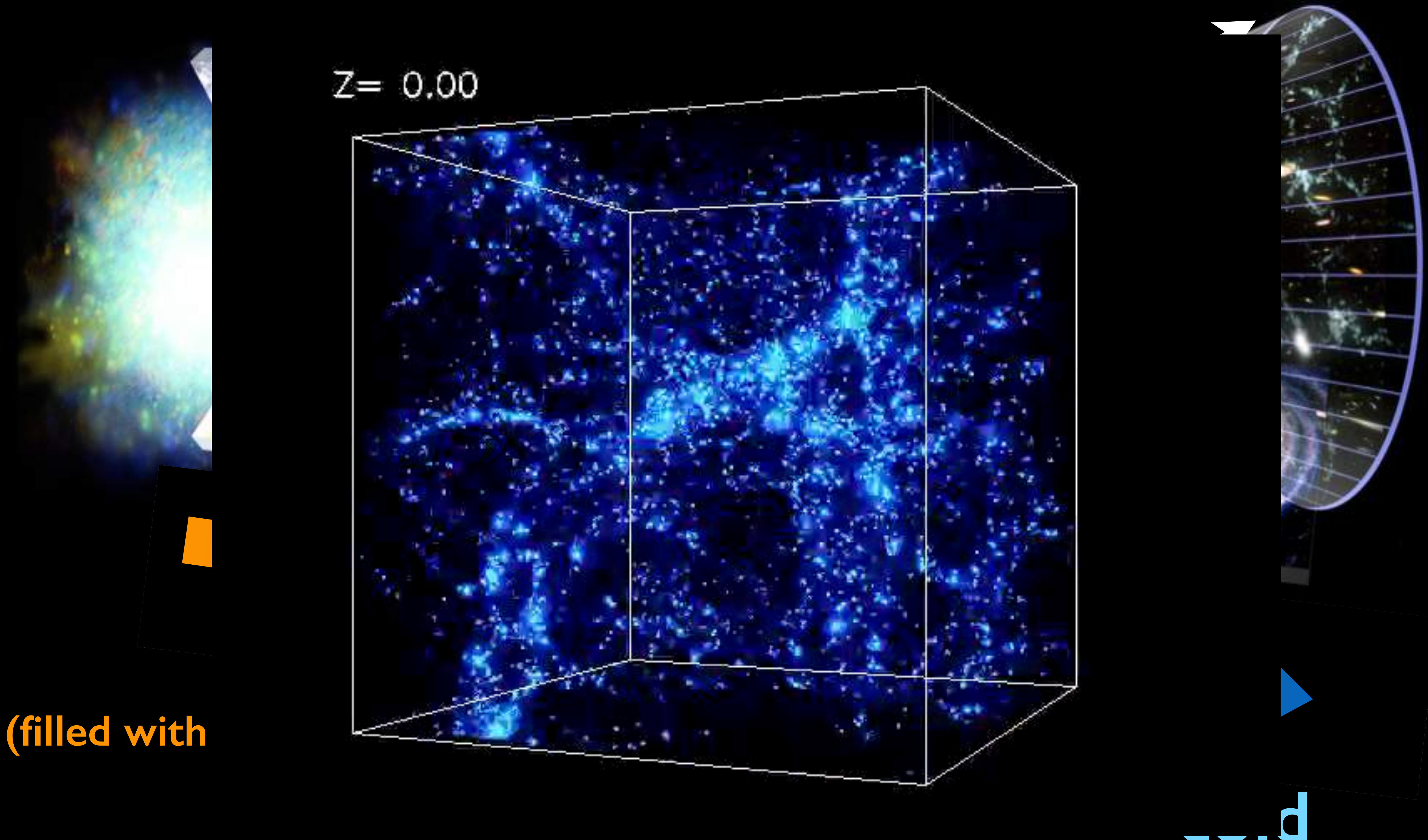
requires seemingly causal correlations

$$T = 2.726 \pm 0.001\,\mathrm{K}$$

COBE/FIRAS
Mather et. al 1994
Fixen et. al 1996

correct “initial” conditions?

requires seemingly causal correlations



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credit: adapted from NASA

you will learn:

- how we know what we know about the cosmos

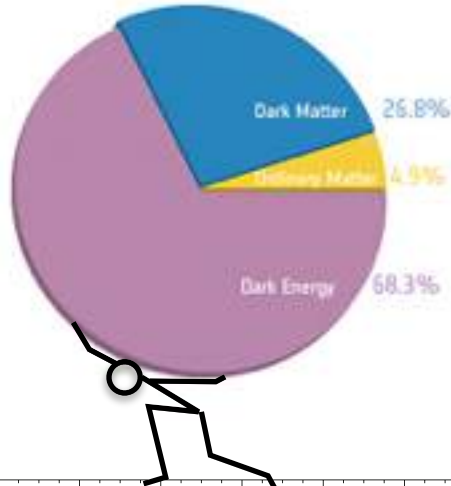
- our best physical model for our universe

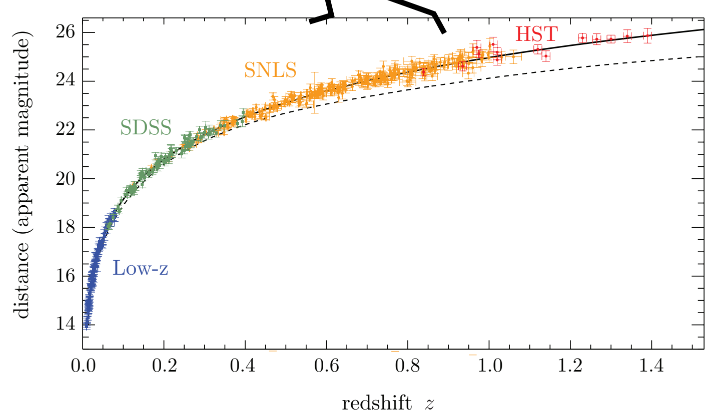
along with "known" physics, we will need to add

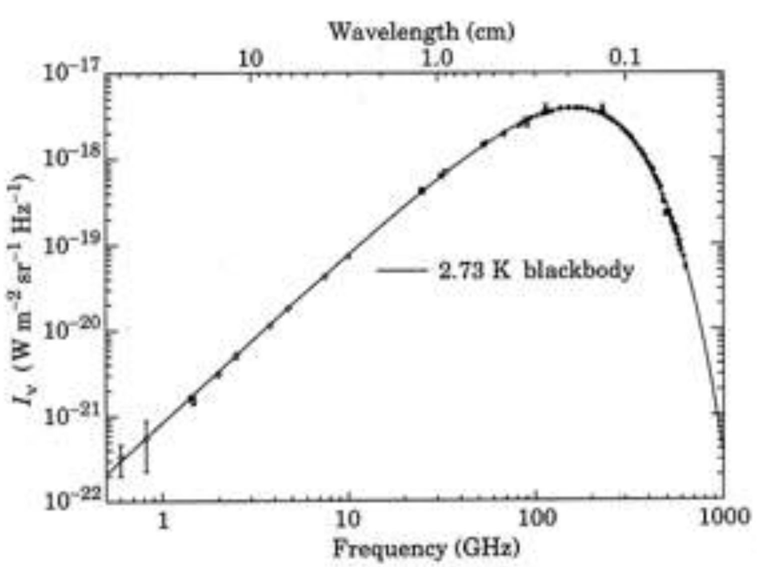
- dark matter
- dark energy
- inflation

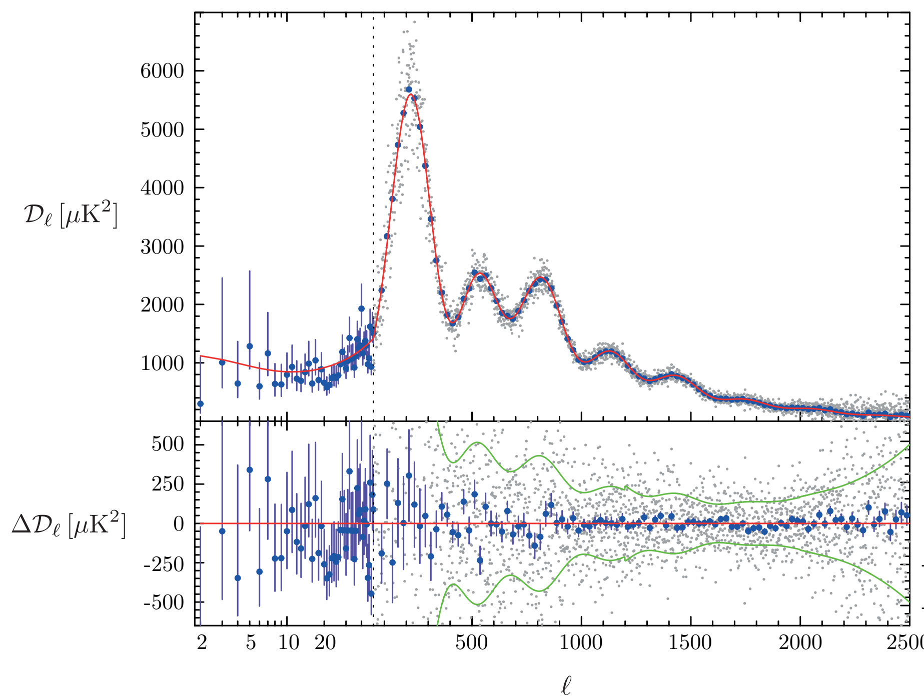
amongst other things, also matter-antimatter asymmetry

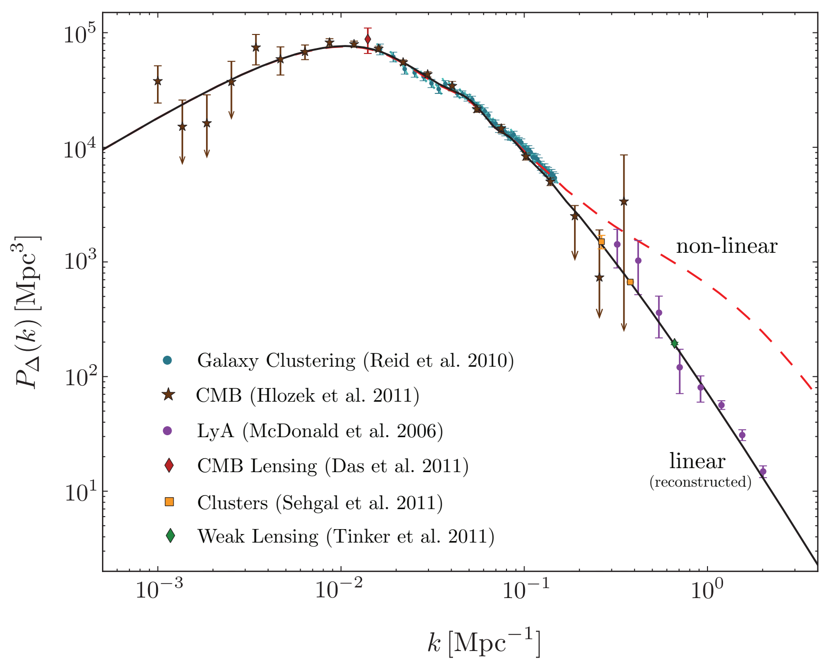
precision cosmology !

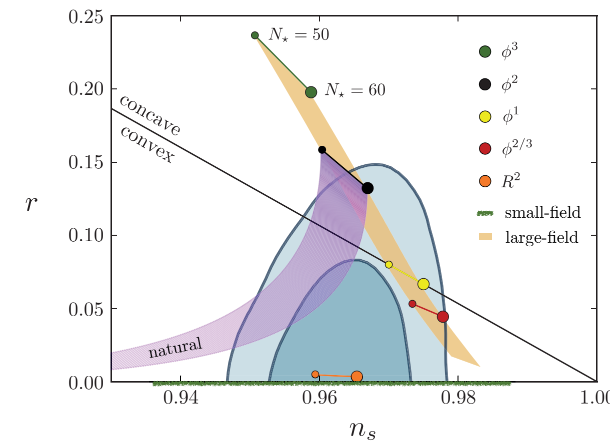


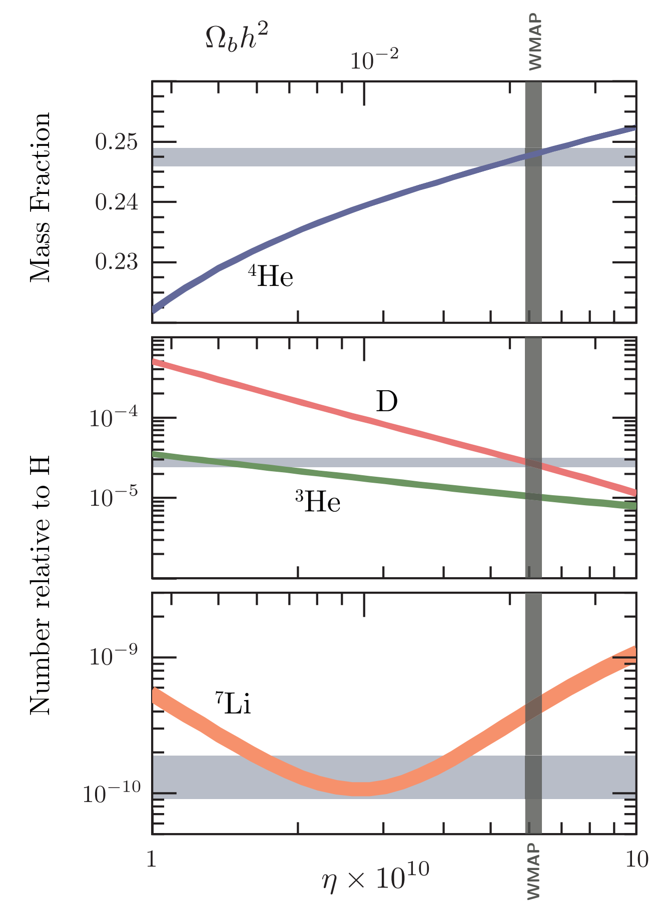




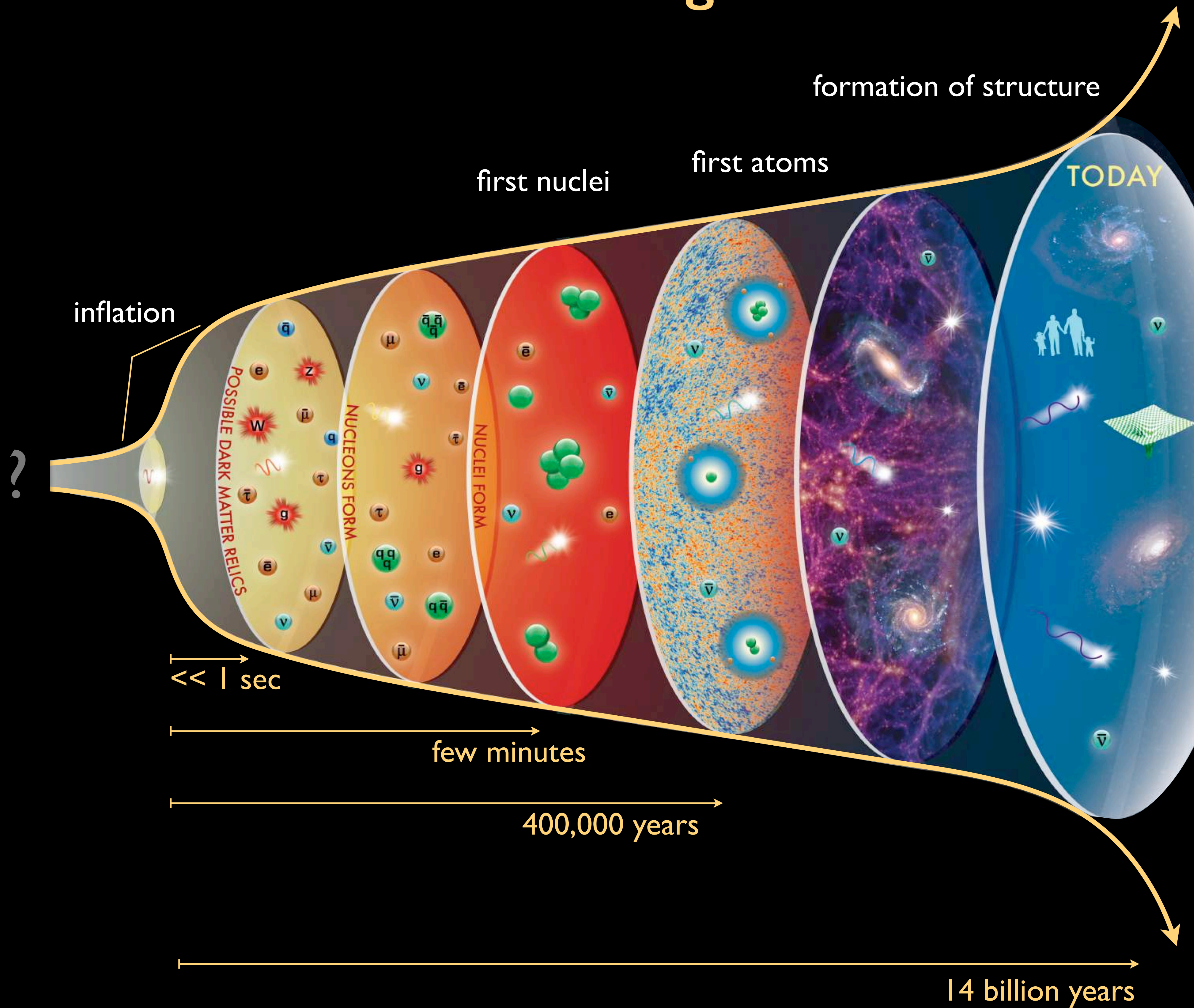








how did we get here ?



Lecture 1

Plan : 1) An orientation an invitation to cosmology

2) Spacetime Geometry - homogeneous & isotropic universe.

Read : pg 1-8 of Baumann's Notes.

1). An orientation and invitation to cosmology
- see slides.

2) Spacetime Geometry

Spacetime = collection of events with a rule for measuring intervals between nearby events.

Co-ordinates = labels = 4 numbers for each spacetime event

Example:
Let Q be an event. In a small neighborhood of this event we label all the events by $(\underbrace{x^0}_{\text{time}}, \underbrace{x^1, x^2, x^3}_{\text{space}})$. We will use the

short hand x^μ where $\mu = 0,1,2,3$ with

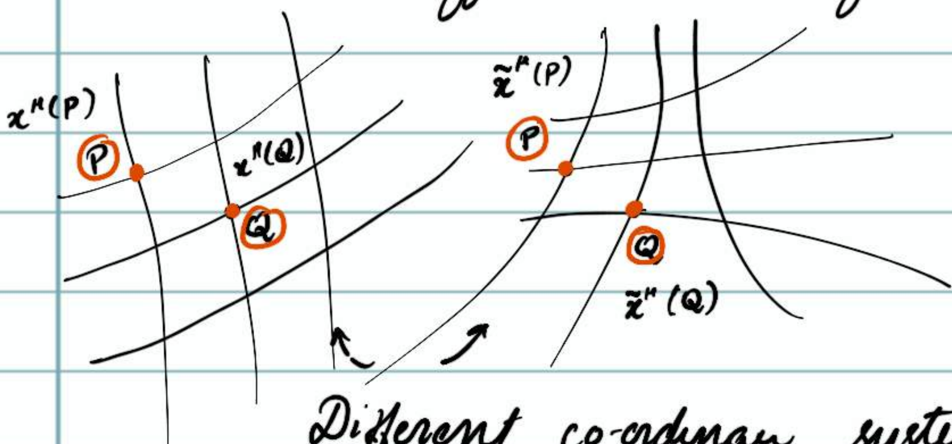
x^0 = time co-ordinate

x^1, x^2, x^3 = spatial co-ordinates

[need not be cartesian]

The co-ordinates of the event $Q \doteq x^\mu(Q)$.

Nothing special about labelling. We could chose a different set of labels of $Q \doteq \tilde{x}^\mu(Q)$.



Different co-ordinate systems = grid imposed on spacetime

To understand the Geometric structure of spacetime, we need Interval between events : (infinitesimal)

$$ds^2 = \underbrace{g_{\mu\nu}(x^\alpha)}_{\boxed{\text{metric tensor}}} dx^\mu dx^\nu \qquad \mu, \nu = 0,1,2,3.$$

Repeated upstairs & downstairs indices are summed.

$$ds^2 = g_{00}\, dx^0 dx^0 + g_{01}\, dx^0 dx^1 + g_{10}\, dx^1 dx^0 + \cdots + \ldots g_{23}\, dx^2 dx^3 + g_{33}\, dx^3 dx^3$$

$$g_{\mu\nu} \sim \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} =$$

components form 4x4 symmetric matrix.

$g_{\mu\nu} = g_{\nu\mu}$.

[Can always symmetrize]

Interval between events does not depend on co-ordinates

$$ds^2 = g_{\mu\nu}(x^\alpha)\, dx^\mu dx^\nu = \tilde{g}_{\mu\nu}(\tilde{x}^\alpha)\, d\tilde{x}^\mu d\tilde{x}^\nu$$

↳ co-ordinate invariant.

$$\left[\tilde{g}_{\mu\nu}(\tilde{x}^\alpha) = g_{\alpha\beta}(x^\alpha) \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \right]$$

how are they related?

The Geometry of spacetime is such that in a small enough region, "it looks" like Minkowski Space. (Flat spacetime)

In terms of co-ordinates, it means that in a small enough regions, one can always find a co-ordinate systems such that the interval takes the form

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = (dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

(cartesian).

$$\eta_{\mu\nu} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

($c = 1$ for this course)

For Minkowski spacetime of special relativity, it is poss. to write the metric in this form everywhere.

For more general spacetimes $g_{\mu\nu}(x^\alpha)$ is a function of spacetime co-ordinates and cannot be made to look like Minkowski spacetime everywhere under the same co-ordinate system.

Questions & Answers

Question: If someone hands you a metric $g_{\mu\nu}(x^\alpha)$ in some co-ordinate system for a region of spacetime and the components depend on the co-ordinates. Is this spacetime necessarily not flat?

Ans: No. Why? Write down the metric for Minkowski space in spherical-polar co-ordinates.

Hence, just because $g_{\mu\nu}$ depends on co-ordinates is not sufficient to garuntee that spacetime in that region is curved.

* A foolproof way of checking is to see if the "Riemann Curvature tensor" $R^{\alpha}{}_{\beta\gamma\delta}$ which is constructed out of (up to) second order derivatives of the metric tensor $(\partial_\gamma g_{\alpha\beta}, \partial_\gamma \partial_\beta g_{\sigma\phi} \dots)$

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta}(x^\alpha) = 0 \iff \text{spacetime is flat at } x^\alpha}$$

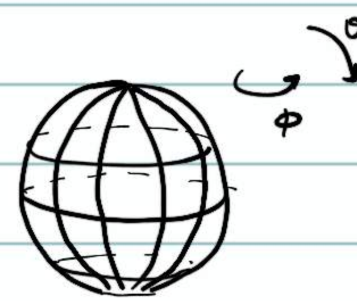
* A more physical way is to let go of a collection of particles and see if the "seperation" between them changes (in absence of any other forces) [Geodesic Deviation].

* But, we said in small enough regions, all spacetimes look like Minkowski Space. Doesn't this mean $R^{\alpha}{}_{\beta\gamma\delta}(x^\alpha) = 0$?

Ans: No. "Looks like" is imprecise; one can calculate ~~still~~ deviation from flat spacetime when one calculates second derivatives of $g_{\mu\nu}$ which is what $R^{\alpha}{}_{\beta\gamma\delta}$ does. [First derivatives can be set to zero by appropriate choice of co-ordinates]

If all this stuff about metric seems too abstract, here is a simple example of a non-trivial 2D space which is curved. [Not spacetime, & not $\underset{\text{space}}{3D} + \underset{\text{time}}{1D}$].

Consider a 2-sphere (surface) of radius $R = 1$. The metric on its surface (in polar co-ordinates) is



$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$= \underbrace{g_{\theta\theta}}_{1} d\theta d\theta + \underbrace{g_{\phi\phi}}_{\sin^2\theta} d\phi d\phi .$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad \mu, \nu \in \{\theta, \phi\}$$

This space is curved. You cannot make this look like $ds^2 = du^2 + dv^2$ everywhere for any choice of co-ordinates (u, v). You will learn ways of checking whether a space is curved or not a bit later (eg, Riemann curvature tensor already mentioned).

Lecture 2

Metric for our universe on large enough scales?

Observations : On large enough scales the
[CMB, LSS] universe is isotropic.

The level of anisotropy is small (1 part in 10^5)
If the universe were inhomogeneous (radially) on large scales, then we would have to be very close to the center to observe such isotropy.
This seems unlikely [or postulate, we are not special]

$\Rightarrow$ The universe is homogeneous also.

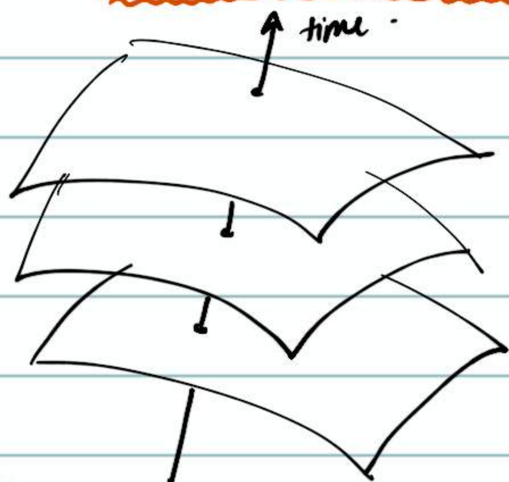
Thus, on large scales, our universe is homogeneous and isotropic.

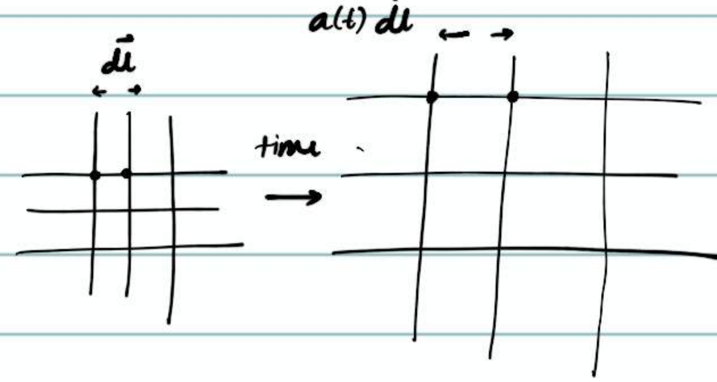
Note : 1) isotropy at one point $\not\Rightarrow$ homogeneity. We need to assume that we are not special.
2) Also note homogeneity $\not\Rightarrow$ isotropy. (Eg. uniform magnetic field).

What is the metric for such a universe?

$$\boxed{ds^2 = dt^2 - a^2(t)\,d\vec{l}^{\,2}}$$

← FRW metric is a certain set of co-ordinates





"Foliation" of space

$a(t)$ = spatial scale factor. ; $H(t) = \frac{\dot{a}(t)}{a(t)}$ = Hubble parameter.

$d\vec{l}^{\,2}$ = interval at some constant time slice.
(when $a(t) = 1$).

- Labels associated with grid points = co-moving [without $a(t)$] co-ordinates.
- If you flow with the expansion, ie remain fixed with respect to the expanding grid, your co-moving co-ordinates do not change

If you remain at rest w.r.t the grid, then the time measured by you is the cosmic time t in the metric.

There is another definition of time in an expanding universe that is quite useful.

Conformal time:

$$d\tau \equiv \frac{dt}{a(t)} \qquad \text{or} \qquad \tau - \tau_0 = \int_{t_0}^{t} \frac{dt}{a(t)}.$$

Note that
$$ds^2 = dt^2 - a^2(t)\, d\vec{l}^{\,2}$$
$$= a^2(\tau)\left[d\tau^2 - d\vec{l}^{\,2}\right]$$

scale factor in terms of conformal time.

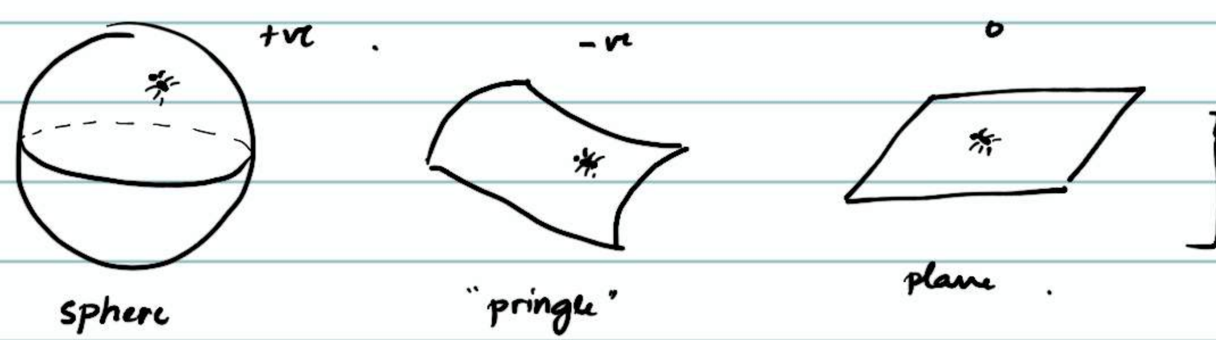
Conformal?

If $d\vec{l}^{\,2}$ = 3-euclidean interval $\Rightarrow$ the FRW metric is "conformal" to Minkowski space. That is $a(\tau)$ stretches the metric uniformly, but preserves angles.

Assume Homogeneity & isotropy,
what is the general form of $d\vec{l}^2$?

3 options - +ve curved spatial slice. (3D). (Spherical)
-ve " " " (Hyperbolic)
0 flat spatial slice (Euclidean)

{Lower dimensional (2D) analogs.



]

$$d\vec{l}^2 = d\vec{x}^2 + k\left(\frac{\vec{x}\cdot d\vec{x}}{1-k\vec{x}^2}\right)$$

$\vec{x} = (x,y,z)$ = cartesian
$k=0$ flat
$k=-ve$ pringle
$k=+ve$ sphere.

$$= \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

(r,θ,ϕ) = spherical
$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

co-moving distance

$$= \quad d\chi^2 + S_k^2(\chi)\, d\Omega^2 \qquad (\chi, \theta, \phi)$$

$$d\chi = \frac{dr}{1-kr^2}$$

$$\text{or } \chi = \int \frac{dr}{1-kr^2}$$

$$S_k(\chi) = \begin{cases} \chi^2 & k=0 \\ \sin^2\chi & k=+ve \\ \sinh^2\chi & k=-ve \end{cases}$$

Note that the 3 forms correspond to different choice of co-ordinate systems.

Thus the metric for a homogeneous & isotropic universe is

$$ds^2 = dt^2 - a^2(t)\, d\vec{l}^{\,2} \qquad (\text{with } d\vec{l}^{\,2} \text{ given above})$$

For example:

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)\, d\Omega^2 \right]$$

We will restrict our attention to flat spatial slices $k=0$ from now on since observations are consistent with this.

$$\therefore \quad ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \chi^2 d\Omega^2\right]$$

Question: But space is flat, so why not do Special Relativity.

Ans: Space is flat, not spacetime!

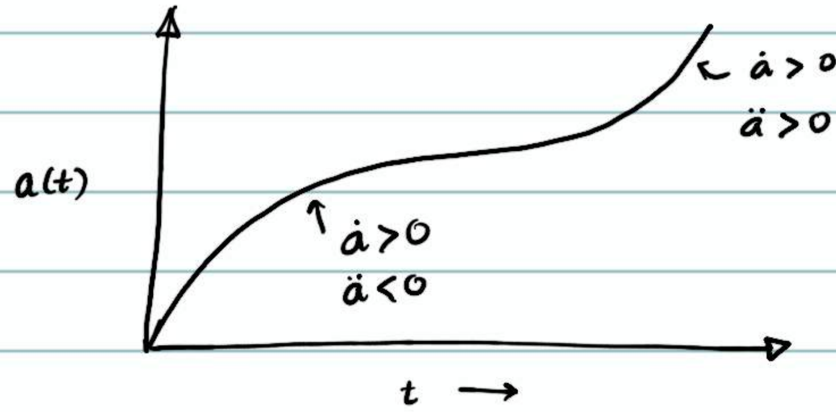
Note:

Using conformal time:

$$ds^2 = \overbrace{a^2(\tau)}^{\text{"conformal" factor}} \overbrace{\left[d\tau^2 - (d\chi^2 + \chi^2 d\Omega^2)\right]}^{\text{Minkowski}}$$

Note that for light, $ds^2 = 0$. For radial trajectories $d\tau^2 = d\chi^2$ or χ = co-moving distance

τ = conformal time



$\dot{a}(t) > 0$ universe is expanding

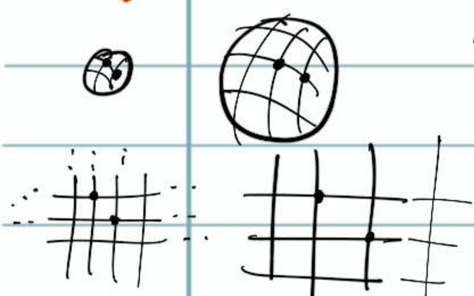
$\dot{a}(t) < 0$ universe is contracting

$\dot{a}(t) > 0$ & $\ddot{a}(t) > 0$ universe is undergoing <u>accelerated</u> expansion.

$\dot{a}(t) > 0$ & $\ddot{a}(t) < 0$ universe is undergoing decelerated expansion.

Question: a) What is the universe expanding into?

b) If we can't "see" the surface of a balloon expanding into the 3D space, how do we know it is expanding?



<u>Ans</u>: You can think of expansion as the relative separation of points. There is no need for an embedding in higher dimensional spaces (though it helps in visualization).

(2) Calculus on Curved spaces

To be able to understand how stuff moves in general spacetimes, and better understand the influence of matter on the curvature of spacetime, we need ways of doing calculus (eg. understand derivatives) on curved spacetimes.

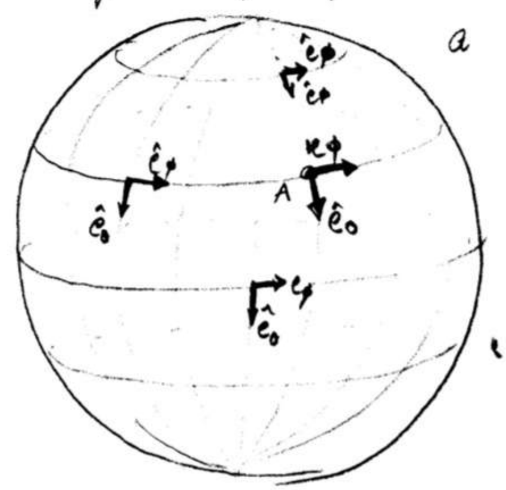
Let us start with a curved 2D space as an example. We will eventually get to $\underset{\text{space}}{3} + \underset{\text{time}}{1}$ dimensions. But starting with a simple "familiar" example helps.

[Derivatives on curved spaces]

<u>Caution</u>: [I make several egregious heuristic jumps * ← means I am doing that]

Consider the surface of a unit-radius = 1 sphere. You are a an ant on this surface and are not allowed to move "out" or "in" from this surface.

We will put a grid (co-ordinate system) on this surface. For concreteness this co-ordinate system is given by latitudes & longitudes. Each point on the sphere (apart from the poles) is labelled by a unique (θ, ϕ)



θ increases from the north pole to the south and ϕ increases from west to east.

At each point let us define a set of basis vectors $\hat{e}_\alpha = (\hat{e}_\theta, \hat{e}_\phi)$. Actually defining them is a bit non-trivial*, but we will think of them as little arrows at each point. See fig.

From the figure [at least as seen by being who can access the extra dimensions] $\hat{e}_\theta$ and $\hat{e}_\phi$ are function of position ie they vary from point to point on the sphere.

Let us consider the derivative of such basis vectors at the point A

$\partial_\alpha \hat{e}_\beta(A)$ [eg $\partial_\theta \hat{e}_\phi(A)$ $\partial_\phi \hat{e}_\theta(A)$ $\partial_\phi \hat{e}_\phi(A)$ $\partial_\theta \hat{e}_\theta(A)$]

This represent how the vectors change as we change as we move along a curve of latitude or longitude. If this derivative is to be a vector at the point A ** then it better be possible to write it in terms of the basis vectors at point A. For example

$$\partial_\theta \hat{e}_\phi(A) = [\quad]\hat{e}_\theta(A) + [\quad]\hat{e}_\phi(A)$$

$$\equiv \Gamma^\theta_{\theta\phi}\hat{e}_\theta + \Gamma^\phi_{\theta\phi}\hat{e}_\phi$$

[I will drop the A dependence]

just symbols at this point which depend on position

More generally $\partial_\alpha \hat{e}_\beta = \Gamma^\lambda_{\alpha\beta}\hat{e}_\lambda$ summation implied.

$\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$ is assumed!

But what are these $\Gamma_{\alpha\beta}^{\lambda}$?

Let us consider a small vector connecting connecting near by points with its foot at A.

$$\vec{ds} = \hat{e}_\theta \, d\theta + \hat{e}_\phi \, d\phi$$

$$= \hat{e}_\alpha \, dx^\alpha$$

$$\vec{ds} \cdot \vec{ds} = ds^2 = \left(\hat{e}_\alpha \cdot \hat{e}_\beta\right) dx^\alpha dx^\beta$$

Compare with $ds^2 = g_{\alpha\beta} \, dx^\alpha dx^\beta$

$$\therefore \quad \underline{\hat{e}_\alpha \cdot \hat{e}_\beta = g_{\alpha\beta}} \qquad !$$

$$\therefore \quad \partial_\gamma g_{\alpha\beta} = \left(\partial_\gamma \hat{e}_\alpha\right) \cdot \hat{e}_\beta + \hat{e}_\alpha \cdot \partial_\gamma \hat{e}_\beta$$

$$= \Gamma_{\gamma\alpha}^{\lambda} \left(\hat{e}_\lambda \cdot \hat{e}_\beta\right) + \hat{e}_\alpha \cdot \hat{e}_\lambda \, \Gamma_{\gamma\beta}^{\lambda}$$

$$= \Gamma_{\gamma\alpha}^{\lambda} \, g_{\lambda\beta} + g_{\alpha\lambda} \, \Gamma_{\gamma\beta}^{\lambda}$$

By cyclically permuting indices γ, α, β & some algebra we get

$$\boxed{\Gamma_{\alpha\beta}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} \left[\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta} \right]}$$

[The christoffels!]

Remember that $\Gamma_{\beta\gamma}^{\alpha}$, $g_{\alpha\beta}$ & $\hat{e}_\alpha$ are all functions of position!

Now let us consider differentiating a vector $\vec{v}(x^\gamma)$

At a given point

$$\vec{v} = v^\alpha \hat{e}_\alpha \qquad [\text{eg} \cdot \ \vec{v} = v^\theta \hat{e}_\theta + v^\phi \hat{e}_\phi]$$

$$\partial_\gamma \vec{v} = (\partial_\gamma v^\alpha)\, \hat{e}_\alpha + v^\alpha (\partial_\gamma \hat{e}_\alpha)$$

∵ the basis change too!

But we know $\partial_\gamma \hat{e}_\alpha = \Gamma^\lambda_{\gamma\alpha}\, \hat{e}_\lambda$

$$\therefore \quad \partial_\gamma \vec{v} = \partial_\gamma v^\alpha\, \hat{e}_\alpha + v^\alpha\, \Gamma^\lambda_{\gamma\alpha}\, \hat{e}_\lambda$$

$$= \underbrace{[\partial_\gamma v^\alpha + \Gamma^\alpha_{\gamma\lambda}\, v^\lambda]}_{|||}\, \hat{e}_\alpha$$

$$\underbrace{[\nabla_\gamma v^\alpha]}_{\text{Covariant derivative}}\, \hat{e}_\alpha$$

$$\therefore \quad \partial_\gamma \vec{v} = (\nabla_\gamma v^\alpha)\, \hat{e}_\alpha \qquad \boxed{\nabla_\gamma v^\alpha = \partial_\gamma v^\alpha + \Gamma^\alpha_{\gamma\lambda}\, v^\lambda}$$

components of the derivative in terms of the co-ordinate basis vectors.

Lecture 3

Plan : 1) Calculus on curved spaces (continued)
- co-variant derivative
- parallel transport
- geodesics.

2) Application to an FRW universe.
- Geodesic Equation (momentum)
- redshift of momentum & energy.

1. (Continued) - Covariant derivative

<u>vectors</u> : $\vec{v} = v^\alpha \hat{e}_\alpha$

$$\partial_\gamma \vec{v} = \left(\nabla_\gamma v^\alpha\right) \hat{e}_\alpha$$

$$\nabla_\gamma v^\alpha = \partial_\gamma v^\alpha + \Gamma^\alpha_{\gamma\lambda} v^\lambda \qquad \text{where } \Gamma^\alpha_{\gamma\lambda} = \frac{1}{2} g^{\alpha\sigma}\left[\partial_\gamma g_{\lambda\sigma} + \partial_\lambda g_{\gamma\sigma} - \partial_\sigma g_{\gamma\lambda}\right]$$

<u>scalars</u> :

$$\partial_\gamma f = \nabla_\gamma f$$

<u>tensors</u> : $\overleftrightarrow{T} = T^{\alpha\beta}\ \hat{e}_\alpha \otimes \hat{e}_\beta$

$\otimes$ ≠ scalar product
≠ vector product
= direct product

$$\partial_\gamma \overleftrightarrow{T} = \left(\nabla_\gamma T^{\alpha\beta}\right) \hat{e}_\alpha \otimes \hat{e}_\beta$$

Think of $\underline{\ \ } \otimes \underline{\ \ }$
Slot to put vectors in.

where

$$\nabla_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha_{\sigma\gamma} T^{\sigma\beta} + \Gamma^\beta_{\sigma\gamma} T^{\alpha\sigma}$$

Define $\hat{e}^{\beta}$ such that $\hat{e}^{\beta} \cdot \hat{e}_{\alpha} = \delta^{\beta}_{\alpha}$.

↑ dual basis vector $\qquad dx^{\alpha} = \hat{e}^{\alpha} \cdot \vec{ds}$

$$\vec{v} = v^{\alpha} \hat{e}_{\alpha} = v_{\alpha} \hat{e}^{\alpha}$$

inverse of $g_{\beta\gamma}$

$$\hat{e}^{\alpha} \cdot \hat{e}^{\beta} = g^{\alpha\beta} \quad \text{where} \quad g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

Useful to note that $g_{\alpha\beta} v^{\beta} = v_{\alpha}$

$$g^{\alpha\beta} v_{\beta} = v^{\alpha}.$$

ie the metric helps in raising and lowering indices.

Also note : $\vec{v} \cdot \vec{w} = g_{\alpha\beta} v^{\alpha} w^{\beta} = g^{\alpha\beta} v_{\alpha} w_{\beta} = v^{\alpha} w_{\alpha}$.

And the co-variant derivative with "lower" components.

$$\nabla_{\alpha} v_{\beta} = \partial_{\alpha} v_{\beta} - \Gamma^{\lambda}_{\alpha\beta} v_{\lambda}$$

$$\nabla_{\gamma} T_{\alpha\beta} = \partial_{\gamma} T_{\alpha\beta} - \Gamma^{\sigma}_{\gamma\alpha} T_{\sigma\beta} - \Gamma^{\sigma}_{\gamma\beta} T_{\alpha\sigma}$$

(note the -ve signs).

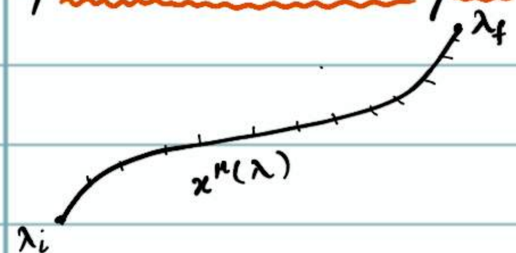
Question : What is a proper generalization of the "straight line" between two points in Euclidean space to curved spaces?

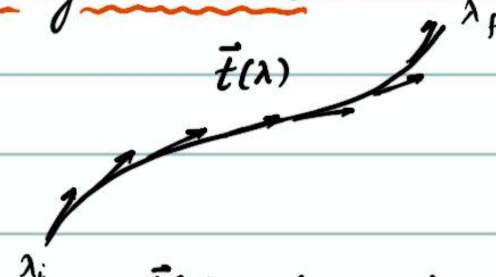
Ans : {Geodesics} = ① curves that parallel transport their own tangent vectors

OR

equivalently ② curves that extremize the interval between those two points.

1) parallel transport of tangent vector :





curve parametrized by λ.

$\vec{t}(\lambda)$ = tangent vectors.

$= \dfrac{dx^\alpha}{d\lambda}\,\hat{e}_\alpha(\lambda)$ ← along curve.

$= t^\alpha \hat{e}_\alpha$

$\dfrac{d\vec{t}}{d\lambda} = 0$ parallel transport

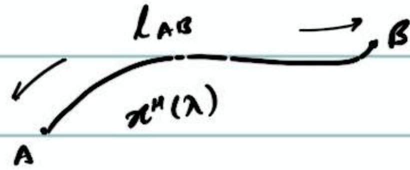
$$\Rightarrow \frac{dx^\mu}{d\lambda}\partial_\mu \vec{t} = \left(t^\mu \nabla_\mu t^\alpha\right)\hat{e}_\alpha = 0 \Rightarrow \boxed{\left(t^\mu \nabla_\mu t^\alpha\right) = 0}$$

Equivalently, $\boxed{\dfrac{dt^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\, t^\alpha t^\beta = 0}$

* More generally; the parallel transport along a curve $\frac{d}{d\lambda}\vec{v} = 0 \Rightarrow \underset{\substack{\uparrow \\ \text{tangent vector.}}}{t^\mu} \nabla_\mu v^\alpha = 0$

Heuristically, the length of the vector $\vec{v}$ and its angle with the tangent vector is preserved as you move along the curve.

2). Geodesics = curves that extremize the distance (interval) between two events.



$$S_{AB} = \int_A^B ds = \int_A^B \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta} = \int_A^B d\lambda \underbrace{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}_{\text{Lagrangian}}$$

Think about S_{AB} as an action and $L = \sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}$ as a Lagrangian. You can extremize this action in the usual way to obtain the equation satisfied by $x^\mu(\lambda)$.

$$\delta S = 0 \quad \underset{\substack{\uparrow \\ \text{Takes some work.}}}{\Rightarrow} \quad \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

Since $\frac{dx^\mu}{d\lambda} = t^\mu$ = components of the tangent vector.

$$\boxed{\frac{d}{d\lambda} t^\mu + \Gamma^\mu_{\alpha\beta} t^\alpha t^\beta = 0}$$

Thus, the two approaches are equivalent.

Note that the form which will be useful to us will be the one above which is written in terms of the components of the tangent vector. Remember that this was derived from $\frac{d}{d\lambda}\vec{t} = 0$ or $\delta S = \delta \int_A^B ds = 0$ which are co-ordinate independent.

Things to keep in mind

* Think about parallel transport geometrically in Euclidean space. - make sure it matches with your intuition of a straight line = geodesic = extremal path length
* The locus of the curve is not enough to define a geodesic. The parametrization is part of the definition of a geodesic. [ie you could take the same locus, possible to change the parametrization, $\lambda \to \lambda'$, then $\frac{d\vec{t}}{d\lambda'} \neq 0$. If $\frac{dt}{d\lambda} = 0$, if $\underbrace{\lambda' = a\lambda + b}_{\text{affine parametrizations}} \Rightarrow \frac{dt}{d\lambda'} = 0$]

* We have assumed that $ds^2 > 0$, ie the path length is +ve definite. This is violated in spacetime where $ds^2 \leq 0$ is also possible. $ds^2 = 0$ (light like) curves are trickier to handle in the derivation, but nevertheless yield the same equation for the geodesic.

* In the HW, you will explore geodesics on a sphere.

Let us now apply what we have learnt to spacetime.

Check:

Minkowski space: $g_{\mu\nu}(x^\alpha) = \eta_{\mu\nu}$; $\Gamma_{\alpha\beta}^{\lambda} = 0$ (cartesian co-ordinates

$$\therefore \quad \frac{dt^\mu}{d\lambda} = 0 \iff \frac{d^2x^\mu}{d\lambda^2} = 0 \iff \text{straight lines} \quad \checkmark$$

4-momentum form of Geodesic eq

It is convenient to chose λ, such that

$\frac{dx^\mu}{d\lambda} = P^\mu$ = components of the four-momentum = tangent vector.

P^0 = energy.

For massive particles $d\lambda = \frac{ds}{m}$ ← proper time, ← mass. ← remains well defined as $ds \to 0$, $m \to 0$ (photons).

ie $m\frac{dx^\mu}{ds} = P^\mu$

$\Rightarrow m \underset{\text{4 velocity}}{U^\mu} = P^\mu$.

Hence the geodesic equation becomes

$$\boxed{\frac{dP^\mu}{d\lambda} + \Gamma_{\alpha\beta}^{\mu} P^\alpha P^\beta = 0}$$

P^μ = 4 momentum

Lecture 4

Plan: 1) Geodesic eq. in FRW spacetimes - implications
2) cosmological redshift
3) Local Hubble Law
4) Distances in cosmology.

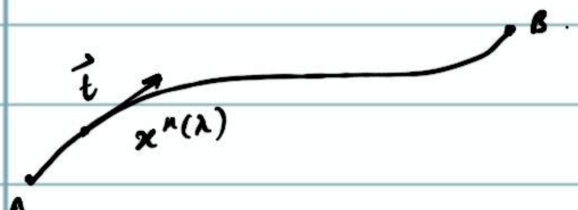
Review

1) Geodesic equation = $\left\{\begin{array}{l}\text{curve that parallel transports its own} \\ \text{tangent vector.}\end{array}\right.$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \iff \frac{dt^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}t^\alpha t^\beta = 0$$

$$\Updownarrow$$

where $t^\mu = \dfrac{dx^\mu}{d\lambda}$. $\qquad t^\alpha \nabla_\alpha t^\mu = 0$



Choose $d\lambda = \dfrac{\overbrace{ds}^{\text{proper time}}}{m} \Rightarrow \dfrac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}P^\alpha P^\beta = 0$

$\Downarrow$

$P^\mu = t^\mu$.

$$\frac{d}{ds}U^\mu = -\Gamma^\mu_{\alpha\beta}U^\alpha \underbrace{U^\beta}_{\text{4 velocity}} \qquad m \neq 0.$$

Let us now put this form to use in an FRW universe

$$ds^2 = dt^2 - a^2(t)\, \delta_{ij}\, dx^i dx^j \qquad \text{(cartesian)}$$
$$= dt^2 - g_{ij}\, dx^i dx^j \qquad (g_{ij} = \text{metric on spatial slice})$$

From Bauman's notes $\Gamma^0_{ij} = a\dot{a}\,\delta_{ij}$ $\quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\,\delta_{ij}$

all others zero.

→ Imagine throwing a particle from a co-moving galaxy. How does the energy & momentum of this particle change? For $\mu = 0$ in the geodesic equation

$$\frac{dP^0}{d\lambda} = -\Gamma^0_{ij} P^i P^j \qquad \Gamma^0_{00} = 0$$

$$= -a\dot{a}\,\delta_{ij} P^i P^j$$

$$= -\frac{\dot{a}}{a} g_{ij} P^i P^j \qquad H \equiv \frac{\dot{a}}{a} \; ; \; g_{ij} = a^2 \delta_{ij}$$

$$= -H p^2 \qquad \text{where } p^2 \equiv g_{ij} P^i P^j$$

"Physical momentum" *

chain rule & $\frac{\partial P^0}{\partial x^i} = 0$ *

$$\Rightarrow \underbrace{\frac{dt}{d\lambda}\frac{dP^0}{dt}}_{\text{chain rule}} = -H p^2$$

$$\Rightarrow P^0 \frac{dP^0}{dt} = -H p^2 \qquad P^0 = \frac{dt}{d\lambda} = E = \text{energy}.$$

$$\Rightarrow E\,dE = -\frac{da}{a} p^2$$

Using $\quad E^2 - m^2 = p^2$

$$E\,dE = p\,dp$$

Hence $\quad \cancel{p}\,dp = -\frac{da}{a}\,p^{\cancel{2}}$

$$\Rightarrow \quad \boxed{\left\{ p(t) \propto \frac{1}{a(t)} \right\}}$$

Thus the physical momentum decays as the universe expands.

For massless particles $\quad p = E$

Hence

massless only $\rightarrow$ $\boxed{\left\{ E(t) \propto \frac{1}{a(t)} \right\}}$ $\leftarrow$ energy of massless particles decays as the universe expands.

These expressions are the energy and momentum of the "particles" as measured by co-moving observers as the particle passes by. As the particle moves along, the universe is expanding. This robs the particle of momentum & energy.

Aside.

A sanity check. If a particle is at rest with respect to the co-moving co-ordinates, it should remain at rest.

$\mu = i$ & $P^i = 0$ initially

$$\frac{dP^i}{d\lambda} = -\Gamma^i_{\alpha\beta} P^\alpha P^\beta = -\underbrace{\Gamma^i_{00}}_{= 0} (P^0)^2 = 0$$

$\therefore$ $P^i = 0$ initially $\Rightarrow$ P^i remains zero.

Cosmological Redshift ← IMP

Derivation 1 : using QM

Physical momentum of a particle on a geodesic

$p(t) \propto \frac{1}{a(t)}$

Also true for photons

$QM \Rightarrow \quad \lambda = \frac{h}{p} \quad \Rightarrow \quad \lambda \propto a(t)$

$\boxed{\text{Redshift}}$

$$z_{em} \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{a(t_0) - a(t_{em})}{a(t_{em})}$$

$$= \frac{1}{a(t_{em})} - 1 \qquad a(t_0) = 1$$

ie $\boxed{z = \frac{1}{a} - 1}$ OR $\boxed{a = \frac{1}{1+z}}$

Derivation 2 : without QM

$$0 \underset{\uparrow}{=} ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2\right]$$

light only

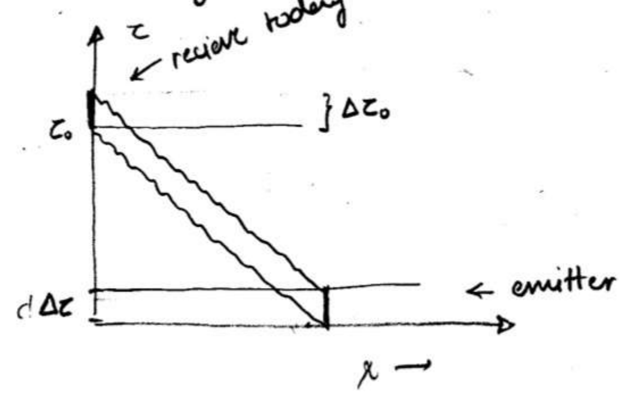
radial light ray from an emitter

$$\Rightarrow \quad d\tau = \pm\, d\chi$$

conformal time ($d\tau$) co-moving distance ($d\chi$)

or $\frac{d\tau}{d\chi} = \pm 1$

$\Rightarrow$ Slopes ± 1 for light in τ, χ plane



See above fig: suppose that N equally spaced in the emitters proper time blips are emitted in a conformal time interval $\Delta\tau$ at the conformal time τ from an emitter at some fixed conformal distance χ.

Since in the τ-χ plane, light travels on straight 45° lines, the N blips will be recieved in an interval $\Delta\tau_0 = \Delta\tau$ [after a time $\tau_0 - \tau = \chi$]

$$\Delta\tau_0 = \Delta\tau$$

$$\Rightarrow \quad \frac{\Delta t_0}{a(t_0)} = \frac{\Delta t}{a(t)} \qquad \text{using } d\tau = \frac{dt}{a(t)}.$$

N blips in $\Delta t_0 \Rightarrow \quad \nu_0 = \frac{\Delta t_0}{N}$

N blips in $\Delta t \Rightarrow \quad \nu = \frac{\Delta t}{N}$

$$\therefore \quad \frac{N}{\nu_0\, a(t_0)} = \frac{N}{\nu \cdot a(t)}$$

$$\Rightarrow \quad \frac{\nu_0}{\nu} = a(t) \qquad a(t_0) = 1.$$

$$\text{eq} \quad \frac{\lambda}{\lambda_0} = a(t)$$

<u>Redshift</u>. $\boxed{z \equiv \frac{\lambda_0 - \lambda}{\lambda} = \frac{1}{a} - 1}$

"Local" Hubble Law

For $t - t_0 \ll t_0$.

$$a(t) = a(t_0)\left[1 + (t-t_0)\frac{\dot a(t_0)}{a(t_0)} + \frac{1}{2}(t-t_0)^2\frac{\ddot a(t_0)}{a(t_0)}\right]$$

$$= a(t_0)\left[1 + (t-t_0)H_0 - \frac{1}{2}(t-t_0)^2 H_0^2 q_0 + \cdots\right]$$

$H_0 \equiv \frac{\dot a(t_0)}{a(t_0)}$ = Hubble constant ← Expansion rate today.
$\approx 67\ km\,s^{-1}\,Mpc^{-1}$

q_0 = deceleration parameter

Using $\underset{\uparrow}{-c}(t-t_0)$ = "distance" travelled by light = d.
[c = 1 in our units]

$$a(t) - 1 = -\frac{H_0}{c}d + \ldots \qquad (t-t_0)H_0 \ll 1$$

$$\Rightarrow \quad -\frac{z}{1+z} \approx -\frac{H_0 d}{c} + \ldots$$

$$\boxed{z \approx \frac{H_0 d}{c}} + \ldots \qquad \left(z \ll 1 \text{ since } a(t_0) \approx a(t) \text{ for } t_0 - t \ll t_0\right)$$

↑ Hubble law ↖ $d = c(t_0 - t)$.

Note 1: $t_0 - t$ = "lookback time".

Note 2: interpreting the redshift as being a doppler shift due to the receding galaxies $\underbrace{\frac{v}{c}}_{\text{Doppler}} = \underbrace{\frac{\lambda_0 - \lambda}{\lambda} = z}_{\text{cosmological}} \Rightarrow \boxed{v = H_0 d}$

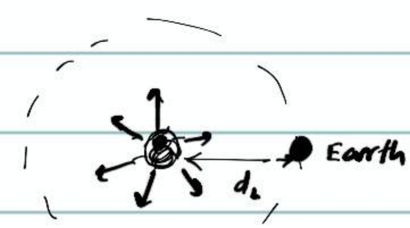
IMP: small distances & v only.

Distances in cosmology

In Minkowski space, we use have different ways of inferring the distance to far away objects (when it is not feasible to lay down small rulers back to back to the object).

For example if we know the intrinsic luminosity L of a star nearby, we can infer the distance to it by measuring the flux F on earth.

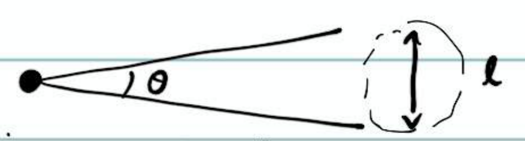
$$d_L = \sqrt{\frac{L}{4\pi F}}$$ ← luminosity distance



(assume isotropic rad)

Similarly if we knew the physical size of some star nearby, we can measure its angular extent in the sky $\Delta\theta$ and hence infer the distance to the star using

$$d_A \approx \frac{l}{\Delta\theta}.$$ ← angular diameter distance



assume small $\Delta\theta$.

Lecture 5

Plan : 1) Distances in cosmology
2) The Energy momentum tensor & conservation equations.

Review :

Geodesic equation : $\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta = 0$

$$\Downarrow$$

FRW universe : $\Rightarrow p(t) \propto \frac{1}{a(t)}$

$ds^2 = dt^2 - a^2(t)\,\delta_{ij}\,dx^i dx^j$

$E(t) \propto \frac{1}{a(t)}$ $\qquad m=0.$

Redshift : $z \equiv \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{1}{a} - 1 \Leftrightarrow a = \frac{1}{1+z}$

Local Hubble law : $z \approx H_0 d$ $\qquad d = c(t_0 - t) \ll ct_0$

$z \ll 1$

$H_0 = 68 \text{ km s}^{-1} \text{Mpc}^{-1}$

Distances in cosmology

In Minkowski space, we can have different ways of inferring the distance to far away objects (when it is not feasible to lay down small rulers back to back to the object).

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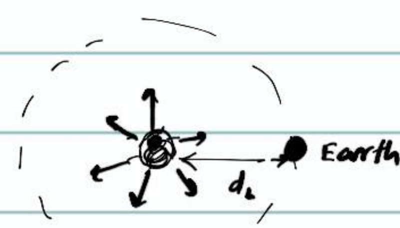

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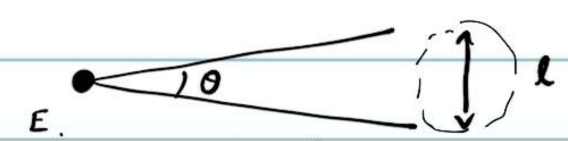


assume small $\Delta\theta$.

In Minkowski space $d_L = d_A$. if we were looking at the same object. [If time of emission & observation is known, $d_A = d_L = -c(t - t_0)$]

Complications from expansion

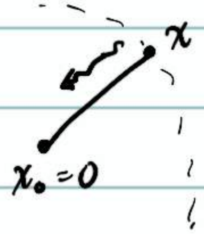
This situation is different when we consider an expanding universe.

Recall our metric for a flat FRW universe

$$ds^2 = c^2 dt^2 - a^2(t)\left(d\chi^2 + \chi^2 d\Omega^2\right)$$

If light was emitted at the time t and observed by us on earth at t_0, then the co-moving co-ordinate (or co-moving distance, if $\chi_0 = 0$) is given by [$ds^2 = 0$

$$\boxed{\chi = c\int_t^{t_0} \frac{dt}{a(t)}} \neq c(t_0 - t)$$



co-moving distance.

This is not a good distance, because I haven't given you a way to measure it.

In an expanding universe, in general,

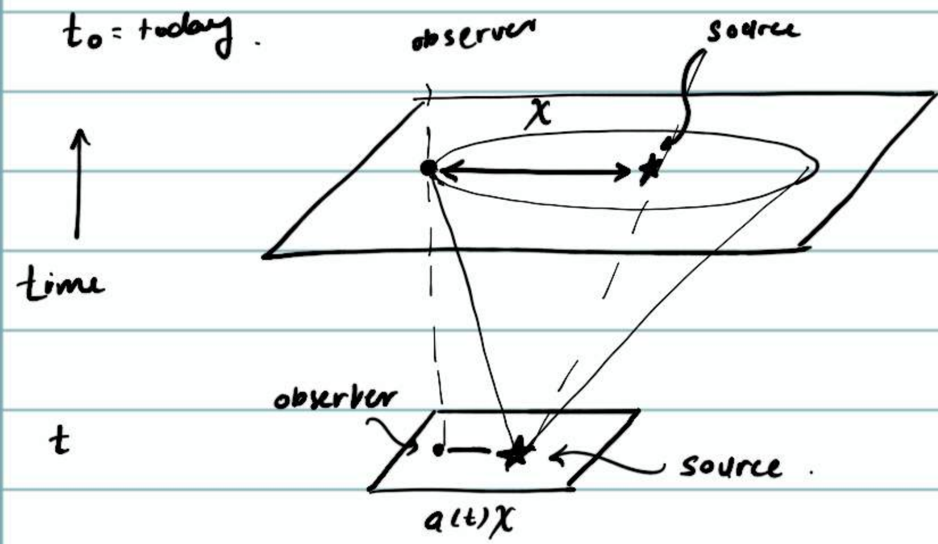
luminosity dist. $d_L \neq d_A$ angular diameter dist.

$\neq \chi$ co-moving dist

$\neq c(t-t_0)$. look back time dist.

Why?

Let us look carefully at the luminosity Distance:



If the physical separation between the source S and observer O is $a(t_0)\chi = \chi$ today $[a(t_0)=1]$, it was $a(t)\chi$ at time t in the past.

Let us assume that the source emits ΔN photons in a time interval Δt_e and with energy ε_e.

$$\therefore \quad L = \frac{\Delta N}{\Delta t_e} \varepsilon_e = \text{luminosity}$$

The photons are released isotropically.
They reach the observer at time t_0

(1) The source and observer are seperated by χ at time of observation.

(2) The energy per photon will be redshifted

(3) The photons arrive at a rate which is slower than the ones they are emitted at

Note that (3) and (2) come from the same argument. (2) is about interval between peaks of of a wave whereas (3) is about the interval between release of photons.

2 factors of a from (2) & (3)

$$\therefore \text{ The flux } F = \frac{L}{\underbrace{4\pi \chi^2}_{\text{from (1)}}} a^2(t_e) = \frac{L}{4\pi \chi^2} a^2(t_e)$$

$$\Rightarrow \quad \frac{\chi}{a(t_e)} = \sqrt{\frac{L}{4\pi F}} = d_L$$

Thus the Luminosity distance

$$\boxed{d_L = \frac{\chi}{a}} = \frac{1}{a(t)} \int_t^{t_0} \frac{dt'}{a(t')}$$

Note: $a(t)$ is the scale factor at time of emission.

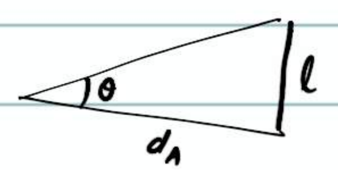
Equivalently.

$$d_L(a) = \frac{1}{a} \int_a^1 \frac{da'}{H(a')a'^2} \qquad H = \frac{\dot{a}}{a} \text{ : Hubble "parameter"}$$

$$d_L(z) = 1+z \int_0^z \frac{dz'}{H(z')} \qquad a = \frac{1}{1+z}.$$

What about the angular diameter distance?

$$d_A = \frac{l}{\Delta\theta}.$$



For a fixed physical size l

$$l = a(t)\chi\,\Delta\theta$$

[Note: $ds^2 = dt^2 - a^2(d\chi^2 + \chi^2 d\Omega^2)$]

Think about $d\chi = 0$, $d\phi = 0$.

$$\therefore \boxed{d_A = a\chi}$$

$$= a(t) \int_t^{t_0} \frac{dt'}{a(t')} = \frac{1}{(1+z)} \int_0^z \frac{dz'}{H(z')}$$

Note that $d_L \neq d_A$

$$a^2 d_L = d_A.$$

As we will see measuring distances is crucial for cosmology. To measure them one typically needs "Standard Candles" where L is known or "Standard Rulers" where l is known.

→ Supernovae Type Ia = "Standardizable" candles

→ Baryon acoustic oscillation scale = "Standardizable" ruler.

Note:

1) · Apart from light, we can also get information from the distant universe using gravitational waves. Like Standard Candles & standard rulers, gravitational wave sources can be used as Standard Sirens!]

2) You should read the history of how we measure distances in cosmology - see Weinberg's book.

3) Also look up "distance ladder".

The Energy Momentum Tensor $T^{\mu\nu}$

Einstein Equation

$$G_{\mu\nu} = 8\pi G\, T^{\mu\nu} = \text{Source of Spacetime curvature}$$

(Einstein tensor) describes curvature of spacetime constructed out of $g_{\mu\nu}$ & its derivatives

" (Energy momentum tensor) energy & momentum densities etc.

Important example : (Perfect fluid)

$$\boxed{T^{\mu\nu} = (\rho + P) u^{\mu} u^{\nu} - P g^{\mu\nu}}$$

$$\begin{array}{cc} \text{energy density} & \text{energy flux} \\ \left(\begin{array}{c|c} T^{00} & T^{0i} \\ \hline T^{i0} & T^{ij} \end{array}\right) & \\ \text{momentum density} & \text{stress tensor} \end{array}$$

ρ = energy density of fluid in its rest frame.

P = pressure of fluid " "

u_μ = bulk 4-velocity of fluid.

* **Note**: No viscosity, heat conduction or anisotropic/shear stress.

Consider the case where we are in a FRW universe and the fluid is at rest compared to the co-moving observers. In this case.

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -a^{-2} & & \\ & & -a^{-2} & \\ & & & -a^{-2} \end{pmatrix} \qquad u^{\mu} = (1, \vec{0})$$

Then

$$T^{00} = \rho$$

$$T^{0i} = T^{i0} = 0$$

$$T^{ij} = \frac{P}{a^2}\delta^{ij}$$

Better yet

$T^{\mu}{}_{\nu} = g_{\lambda\nu} T^{\mu\lambda}$ Note $T^{\mu}{}_{\nu} = T_{\nu}{}^{\mu}$

$\Rightarrow \quad T^0{}_0 = \rho \qquad T^0{}_i = 0 \qquad T^i{}_j = -P\delta^i_j$

$$T^{\mu}{}_{\nu} \approx \begin{pmatrix} \rho & & & \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix}$$ in FRW universe in frame of the co-moving observers.

Q: How do these pressures and densities evolve?

Ans: $\nabla_\mu T^{\mu\nu} = 0$.

Q: Meaning?

Ans: It is a conservation law (local)

- It is equivalent to the continuity & Euler equations for fluids $\dot{\rho} + \nabla\cdot(\rho\vec{v}) = 0$; $\rho(\dot{\vec{v}} + \vec{v}\cdot\nabla)\vec{v} = -\nabla p$
- More generally it expresses the local conservation of energy & momentum in curved space time

Q: What is $\boxed{\nabla_\mu} T^{\mu\nu}$?

Ans: Operationally $\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\lambda\mu} T^{\lambda\nu} + \Gamma^{\nu}_{\lambda\mu} T^{\lambda\mu}$

[$T^{\mu\nu}$ is symmetric]

We will use ∇_μ extensively in the course. Get used to it and come talk to me to get a better understanding (we discussed ∇_μ in the previous lecture]

——— x ———

Back to T^μ_ν for perfect fluids in FRW universe.

$$\nabla_\mu T^\mu_\nu = 0$$

Let us consider $\nu = 0$

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda = 0$$

$$T^i_0 = 0 \ ; \ T^0_0 = \rho$$

$$\Rightarrow \partial_0 \rho + \Gamma^\mu_{\mu 0} \rho - \Gamma^\lambda_{\mu 0} T^\mu_\lambda = 0$$

[check with the $\Gamma^\mu_{\lambda\nu}$ calculated from FRW universe]

$$\Rightarrow \boxed{\dot{\rho} + 3H(\rho + P) = 0}$$

Conservation equation!

$H = \frac{\dot{a}}{a}$.

Note $\quad dU = -PdV \quad$ (1st law of Thermodyn)

$$U = \rho V \qquad V \propto a^3$$

$$\Rightarrow d(\rho a^3) = -P d(a^3)$$

$$\Rightarrow \frac{d}{dt}(\rho a^3) = -P \frac{d(a^3)}{dt}$$

$$\Rightarrow \dot{\rho} + 3H(\rho + P) = 0.$$

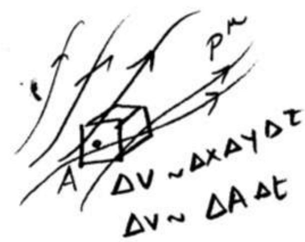
See Jim Hartle's book Chapter 22

Extra :

The Energy Momentum Tensor

Consider a stream of momentum p^μ flowing past an observer at some space time point A. [For simplicity let us assume we are in flat space time]

Let the observer account for all the four momentum Δp^μ in a spatial volume $\Delta V = \Delta x \Delta y \Delta z$ in the observer's rest frame This volume ΔV is of course a directed surface in spacetime $n_\nu \Delta V$ where $n_\nu = (1, \vec{0})$ is the normal to this surface



$$\Delta p^\mu \text{ (proportional to) } n_\nu \Delta V$$

$$= \underbrace{T^{\mu\nu}}_{} n_\nu \Delta V$$

structure imposed by indices.

$$= T^\mu_{\ \nu} n^\nu \Delta V \quad \leftarrow \text{this is not any different}$$

$$\Delta p^\mu = T^{\mu\nu} n_\nu \Delta V$$

$$= T^{\mu 0} \Delta V$$

$$\therefore \quad T^{\mu 0} = \frac{\Delta p^\mu}{\Delta V} = 4 \text{ momentum density}$$

Now imagine a surface $\Delta x \Delta y = (\Delta A)_z$ normal in z direction

or more generally $(\Delta A)_i$

The volume $\Delta V \to (\Delta A)_i \Delta t$ & $n_z = (0,0,0,1)$ ith location

Hence

$$\Delta p^\mu = T^{\mu i} (\Delta A)_i \Delta t$$

$$\therefore T^{\mu i} = \frac{\Delta p^\mu}{(\Delta A)_i \Delta t}$$ = flux of momentum p^μ across ΔA (in the ith direction)

Thus we have.

$$T^{\mu 0} = \frac{\Delta p^\mu}{\Delta V}$$

$$T^{\mu i} = \frac{\Delta p^\mu}{(\Delta A)_i \Delta t}.$$

$$T^{00} = \frac{\Delta p^0}{\Delta V}$$ = energy density

$$T^{i0} = \frac{\Delta p^i}{\Delta V}$$ = density of ith component of momentum.

$$T^{0i} = \frac{\Delta p^0}{(\Delta A)_i \Delta t}$$ = energy flux in ith direction.

$$T^{ji} = \frac{\Delta p^j}{(\Delta A)_i \Delta t}$$ = j component momentum flux in ith direction.

Also $T^{ji} = \dfrac{\Delta F^j}{\Delta A_i}$ = j-component of force in the ith direction.

= stress.

T^{ii} = pressure!

Lecture 8

Lecture 8

Goal (1) Physics of Inflation & Reheating

Inflation & reheating

To solve the horizon problem we need an early phase in the history of our universe where $w < -\frac{1}{3}$ (ie $\ddot{a} > 0$).

Note: If you look at the derivation more carefully, we need accelerated "expansion" for an expanding universe. However, accelerated contraction, for a contracting universe would work too!]

What kind of component would yield $w = \frac{P}{\rho} < -\frac{1}{3}$?

An example:

Consider a scalar field $\varphi(\vec{x}, t)$. [Higgs field is an example of a fundamental scalar field].

Its energy momentum tensor.

$$T^{\mu}_{\ \nu} = \partial^{\mu}\varphi\, \partial_{\nu}\varphi - g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta}\partial_{\alpha}\varphi\partial_{\beta}\varphi - V(\varphi)\right]$$

$V(\varphi)$ = potential which governs the dynamics.

[Compare with perfect fluid EM tensor to get P_φ & ρ_φ]

For a homogeneous field $\varphi(\bar{x},t) = \varphi(t)$, the energy density & pressure

$$\rho_\varphi = \boxed{\frac{1}{2}\dot{\varphi}^2} + V(\varphi)$$ potential energy.

Kinetic energy of field

$$P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

If $\dot{\varphi}^2 << V(\varphi)$ ["slow roll"]

$$\rho_\varphi \approx V(\varphi)$$
$$P_\varphi \approx -V(\varphi)$$

$$\therefore\ w_\varphi = \frac{P_\varphi}{\rho_\varphi} \approx -1 < -\frac{1}{3}\ !$$

Thus a slowly rolling scalar field is a good candidate for $w < -\frac{1}{3}$ stuff!

[, If $V(\varphi)$ = constant, $\dot{\varphi} = 0$ this would be identical to Λ]

How do we get $\varphi(t)$ to roll slowly?

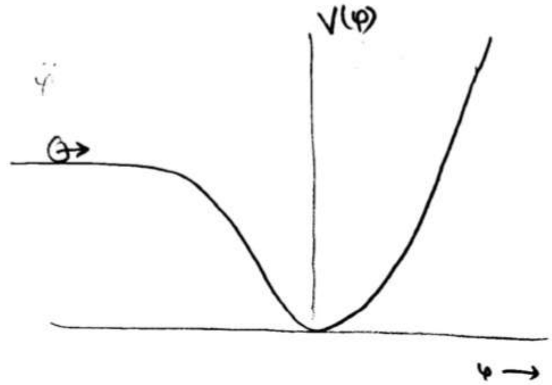
Let us look at the equation of motion.

$$\Box\varphi = V'(\varphi)$$ [Derive from $\mathcal{L}_\varphi = \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi) + V(\varphi)$.

$$\Rightarrow \underbrace{\ddot{\varphi}}_{\text{like acceleration}} + \underbrace{3H\dot{\varphi}}_{\text{like friction}} + \underbrace{V'(\varphi)}_{\text{like gradient of potential}} = 0$$

$H = \frac{\dot{a}}{a}$. $\varphi(\bar{x},t) \to \varphi(t)$ Homogeneous.

$$\ddot{x} + b\dot{x} + \frac{dV}{dx} = 0$$



Inflationary solutions are attractors ÷ ie small perturbations away from it decay away [not all cases though].

Think of $\varphi(t)$ as the co-ordinate of a ball rolling in a potential $V(\varphi)$.

—x—

Note that $H^2 = \frac{\rho_\varphi}{3 m_{pl}^2} = \frac{1}{3 m_{pl}^2}\left[\frac{\dot\varphi^2}{2} + V(\varphi)\right]$ $\qquad \frac{8\pi G}{3} = \frac{1}{3 m_{pl}^2}$

We need $\dot\varphi^2 \ll \frac{\dot\varphi^2}{2} + V(\varphi) = 3 m_{pl}^2 H^2$ for slow roll & $\ddot\varphi \ll H\dot\varphi$ to make sure such a period lasts long enough. A way of quantifying this

$$\varepsilon = \frac{\frac{1}{2}\dot\varphi^2}{m_{pl}^2 H^2} \ll 1$$

$$\eta = \frac{\dot\varepsilon}{H\varepsilon} \ll 1$$

[Note $\varepsilon = -\frac{\dot H}{H^2} < 1$ was a condition for $\ddot a > 0$]

As long as $\varepsilon \ll 1$ & $\eta \ll 1$ slow roll is satisfied.

In terms of the potential.

$$\varepsilon \approx \frac{m_{pl}^2}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 \ll 1 \;;\; \eta \approx m_{pl}^2\left(\frac{V''}{V}\right) \ll 1$$

So the potential should be sufficiently flat & not changing too rapidly.

Caution: flat ≠ visually flat. $V(\varphi) \sim \frac{\lambda}{n}\varphi^n$ $n>0$ can be flat enough for even $n \gg 1$

$$\varepsilon \approx \frac{m_{pl}^2}{2}\frac{1}{\varphi^2} \ll 1 \qquad \eta = \frac{m_{pl}^2(n-1)}{\varphi^2} \ll 1$$

As long as $\varphi \gg m_{pl}$ during inflation!

Classic & very important example $V(\varphi) = \frac{1}{2}m^2\varphi^2$.

<u>How much inflation?</u>

Recall that we wanted $\frac{a_{end}}{a_{beg}} \gtrsim e^{60}$ ← # of efolds.

$$\therefore \quad N \equiv \int_{a_{beg}}^{a_{end}} d\ln a = \int_{t_{beg}}^{t_{end}} H\,dt \qquad \frac{d\ln a}{dt}dt = H dt$$

$$H\,dt = \frac{H}{\dot\varphi}d\varphi \approx \frac{1}{\sqrt{2\varepsilon}}\frac{d\varphi}{m_{pl}}.$$

$$\therefore \text{ for } N > 60 \Rightarrow \int_{\varphi_{end}}^{\varphi_{beg}} \frac{1}{\sqrt{2\varepsilon}}\frac{d\varphi}{m_{pl}} \approx 60.$$

for $V(\varphi) \approx \frac{1}{2}m^2\varphi^2$, we get $\varphi_{beg} - \varphi_{end} = \Delta\varphi \approx 15\, m_{pl}$!

Reheating : "Populating" the universe after inflation

Inflation : a period of accelerated expansion solves the horizon problem. However it leads to a universe devoid of ~~most~~ regular matter & radiation. We know from observations that the early universe (at least by the time of production of light elements) ~~it~~ was radiation dominated & thermal. How did we go ~~from~~ from a "inflaton" dominated to a radiation dominated universe?

This transition is called "reheating"

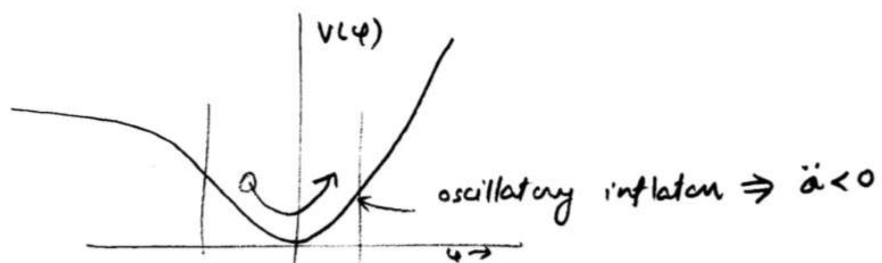
It is one of the most fascinating areas of early universe cosmology. The transfer of energy can be explosive → - gravitational waves
- solitons
- BH etc.

Understanding this period is also important ~~bc~~ because it serves as a bridge between ~~ea~~ inflation & better understood low energy physics.

– Ending inflation :

Recall : $\dot{\varphi}^2 << V(\varphi) \Rightarrow \ddot{a} > 0$

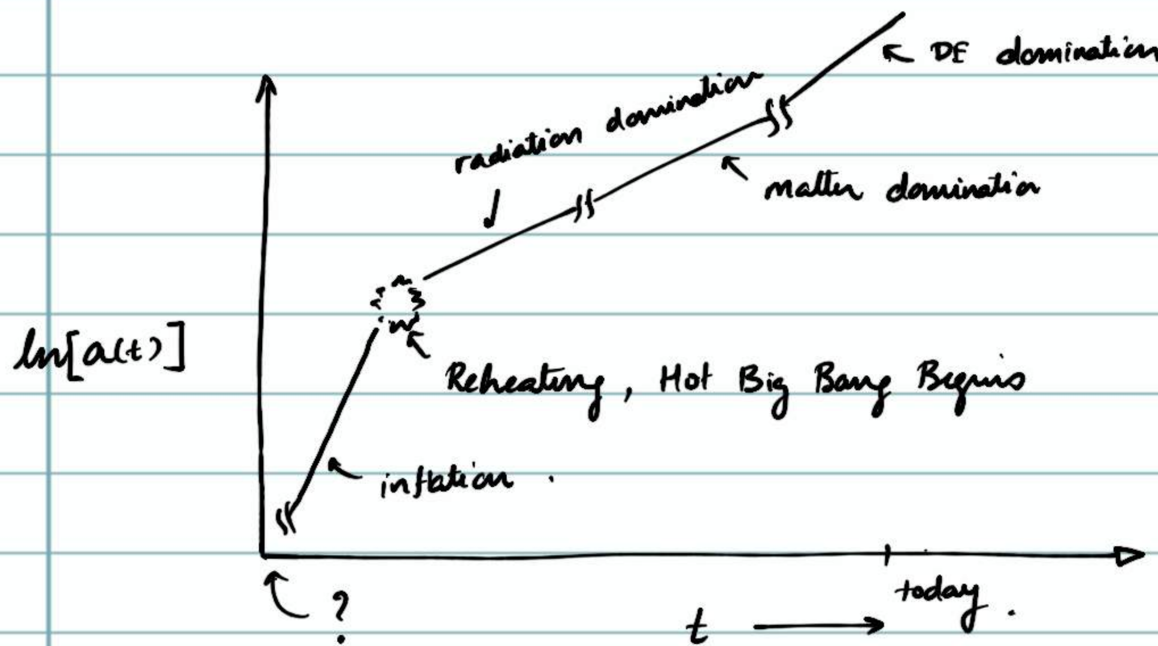
So if $\dot{\varphi}^2 \not< V(\varphi)$ then we can get $\ddot{a} < 0$



So inflation can end if the field starts oscillating. This naturally happens in potentials with a minimum.

At this stage if φ is coupled to other fields there is an explosive transfer of energy $\varphi \rightarrow$ other fields.. We do not have time to go into this, but ask me!

The transfer of energy must eventually lead to a thermal universe.. This thermal hot universe is the beginning of the "Hot Big Bang!" which we turn to next



$\ln[a(t)]$
radiation domination
DE domination
matter domination
Reheating, Hot Big Bang Begins
inflation.
?
today.
$t \longrightarrow$

Lecture 9

Plan: 1) Thermal universe timeline
2) Important concepts
3) Review Eq. Thermodynamics.

thermal history

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
✓ Dark matter freeze-out	?	?	?
✓ Neutrino decoupling	1 s	6×10^9	1 MeV
✓ Electron-positron annihilation	6 s	2×10^9	500 keV
✓ Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
✓ Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
✓ Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

$\boxed{\text{Imp}}$

$\Gamma \gg H$ (eq.)

$\Gamma \sim H$ (dec.)

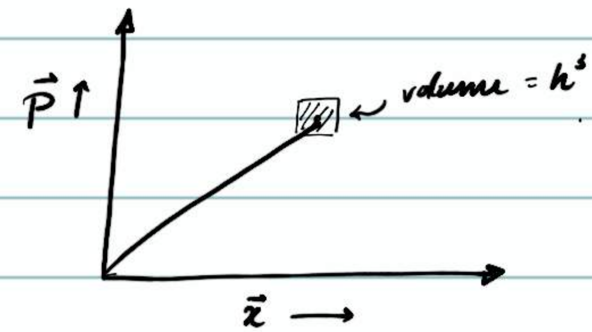
$\Gamma \ll H$ (freezout).

Table From DB Notes..

Review of Statistical Mechanics

Consider a gas of N identical particles, each with "g" internal degrees of freedom (d.o.f).

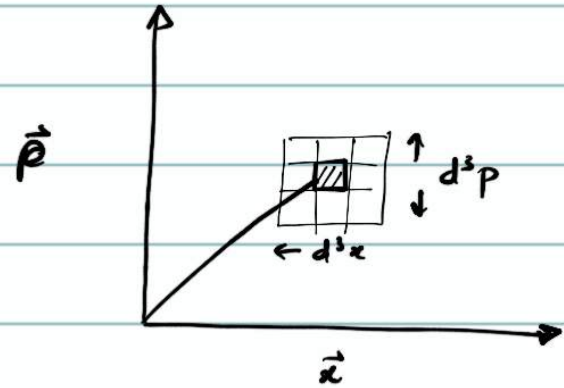
Consider the single particle phase space.



The state of a particle is given by $(\vec{p}, \vec{x}, s)$ where $s = 1, \ldots, g$ (for photons $g = 2$ and corresponds to two polarizations)
$\pm\Delta p$ $\pm\Delta x$

Note that Heisenberg's uncertainty principle tells us that phase space is coarse grained. That is the state occupies a volume $= h^3$. We cannot specify the position and momentum to better than $\Delta p \Delta x \sim h$.

[For more discussion on why the volume is h^3, see the end of this lecture]



Now consider a volume $d^3p\,d^3x$ in phase space.

Q: How many single particle states are contained in this volume.

Ans $\quad g\,\dfrac{d^3p\,d^3x}{h^3}$

Q: How many particles (typically) occupy this volume of phase space at time t?

Ans $\quad \underbrace{f(\vec{p},\vec{x};t)}_{\text{prob. of \# of particles/state}} \times \underbrace{g\,\dfrac{d^3p\,d^3x}{h^3}}_{\text{\# of states}}$

$f(\vec{p},\vec{x};t) \equiv$ distribution function
$\qquad\qquad = $ occupation number

Note that we are assuming that f does not depend on $s = 1, \ldots g$.

Homogeneity + isotropy $\Rightarrow f(\vec{p},\vec{x};t) \longrightarrow f(p;t)$

We will drop the time dependence "t" to reduce clutter.

Note the total # of particles in a volume V

$$N = \frac{g}{h^3}\int d^3x\, d^3p\, f(\vec{p},\vec{x};t) \overset{\hbar=1}{=} \frac{g}{(2\pi)^3}\int d^3x\, d^3p\, f(\vec{p},\vec{x},t)$$

$$\overset{\text{homogeneity \& isotropy}}{=} \frac{g}{(2\pi)^3} V \int d^3p\, f(p)$$

$\therefore$ Number density

$$n = \frac{g}{(2\pi)^3}\int d^3p\, f(p)$$

If the energy of a single particle momentum state $E(\vec{p}) = E(p)$, then the energy density

$$\rho = \frac{g}{(2\pi)^3}\int d^3p\; E(p) f(p)$$ Energy density

Similarly pressure.

$$P = \frac{g}{(2\pi)^3}\int d^3p\, \frac{p^2}{3E(p)}\, f(p).$$

For small interaction energies

$$E(p) = \sqrt{p^2+m^2} + \cancel{\epsilon_{int}}^{\text{small} \approx 0}$$

What is $f(p)$? Depends on the situation.
Important case:

Thermal Equilibrium: $(\hbar = k_B = c = 1)$.

$$f(p) = \frac{1}{\exp\left[\frac{E(p)-\mu(T)}{T}\right] \pm 1}$$

+ for fermions
− for bosons.

$T(t)$ = temperature
$\mu(T)$ = chemical potential
[Recall: $dU = TdS - pdV + \mu dN$.

If there are different species, each will have its own m_i (hence $E_i(p)$), μ_i etc. Hence each has a different $f_i(p)$.

However in thermal equilibrium.
T = same for all species.
μ_i's are related to each other.

In particular

$$A + B \rightleftarrows C + D \quad \Rightarrow \quad \mu_A + \mu_B = \mu_C + \mu_D.$$

Useful to note that:

$\mu_\gamma = 0$ photons. (see why?)

$\mu_A = -\mu_{\bar{A}}$ $A, \bar{A}$ are particle/antiparticle pair

Also at early enough times $\frac{\mu}{T} \to 0$, so for now we will ignore the chemical potential.

(a) From Question by Osmond:

* Why is phase space divided into (volume $= h^3$) elements? Why not $\hbar^3$ or $2h^3$?

Ans: Start with the 3rd law of Thermodynamics (Equilibrium)

$$S = k_B \ln \Omega \longrightarrow 0 \quad \text{as } T \to 0$$

entropy # of states in phase space $= \dfrac{\text{Volume of phase space}}{(?)}$

- So the (?) above must be such that $S \to 0$ as $T \to 0$
- We want this (?) to be universal (ie not depend on property of system)

You can carry of the calculation, for say a collection of oscillators and convince yourself that $(?) = h^3$.

Also, you can calculate the area between n and $n+1$ eigenstate of a harmonic oscillator.

[Note: Need to think more about the exp/theoretical underpinning for 3rd law]

(b) Related to question by Laura & Jake.

Relativistic species in equilibrium ($k_B = c = \hbar = 1$)

$$n = \frac{\xi(3)}{\pi^2} g T^3 \qquad \text{bosons.}$$

→ Only dimensionful parameter is T, must be T^3 to get right dims.

Re-introducing factors of $\hbar, c, k_B$.

$[k_B T]$ = energy

$[\hbar c]$ = energy × length

$$\therefore \quad n = \frac{\xi(3)}{\pi^2} g \frac{(k_B T)^3}{(\hbar c)^3}$$

which has units of $(\text{length}^{-1})^3$.

Lecture 10

Plan: - Equilibrium — n, ρ, P
- Effective relativistic d.o.f
- Entropy conservation.

* Application - Neutrino decoupling

Equilibrium expressions for n, ρ, P

$$n = \frac{g}{(2\pi)^3} \int d^3p \, \frac{1}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

$$= \frac{g}{2\pi^2} \int dp \, \frac{p^2}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

Relativistic : $m >> T$

non-relativistic : $m << T$.

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases}$$

$$n \simeq g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

T^3 available from dim. analysis

Boltzmann supressed

Similarly, for the energy density.

$$\rho = \frac{g}{(2\pi)^3} \int d^3p \, \frac{\sqrt{p^2+m^2}}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

$m \ll T$ (Relativistic) $\qquad\qquad$ $m \gg T$. (non relativistic)

$$\rho \approx \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases} \qquad\qquad \rho \approx mn = g m \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

Finally, for the pressure.

$$P = \frac{g}{(2\pi)^3} \int d^3p \, \frac{p^2}{3\sqrt{p^2+m^2}} \, \frac{1}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

$m \ll T$ (Relativistic). $\qquad\qquad$ $m \gg T$ (non-relativistic)

$$P \approx \frac{1}{3}\rho \qquad\qquad P \approx nT \ll \rho = mn$$

		# density n	energy density ρ	Pressure P
$T \gg m$	Bosons	$\frac{\zeta(3)}{\pi^2} g T^3$	$\frac{\pi^2}{30} g T^4$	$\frac{1}{3}\rho$
	Fermions	$\frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3$	$\frac{7\pi^2}{240} g T^4$	$\frac{1}{3}\rho$
			↑ relativistic / ↓ non-relativistic	
$T \ll m$	Bosons OR Fermions	$g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$	mn	nT $(\ll \rho)$

Consider two species of bosonic particles with masses m_1 & m_2, in thermal equilibrium at a temperature T. Consider $m_1 \ll T$ & $m_2 \gg T$ then

$$\rho_{tot} = \rho_1 + \rho_2 \approx \frac{\pi^2}{30} g T^4 + m_2 g \left(\frac{m_2 T}{2\pi}\right)^{3/2} \underbrace{e^{-m_2/T}}_{\text{small!}}$$

$$\approx \frac{\pi^2}{30} g T^4$$

Hence the energy density is dominated by relativistic ($m \ll T$) particles.

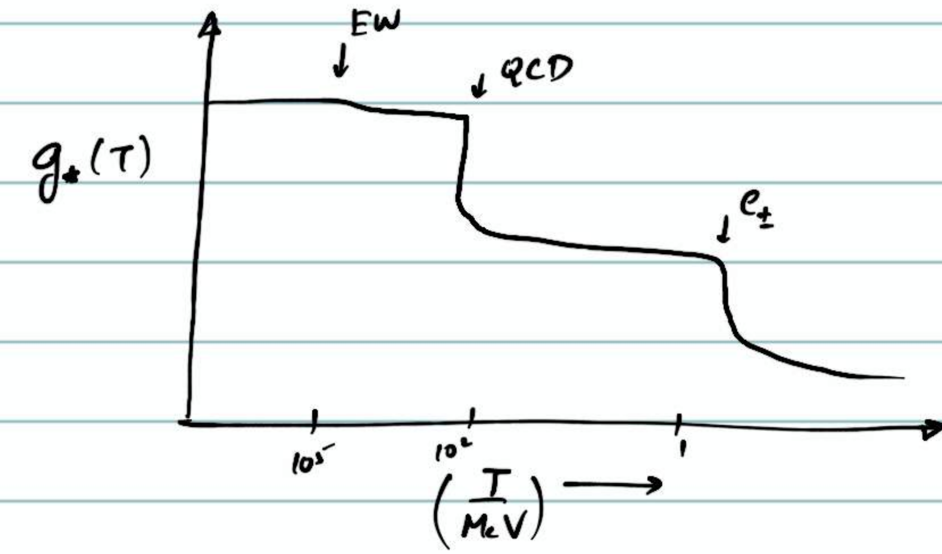
- Note that if $T \gg m_1, m_2$ then at that temperature both would contribute
- If $m_1 \ll m_2$, then as the temperature falls in an expanding universe, there will be a temperature where m_1 contributes but m_2 does not.

It is useful and convenient to define the "effective relativistic degrees of freedom"

$$\rho(T) = \sum_{m \ll T} \rho_i + \sum_{m \gg T} \rho_i \approx \sum_{m \ll T} \rho_i \equiv \frac{\pi^2}{30} g_*(T)\, T^4$$

where $g_*(T) \equiv \underset{\substack{\uparrow \\ \text{total bosonic d.o.f.}}}{g_b} + \frac{7}{8} \underset{\substack{\uparrow \\ \text{fermionic d.o.f.}}}{g_f}$ = effective relativistic d.o.f.

Why is $g_*(T)$? because as the temperature falls, a given species might become non-relativistic and stop contributing to the energy density.



for standard model. (SM)

For $T > 10^5$ MeV, all SM particles are relativistic.

$g_b = 28$ $\Rightarrow$ $g_* \approx 106.75$
$g_f = 90$

Today ($T_0 \sim 10^{-4}$ eV) $g_* \approx$ few [photons & maybe neutrinos].

Note if species decouple from each other ie ($\Gamma \lesssim H$), then they might each still have a FD or BE distribution and could be at different temperatures. (T_i)

More generally

$$g_*(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum g_i \left(\frac{T_i}{T}\right)^4$$

Conservation of Entropy (Equilibrium).

Want to show $\frac{dS}{dt} = 0$.

For $\frac{\mu}{T} \to 0$, note that since the distribution function only depends on E/T, we get

① $$\frac{\partial P}{\partial T} = \frac{\rho + P}{T} \qquad \text{(see proof later).}$$

Second, using the 2nd law of thermodynamics & ①, we get

$$dS = \frac{dU + PdV}{T}.$$

② $$dS = d\left[\frac{(\rho + P)V}{T}\right]$$

Finally using ①, ② & the continuity eq. $\frac{d\rho}{dt} = -3H(\rho+P)$ we get

✓ ③ $$\frac{dS}{dt} = 0$$

Proof of ①, ② & ③ next.

Proving $\frac{\partial P}{\partial T} = \frac{\rho + P}{T}$

$$P = \frac{g}{(2\pi)^3} \int d^3p \, f(p,T) \frac{p^2}{3E(p)} \qquad f(p) = \left[e^{\frac{\sqrt{p^2+m^2}}{T}} \pm 1 \right]^{-1}$$

$$E(p) = \sqrt{p^2 + m^2}$$

$$\rho = \frac{g}{(2\pi)^3} \int d^3p \, f(p,T) E(p)$$

Note $d^3p = p^2 dp \, d\Omega$

Since f & E are independent of angles

$$\int d^3p \rightarrow 4\pi \int dp \, p^2$$

$$\therefore \quad P = \frac{g}{2\pi^2} \int_0^\infty dp \, p^2 f(p,T) \frac{p^2}{3E} \qquad \rho = \frac{g}{2\pi^2} \int dp \, p^2 f(p,T) E(p)$$

Also note that

$$\frac{\partial f}{\partial T} = -\frac{E}{T} \frac{\partial f}{\partial p} \frac{dp}{dE} = -\frac{E^2}{Tp} \frac{\partial f}{\partial p} \qquad \because E dE = p dp$$

$$\therefore \quad \frac{\partial P}{\partial T} = \frac{g}{2\pi^2} \int_0^\infty dp \, p^2 \, \frac{-E^2}{Tp} \left(\frac{\partial f}{\partial p} \right) \frac{p^2}{3E}$$

$$= -\frac{g}{6\pi^2} \frac{1}{T} \int_0^\infty dp \, p^3 E(p) \frac{\partial f}{\partial p}$$

$$\frac{\partial}{\partial p} \left[p^3 E f \right] = p^3 E(p) \frac{\partial f}{\partial p} + \frac{\partial}{\partial p}(p^3 E) f$$

$$\therefore \quad \frac{\partial P}{\partial T} = -\frac{g}{6\pi^2} \frac{1}{T} \cancel{\left[p^3 E \overset{0}{f} \right]_0^\infty} + \frac{g}{6\pi^2} \frac{1}{T} \int dp \left(3p^2 E + \frac{p^4}{E} \right) f$$

$$= \frac{1}{T} \left[\frac{g}{2\pi^2} \int dp \, p^2 E f + \frac{g}{2\pi^2} \int dp \, p^2 \frac{p^2}{3E} f \right]$$

$$\therefore \quad \boxed{\frac{\partial P}{\partial T} = \frac{1}{T}(\rho + P)}$$

② Proof of $dS = d\left[\frac{(\rho+P)}{T}\right]$

$$dS = \frac{dU + PdV}{T}$$

$$= \frac{d(\rho V) + PdV}{T}$$

$$= \frac{d[(\rho+P)V]}{T} - V\frac{dP}{T}$$

$$= \frac{d[(\rho+P)V]}{T} - \frac{V}{T^2}(\rho+P)dT \qquad \text{using } \frac{\partial P}{\partial T} = \frac{(\rho+P)}{T}.$$

$$= d\left[\frac{(\rho+P)V}{T}\right]$$

③ Proof of $\frac{dS}{dt} = 0$

$$\frac{dS}{dt} = \frac{d}{dt}\left[\frac{(\rho+P)V}{T}\right] \qquad V \propto a^3.$$

$$= \underbrace{3H\cancel{\frac{(\rho+P)V}{T}} + \cancel{\frac{\dot\rho}{T}}V}_{0} + \frac{\dot P V}{T} - \frac{(\rho+P)V\dot T}{T^2}$$

$[\because \quad \dot\rho = 3H(\rho+P)]$

$$= \frac{V}{T}\Big[\underbrace{\dot P - \rho+P\frac{\dot T}{T}}_{0 \text{ because of ①}: \ \frac{\partial P}{\partial T} = \frac{\rho+P}{T}.}\Big] = 0$$

Implications of $\frac{d}{dt}\Big[\overbrace{\frac{(\rho+P)V}{T}}^{S}\Big] = 0$

Useful to define entropy density

$$s = \frac{S}{V} = \frac{\rho+P}{T}$$

$$\frac{dS}{dt} = 0 \Rightarrow \boxed{s \propto a^{-3}} \qquad (\because V \propto a^3)$$

↑ IMPORTANT.

It turns out that this relationship is true for each species, even if they are at different temperatures T_i

some might have decoupled.

ie $$s_i = \frac{\rho_i + P_i}{T_i} \propto a^{-3}$$

& $$s = \sum_i \left(\frac{\rho_i + P_i}{T_i}\right) \propto a^{-3}.$$

$\frac{\rho+P}{T}$ is dominated by relativistic species. Upon evaluation of the $\rho+P$ integrals we get.

$$s = \frac{2\pi^2}{45} g_*(T) T^3$$

effective relativistic d.o.f.

Implications of Entropy Conservation

$s \propto a^{-3} \Rightarrow \qquad g_{*}(T)\, T^3 a^3 = \text{const}.$

If $g_{*s}(T)$ is not changing the temperature

$$\boxed{T \propto \frac{1}{a}} \quad \checkmark$$

Putting all this formalism to use:

(a) Neutrino decoupling.

$\sigma \sim G_F^2 T^2$

$\Gamma = n\sigma v \sim n\sigma \quad G_F^2 T^5$

Fermi's constant.

$\downarrow$

$G_F = 10^{-5} \ GeV^{-2}$

$H \simeq \sqrt{\frac{g_* \pi^2}{90 m_{pl}^2}} \, T^2 \sim \frac{T^2}{m_{pl}}$ $\leftarrow$ (get by dim analysis)

Decoupling happens when $\frac{\Gamma}{H} \lesssim 1$

To find when this happens.

$$\frac{\Gamma}{H} \sim 1 \Rightarrow \frac{G_F^2 T^5}{T^2/m_{pl}} \sim \left(\frac{T}{1\,MeV}\right)^3$$

$$\therefore \boxed{T_{\nu\,dec} \simeq 1\,MeV} \iff \boxed{t_{\nu\,dec} \simeq 1\,sec.}$$

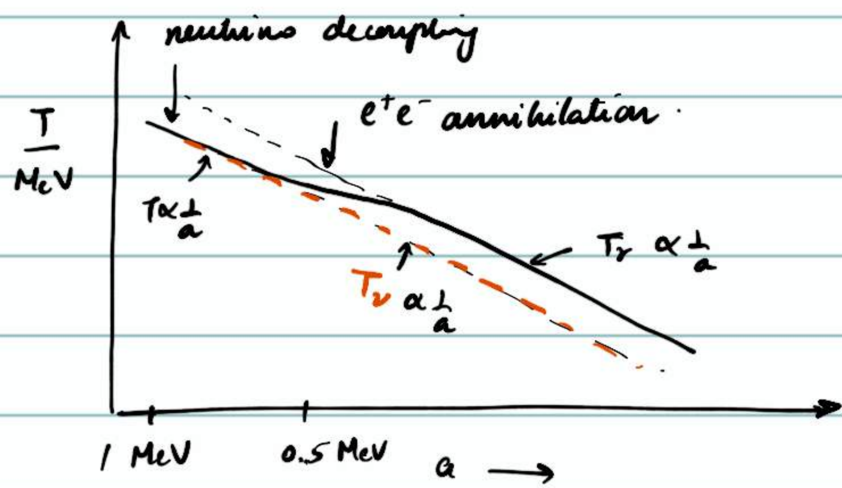
Lecture 11

2. <u>e^+e^- annihilation</u>.

Shortly after neutrino decoupling
e^+e^- annihilate into photons efficiently.
(However photons can no longer efficiently generate e^+e^- pairs). This is because at $T \sim 0.5\,\text{MeV}$
$m_e \sim T$.
electron mass
or positron

- The extra photons from the annihilation contribute to an increase in temperature of the photons.

- However the neutrinos have decoupled, and no longer get a boost in temperature!
  Hence we should expect



To actually calculate the difference in temperature, we will use the conservation of entropy:

$$[Sa^3]_{before} = [sa^3]_{after}.$$

before = before e^+e^- annihilation.

$$[sa^3]_{before} = \Big[\underbrace{(\tilde{s}a^3)}_{\text{everything relativistic other than } \nu} + s_\nu a^3 \Big]_{before}.$$

because decoupled so lets count it separately.

$$= \left[\frac{2\pi^2}{15} g_* T^3 a^3 + s_\nu a^3 \right]_{before}.$$

where $g_* = \underset{\gamma}{2} + \frac{7}{8}\underbrace{(\underset{e^+}{2} + \underset{e^-}{2})}_{\text{fermions}} + \dots$

Hence

$$[sa^3]_{before} = \left[\frac{2\pi^2}{15}\left(\frac{11}{2}\right) T^3 a^3 \right]_{before} + [s_\nu a^3]_{before}.$$

Similarly

$$[sa^3]_{after} = \Big[\frac{2\pi^2}{15}(2 + \underset{e^+, e^- \text{ are non relativistic}}{0} + \dots) T^3 a^3 \Big]_{after} + [s_\nu a^3]_{after}.$$

$\therefore$

$\therefore \quad T_{after} = \left(\frac{11}{4}\right)^{1/3} T_{before}.$

$$\Rightarrow \boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$$

after

where we used the fact that after decoupling
$[S_\nu a^3]$ is conserved by it-self (free streaming).

& that before e^+e^- annihilation $T_\gamma = T_\nu = T$

* Notes: - When referring to T, we are using the photon temperature.
- the e^+e^- transition as well as decoupling of neutrinos is not quite instantaneous.

IMP

$\Gamma \sim H$: decoupling of ν = ν stop interacting with the rest of the plasma.

$T \sim m_e$: e^+e^- annihilation = not enough energy in plasma to regenerate e^+e^- pairs

Beyond Equilibrium.

We can no longer use $f(p) = \frac{1}{e^{\frac{E}{T}-\frac{\mu}{T}} \pm 1}$

Instead we have to solve the Boltzmann equation. Still assuming homogeneity and isotropy, the Boltzmann equation is given by

The RHS is due to interactions.

Instead of dealing with f, it is convenient (and sufficient for our purposes) to deal with the number density.

In absence of interactions the number density of a species i, $n_i \propto a^{-3}$

$$\therefore \quad \frac{1}{a^3}\frac{d}{dt}(n_i a^3) = 0 .$$

In presence of interactions

$$\frac{1}{a^3}\frac{d}{dt}(n_i a^3) = C[\{n_j\}]$$

$\uparrow$ interactions

Let us consider $1 + 2 \rightleftarrows 3+4$
type interactions (those involving higher # of particles are less likely).
Focus on 1.

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = C[\{n_1, n_2, n_3, n_4\}]$$

What is the form of C?

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = \underset{\substack{\uparrow \\ \text{destroys } 1}}{-\alpha n_1 n_2} + \underset{\substack{\uparrow \\ \text{Generates } 1.}}{\beta n_3 n_4}$$

One can arrive at the above general form, as well as determine α & β by considering a more detailed derivation (see for example, Dodelson's Textbook).
I will state the answer here, and then justify that it is reasonable.
$\alpha = \langle \sigma v \rangle$ = thermally averaged crosssection $1+2 \to 3+4$.
$\beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}$ where "eq" stands for the equilibrium expressions for the respective # densities.
Note that the chemical potentials $\mu_1 + \mu_2 = \mu_3 + \mu_4$.

i.e.

$$\frac{1}{a^3}\frac{d}{dt}(n_1 a^3) = -\langle\sigma v\rangle n_2 n_1 \left[1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \frac{n_3 n_4}{n_1 n_2}\right]$$

Now $\Gamma_1 \equiv n_2\langle\sigma v\rangle$

$$\therefore \frac{1}{a^3}\frac{d}{dt}(n_1 a^3) = -\Gamma_1 n_1 \left[1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \frac{n_3 n_4}{n_1 n_2}\right]$$

Let us make sure that this equation makes sense.
We expect that if $\Gamma_1 \gg H$, then $n_i \approx n_i^{eq}$ should be an "approximate" solution of the above equation.
This is indeed the case.
You should check that regardless of $m_1 \ll T$ or $m_1 \gg T$, as long as $\frac{\Gamma_1}{H}$ is large enough, $n_i \sim n_i^{eq}$ will be an approximate solution to the above equation.

A convenient re-writing of the above equation is

using $\frac{d\ln a}{dt} = H$

Hence

$$\frac{d\ln(a^3 n_1)}{d\ln a} = \frac{-\Gamma_1}{H}\left[1-\left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}\frac{n_3 n_4}{n_1 n_2}\right]$$

It is convenient to define

$$N_i = \frac{n_i}{s} \propto n_i a^3 \quad (\text{Recall } s \propto a^{-3})$$
$$\propto n_i / T^3$$

Then we have.

$$\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1-\left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq}\frac{N_3 N_4}{N_1 N_2}\right]$$

Another sanity check for this equation.

Case 1 : $\Gamma_1 >> H$

If $N_1 >> N_{1eq}$, $N_i \sim N_{i\,eq}$ $i = 2, 3, 4$.

then $\frac{d\ln N_1}{d\ln a} < 0 \Rightarrow N_1$ decreases (towards the eq value)

If $N_1 << N_{eq}$

$\frac{d\ln N_1}{d\ln a} > 0 \Rightarrow N_1$ increase (towards the eq. value)

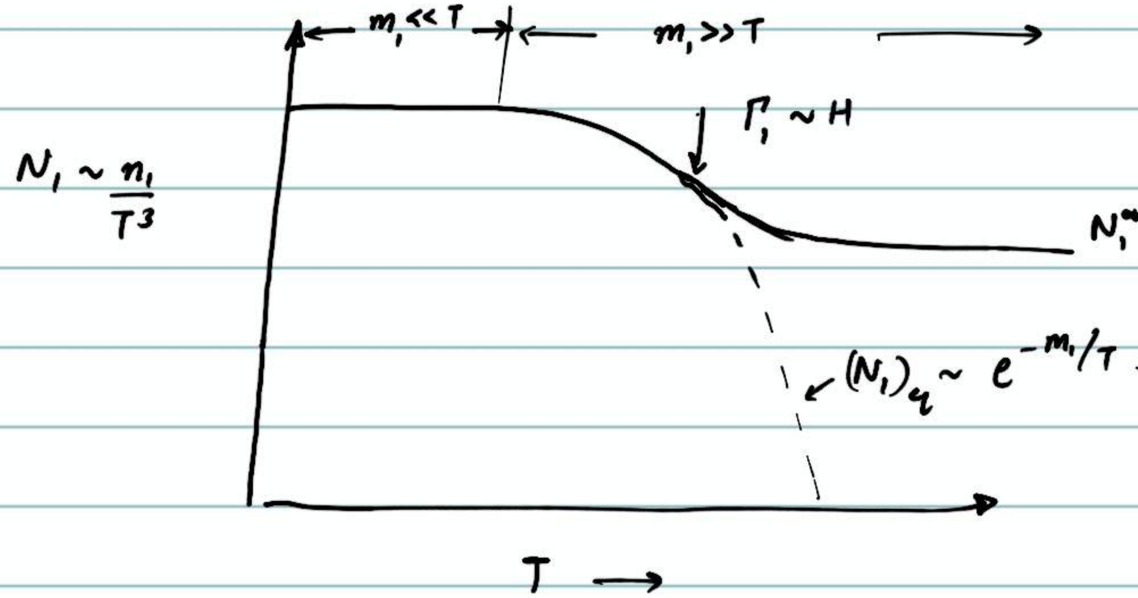
<u>Case 2</u> $\Gamma_1 \ll H$.

$$\frac{d\ln N_1}{d\ln a} \approx 0 \Rightarrow N_1 \approx \text{constant} \leftarrow \text{Freezout}.$$

Thus if $\Gamma_1 \gg H$, $N_1 \to (N_1)_{eq}$.

$\Gamma_1 \ll H$, $N_1 \to$ constant.

Here is a typical behavior.



IMPORTANT.

Applications 1) Dark matter freezout
2) BBN
3) Recombination & Decoupling.

1) Dark matter freezout.

Consider a neutral dark matter particle mass M_X which can annihilate to massless leptons.

$$X + \bar{X} \rightleftarrows l + \bar{l}$$

antiparticle (of X), antiparticle (of l)

Assume $n_X = n_{\bar{X}}$ initially
$l, \bar{l}$ are charged & in equilibrium with photons
$n_l = (n_l)_{eq} = n_{\bar{l}}$

In this case the Boltzmann eq becomes.

$$\frac{dN_X}{d\ln a} = \frac{-\Gamma}{H}\left[1 - \frac{(N_X^{eq})^2}{N_X^2}\right]$$

Re-arranging, & defining $x = \frac{M_X}{T}$, as well as $H = \frac{H(M_X)}{x^2}$

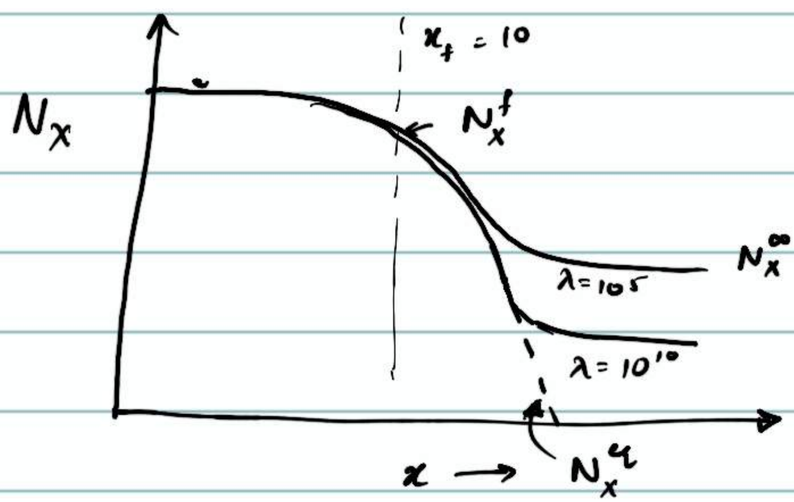
$\because$ we are in radiation domination.

we get

$$\frac{dN_\chi}{dx} = -\frac{\lambda}{x^2}\left[N_\chi^2 - (N_\chi^{eq})^2 \right]$$

$$\text{where} \quad \lambda = \frac{2\pi^2}{45} g_* \frac{\overbrace{M_\chi^3 \langle \sigma v \rangle}^{\text{particle physics}}}{\underbrace{H(M_\chi)}_{\text{cosmology!}}}$$

We can solve the equation numerically, to get
(assuming λ = const).



For $x \gtrsim x_f$, $N_\chi^{eq} \ll N_\chi \Rightarrow \quad \dfrac{dN_\chi}{dx} \approx -\dfrac{\lambda}{x^2} N_\chi^2$

integrate

$$\Rightarrow \quad \frac{1}{N_\chi^\infty} - \frac{1}{N_\chi^f} \approx \frac{\lambda}{x_f}$$

Since $N_\chi^\infty \ll N_\chi^f \Rightarrow \quad N_\chi^\infty \approx \dfrac{x_f}{\lambda}$

What remains, is estimating x_f. You will do this in your homework.
It turns out $x_f \sim 10$ (reasonably independent of λ!).

Thus $N_\chi^\infty \sim \frac{10}{\lambda}$

We can convert N_χ^∞ into Ω_χ = fraction of energy density of the universe in χ today.

$$\Omega_\chi \sim 0.2 \left(\frac{x_f}{10}\right)\left(\frac{10}{g_*(M_\chi)}\right)^{1/2} \frac{10^{-8}\,\text{GeV}^{-2}}{\langle\sigma v\rangle}.$$

Amazingly, if $\langle\sigma v\rangle \sim 10^{-4}\,\text{GeV}^{-2}$ (typical weak interaction cross section
$\sim 0.\sqrt{G_F}$

the abundance of dark matter is roughly what we observe. This is called the WIMP miracle.
(weakly interacting massive particle)

Lecture 12

Plan : 1) Dark - Matter Freezout
2) Big - Bang Nucleosynthesis (First nuclei).
3) Recombination + Decoupling (First atoms)

Review : $1 + 2 \rightleftarrows 3 + 4$.

Boltzmann equation

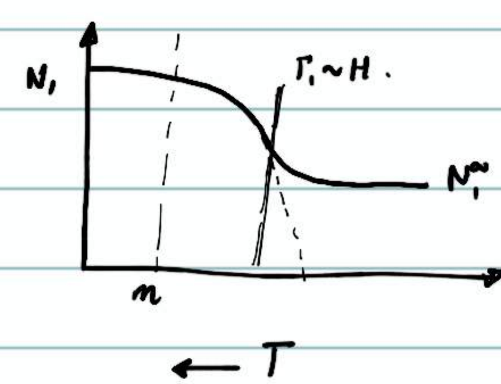
$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle \sigma v \rangle \left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} n_3 n_4 \right].$$

$$\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2}\right]$$

where $N_i = \frac{n_i}{s} \propto n_i a^3$; $\Gamma_1 = \langle \sigma v \rangle n_2$.

$\frac{\Gamma_1}{H} \gg 1 \Rightarrow N_1 \to N_1^{eq}$

$\frac{\Gamma_1}{H} \ll 1 \Rightarrow N_1 \to N_1^{\infty} = \text{const}$.



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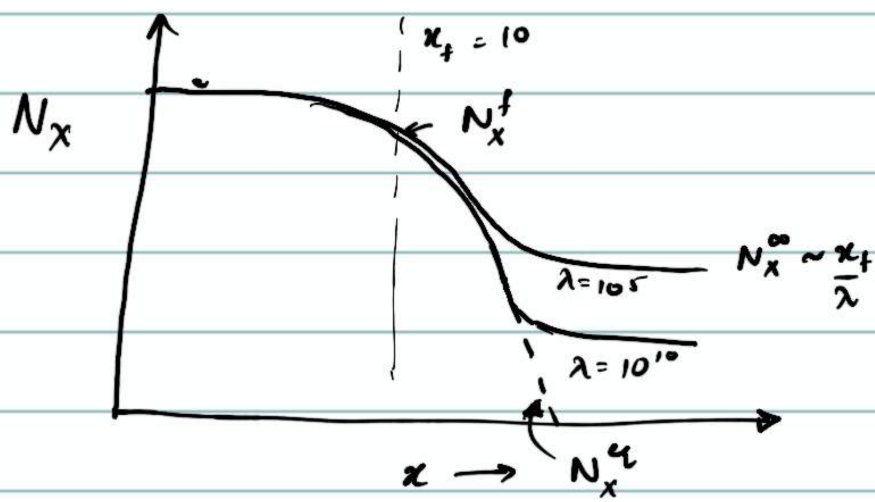

$\Gamma(x_f) \sim H(x_f)$.

For $x \gtrsim x_f$, $N_\chi^{eq} \ll N_\chi \Rightarrow \quad \frac{dN_\chi}{dx} \simeq -\frac{\lambda}{x^2} N_\chi^2$

integrate

$$\Rightarrow \quad \frac{1}{N_\chi^\infty} - \frac{1}{N_\chi^f} \simeq \frac{\lambda}{x_f}$$

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Big Bang Nucleosynthesis ($T \sim 0.1\,\mathrm{MeV}$)

Syntheses of light element : Hydrogen H
Helium . He
Lithium Li
⋮

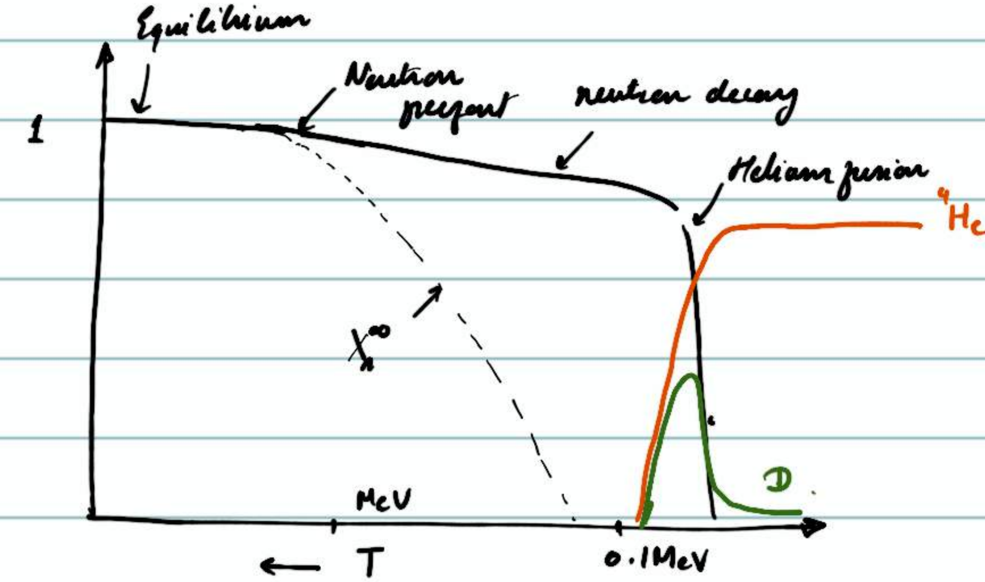
Important steps : (see figure).

1) Neutron Freezout : $n + \nu_e \rightleftharpoons p + e \Rightarrow n_n^\infty \sim \frac{1}{6} n_p^\infty$
2) Neutron decay : $n_n(t) \simeq n_n^\infty e^{-t/\tau_n}$ $\tau_n \approx 900\,\mathrm{sec}$.
3) Helium fusion.

Helium can only form after deuterium
$\Rightarrow$ deuterium "Bottleneck" (Deuterium takes a while because of small binding energy and large photon/baryon ratio).

$$n + p \to D + \gamma$$
$$\downarrow$$
$$D + p \longrightarrow {}^3He + \gamma$$
$$D + {}^3He \longrightarrow {}^4He + p.$$

Higher elements built from lighter elements. Plasma is too dilute for heavier elements to form by > 2 body interactions.



$X_n = \dfrac{n_n}{n_n + n_p}$ = neutron fraction.

Some useful simplifications that make the calculation tractable:

1) only track elements lighter than Helium.
2) for $T > 0.1$ MeV, we only need to track protons & neutrons, all others negligible.

* The binding energy for Deuterium is ~ 1 MeV. Why do we have to wait till 0.1 MeV for it to form.

Ans. Too many photons.

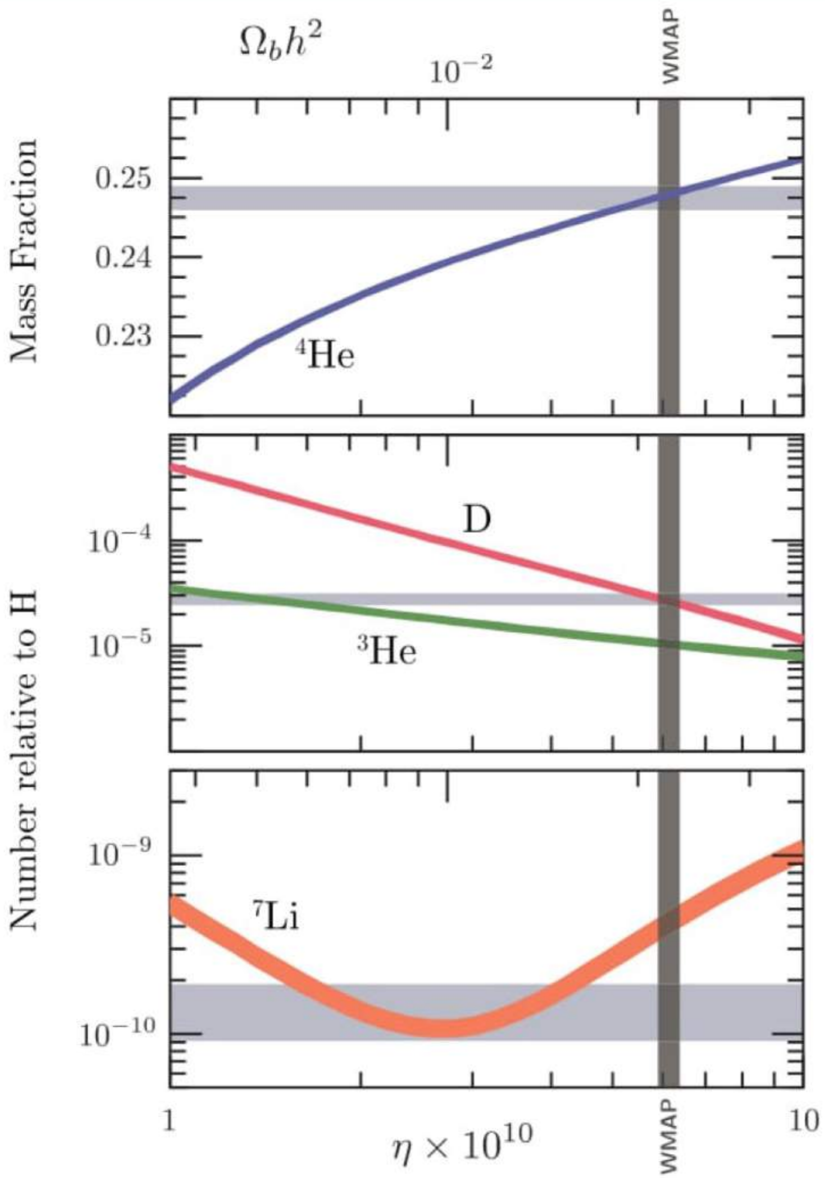
$\dfrac{\text{photons}}{\text{baryon}} \simeq \dfrac{1}{\eta_b} \simeq 10^{10}$ ⇒ there are enough high energy photons in the tail of the distribution that can disrupt the nucleus!

At the end of the day one finds

$$\boxed{\frac{4\, n_{He}}{n_H} \simeq \frac{1}{4}}$$

, agrees beautifully with observations.

(Other light element abundances can also be calculated)



Theoretical predictions (colored bands) and observational constraints (grey bands).

From DB Notes.

Recombination: $(T \sim 0.3\,\text{eV})$. [ignore Helium]

For $T \gtrsim \text{eV}$ $\quad e + p \rightleftharpoons H + \gamma$ (equilibrium).

Since $T < m_i$ $\quad i = \{e, p, H\}$.

$$n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-\frac{m_i}{T}} e^{\mu_i/T}$$ (re-introducing the chemical potential).

In equilibrium. $\mu_e + \mu_p = \mu_H$. $\quad (\because \mu_\gamma = 0)$.

$$\therefore \left(\frac{n_H}{n_e n_p}\right)_{eq} = \frac{g_H}{g_e g_p} \left(\frac{m_H}{m_e m_p} \frac{2\pi}{T}\right)^{3/2} e^{(m_p + m_e - m_H)/T}.$$

$$\Rightarrow \left(\frac{n_H}{n_e^2}\right)_{eq} \approx \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}.$$

where $B_H = m_p + m_e - m_H \approx 13.6\,\text{eV}$ (Binding energy)

$g_H = 4$, $g_e = g_p = 2$. & we used $n_e = n_p$ (neutrality).

Note that it is safe to set $m_p \approx m_H$ in the co-efficient, but not in B_H.

Let us follow the free electron-fraction

$$X_e = \frac{n_e}{n_b} = \frac{n_e}{n_p + n_H}$$

n_b = # density of "baryons".

Assuming $n_b \approx n_p + n_H = n_e + n_H$ (no Helium).

yields

$$\therefore \quad \boxed{\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2}\eta_b \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}.} \quad \leftarrow \text{SAHA Eq.}$$

(Equilibrium).

Note $n_b = \eta_b n_\gamma = \eta_b \frac{2\zeta(3)}{\pi^2} T^3$ where $\eta_b \approx 10^{-10}$

When does $X_e = 10^{-1}$. Solve the Saha eq to get T_{rec} = Hydrogen recombination temperature

$T_{rec} \approx 0.3$ eV

Note again $T_{rec} = 0.3\,\text{eV} \ll B_H = 13.6\,\text{eV}$ because of smallness of η_b.

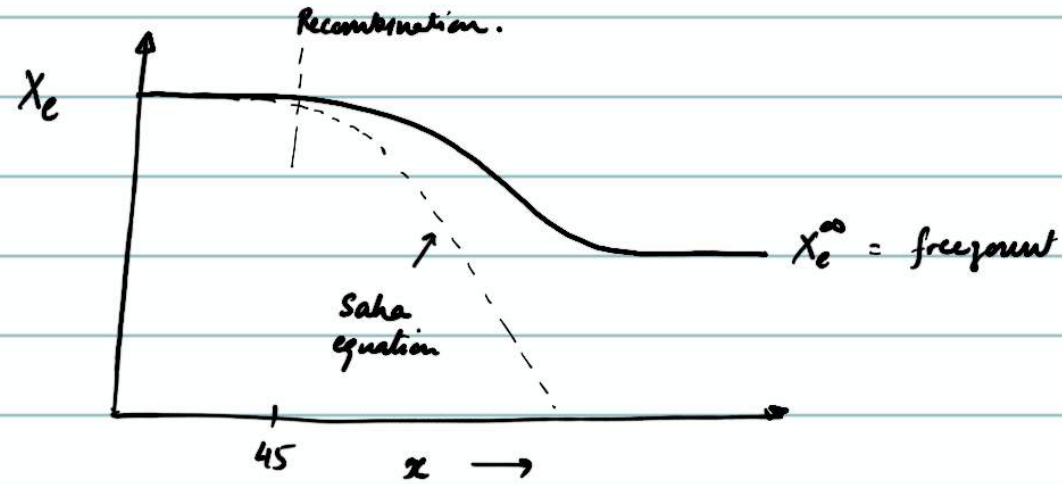
* Redshift of recombination : $z_{rec} \approx 1320$.

This was all done using equilibrium expressions. By now we know that this is an incomplete story. Eventually $p + e \leftarrow H + \gamma$ becomes too inefficient and the free electron density freezes out.

The equation describing the process. (see Baumann's notes for details

$$\frac{dX_e}{dx} = -\frac{\lambda}{x^2}\left[X_e^2 - \left(X_e^{eq}\right)^2 \right]$$

when $x = \frac{B_H}{T}$ $\quad \lambda = \left[\frac{n_b \langle \sigma v \rangle}{xH}\right]_{x=1}$



There is another important event, squeezed in between Hydrogen recombination and free electron freezout.

Photon Decoupling:

Consider the reaction $e + \gamma \rightleftarrows e + \gamma$

$$\Gamma_\gamma = \langle \sigma v \rangle n_e = \sigma_T n_e .$$

$\sigma_T \equiv$ thompson cross section.

Since the free electron fraction is dropping

$\frac{\Gamma_\gamma}{H} \lesssim 1$ will happen! Using the equilibrium abundances, we find $\Gamma_\gamma(T_{dec}) \simeq H(T_{dec})$

$$\Rightarrow \boxed{T_{dec} \simeq 0.27 \text{ eV}}$$

After this point the photons barely interact with the electrons and are free to stream to us. This is the cosmic microwave Background. (CMB) !!!

The redshift $\boxed{z_{dec} \simeq 1100}$.

Note the ordering

① Recombination: $T_{rec} \simeq 0.3\,eV$, $z_{rec} \simeq 1320$, $t_{rec} \simeq 290{,}000$ yrs

② Photon decoupling: $T_{dec} \simeq 0.27\,eV$, $z_{dec} \simeq 1100$, $t_{dec} \simeq 380{,}000$ yrs.

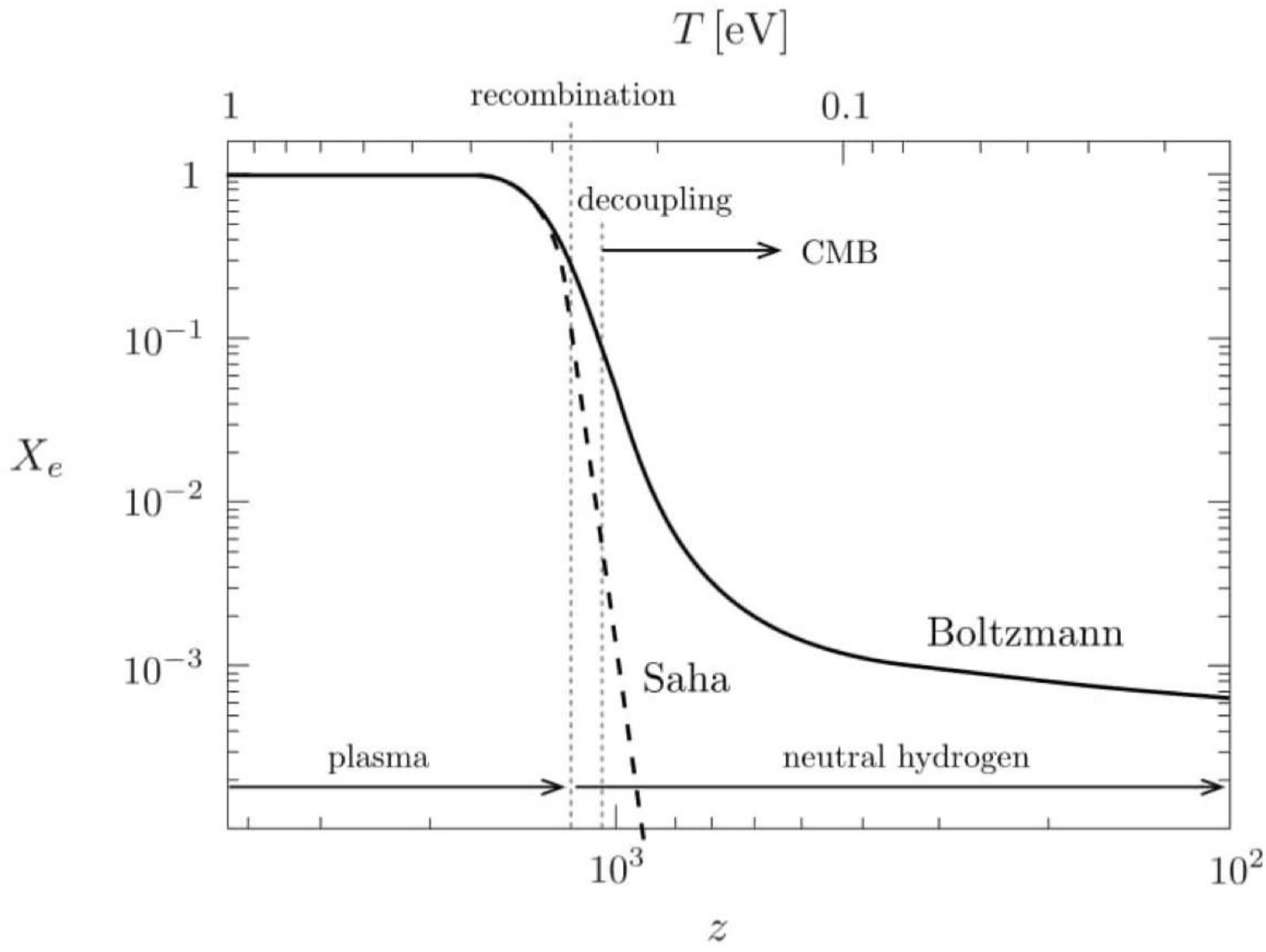


Figure 3.8: *Free electron fraction as a function of redshift.*

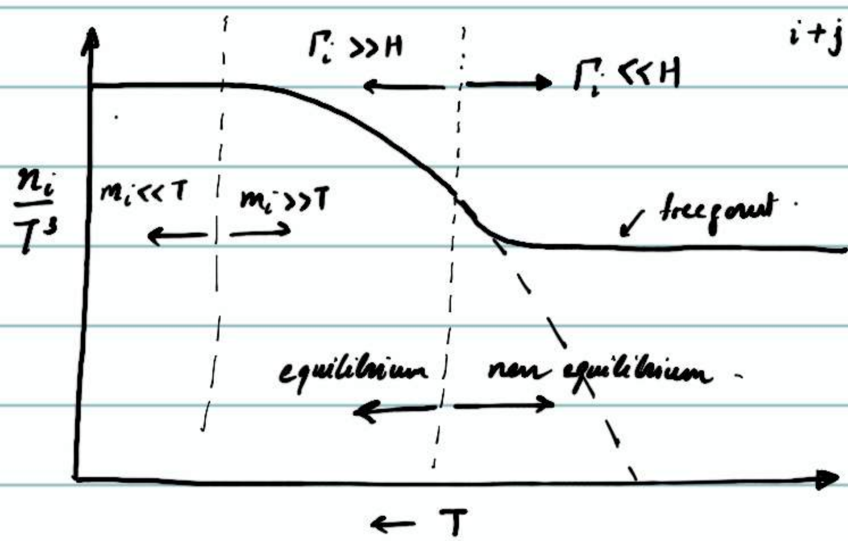
From DB Notes.

Summary of the Chapter

thermal history

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

$i + j \rightleftharpoons k + l$.



Lecture 13

<u>Plan</u> : 1) Photon decoupling ; electron freezout.
2) The Inhomogeneous universe
- The perturbed metric.

<u>Review</u> :

The Happenings around $T \simeq 0.3\ eV$.

$$e + p \rightleftharpoons H + \gamma .$$

In equilibrium.

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} \sim \eta_b \left(\frac{T}{m_e}\right)^{3/2} e^{B_H/T} \qquad \begin{array}{l} B_H = 13.6\ eV \\ \eta_b \sim 10^{-10} \end{array}$$

$X_e \approx$ free electron fraction.

$X_e \approx 10^{-1} \Rightarrow T_{rec} \approx 0.3\ eV \quad , z_{rec} = 1380 .$

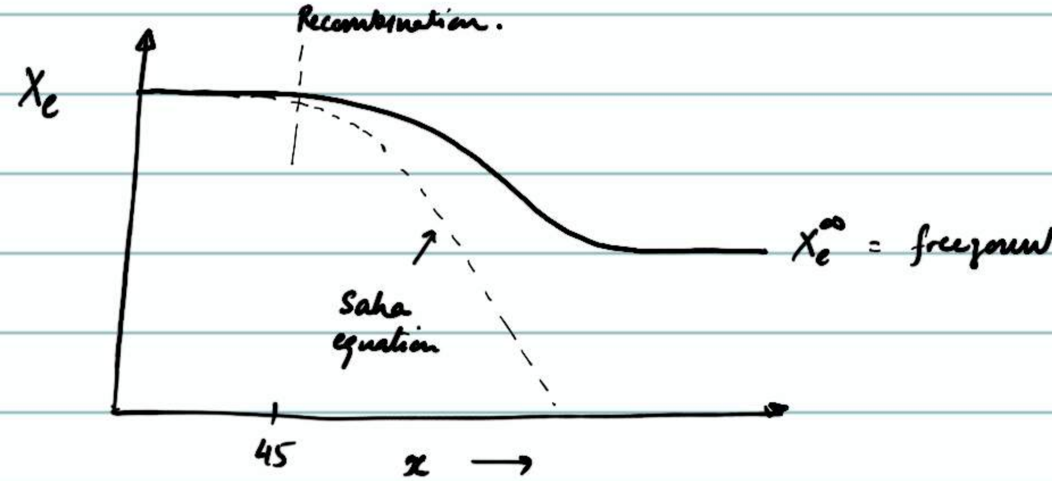
This was all done using equilibrium expressions. By now we know that this is an incomplete story. Eventually

$p + e \leftarrow H + \gamma$ becomes too inefficient and the free electron density freezes out.

The equation describing the process. (see Baumann's notes for details

$$\frac{dX_e}{dx} = -\frac{\lambda}{x^2}\left[X_e^2 - \left(X_e^{eq}\right)^2\right]$$

when $\quad x = \frac{B_H}{T} \qquad \lambda = \left[\frac{n_b \langle \sigma v \rangle}{xH}\right]_{x=1}$



There is another important event, squeezed in between Hydrogen recombination and free electron freezout.

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Consider the reaction $e + \gamma \rightleftarrows e + \gamma$

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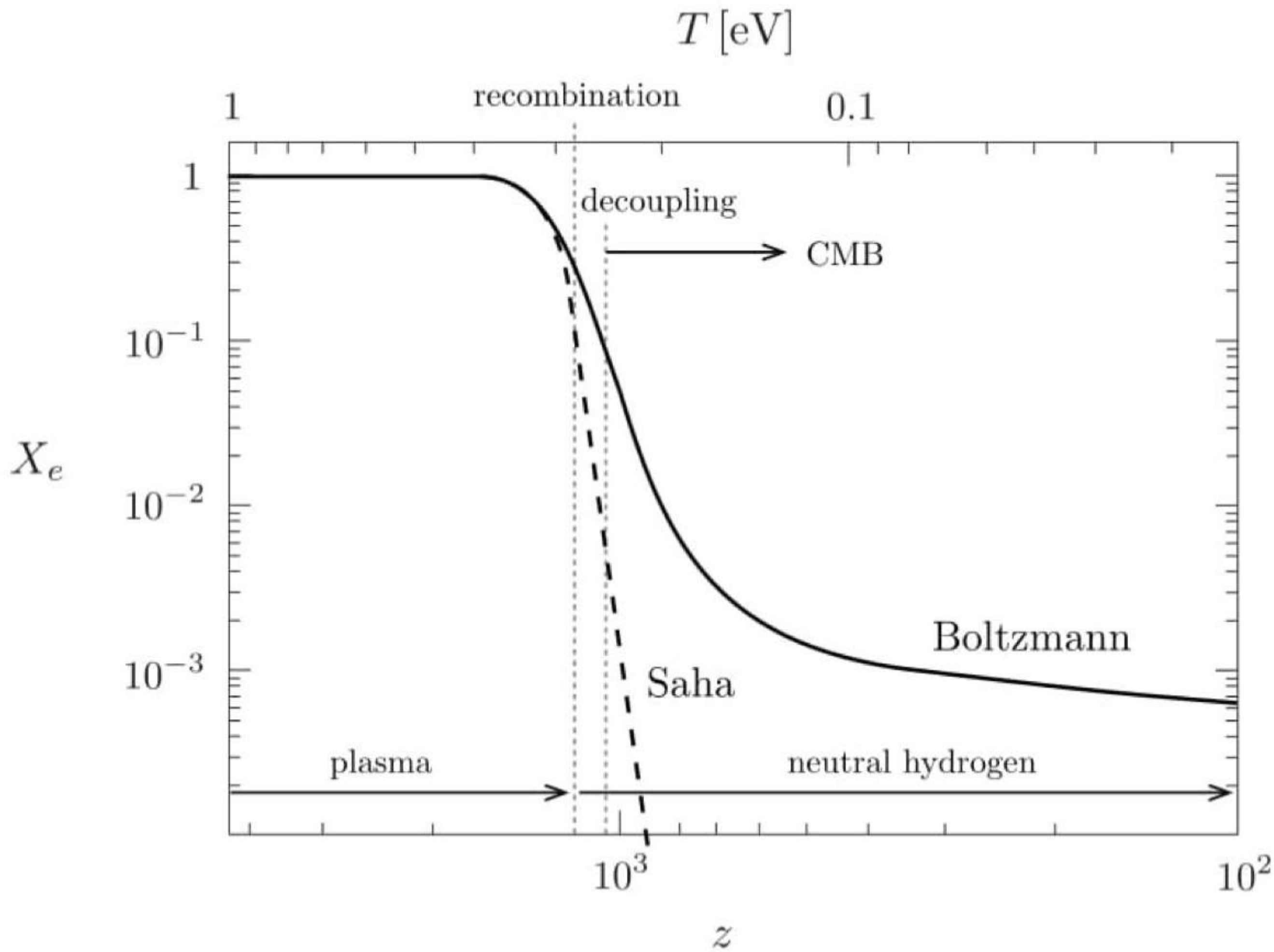

Figure 3.8: *Free electron fraction as a function of redshift.*

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thermal history

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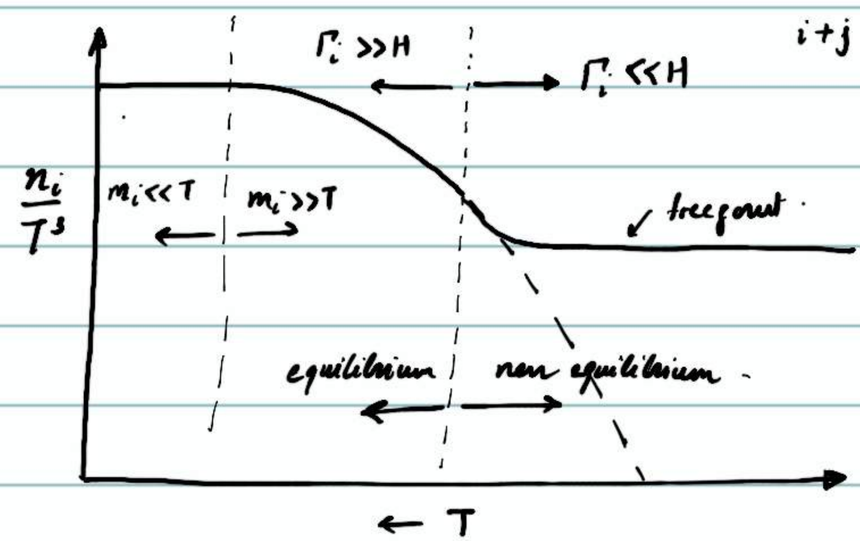
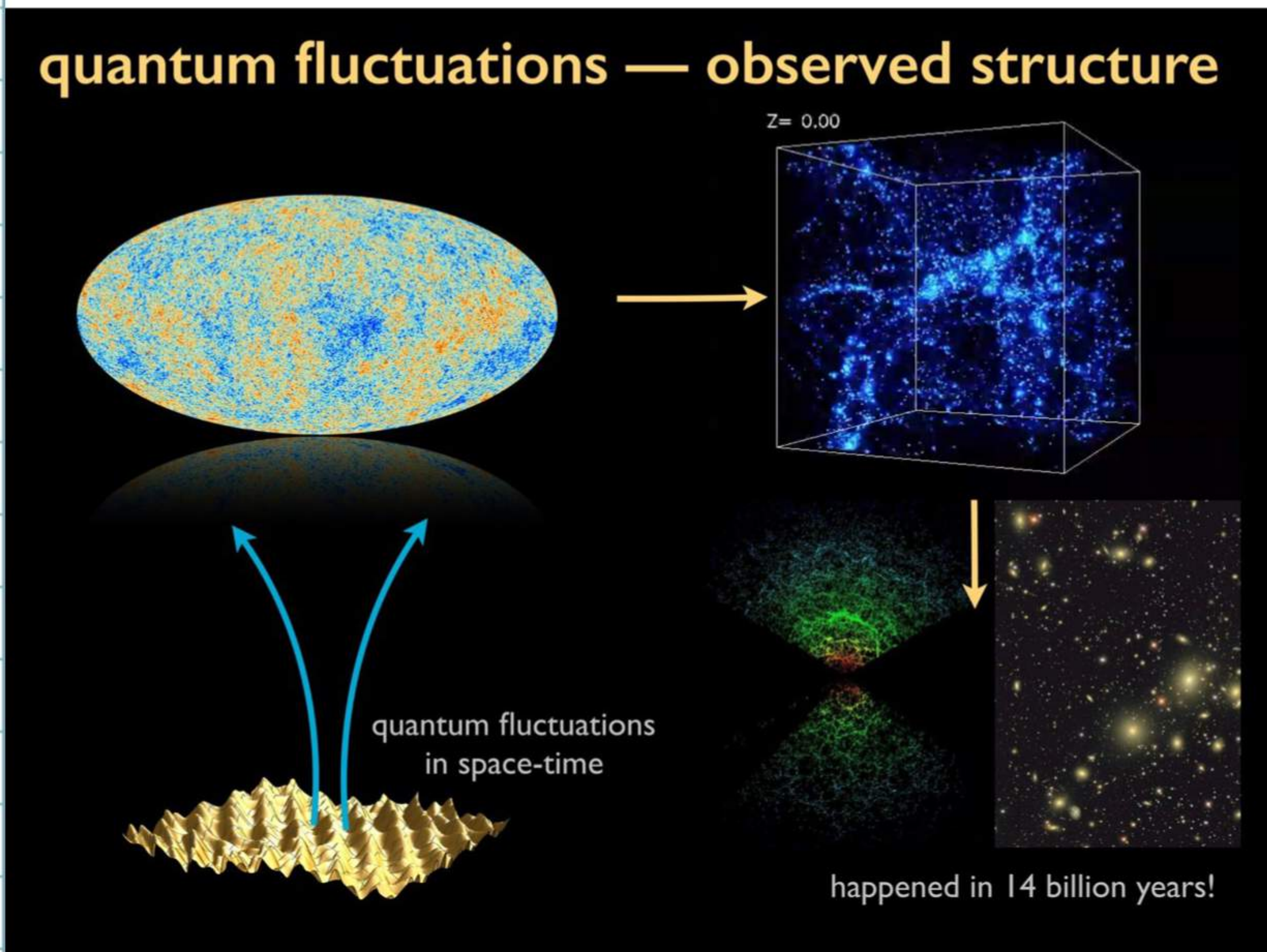


The Inhomogeneous Universe

Overall goal:



Focus on small deviations from homogeneity

$$g_{\mu\nu}(\tau,\vec{x}) = \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau,\vec{x}) \; ; \quad |\delta g_{\mu\nu}| \ll \bar{g}_{\mu\nu}$$

$$T_{\mu\nu}(\tau,\vec{x}) = \bar{T}_{\mu\nu}(\tau) + \delta T_{\mu\nu}(\tau,\vec{x}) \; ; \quad |\delta T_{\mu\nu}| \ll \bar{T}_{\mu\nu}.$$

Linearize $\Rightarrow$ ignore $(\delta g_{\mu\nu})^2$, $(\delta T_{\mu\nu})^2$ etc.

Perturbations of the metric

$$ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - 2B_i\, dx^i d\eta - (\delta_{ij}+h_{ij})dx^i dx^j\right]$$

(Ψ, B_i etc are all functions of $(\eta, \vec{x})$.
Ψ, B_i etc $=0$ for a perfectly homogeneous universe)

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}}.$$

$\varepsilon_{ijk}\partial^i\partial^j B = 0 \qquad \partial_i \hat{B}^i = 0$

$$h_{ij} = \underbrace{-2\Phi\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle} h}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{h}_{j)}}_{\text{vector}} + \underbrace{2\hat{h}_{ij}}_{\text{tensor.}}$$

trace (for $-2\Phi\delta_{ij}$)   trace free (for $2\partial_{\langle i}\partial_{j\rangle} h$). $\partial_i \hat{h}^i = 0$ $\partial^i \hat{h}_{ij} = \hat{h}^i_i = 0$

$$\partial_{\langle i}\partial_{j\rangle} h \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)h \qquad \text{trace free}$$

$$\partial_{(i}\hat{h}_{j)} \equiv \frac{1}{2}\left(\partial_i\hat{h}_j + \partial_j\hat{h}_i\right) \qquad \text{divergence free}$$

& $ij \leftrightarrow ji$.

Counting the # of free functions (d.o.f).

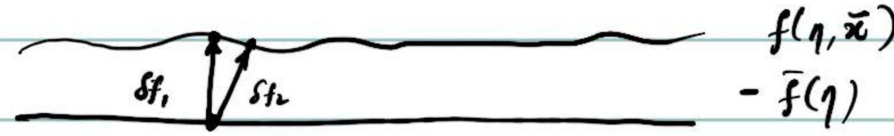
	Ψ, B, Φ, h	$= 4$	"scalar"	
+	$\hat{B}_i, \hat{h}_i$	$= 4$	"vector"	$(\because \partial_i\hat{B}^i = \partial_i\hat{h}^i = 0)$
+	$\hat{h}_{ij}$	$= 2$	"tensor"	$(\because \partial_i\hat{h}^{ij} = 0 \text{ ; } \hat{h}^i_i = 0)$
		10	independent d.o.f.	

Why split things this way? (SVT decomposition).

Because the evolution equations do not couple the scalar, vector and tensor parts at linear order in the perturbations! Each group evolves independently.

Gauge Freedom.

Perturbation ≡ −



But no unique way to identify which points the difference is calculated. This is nothing but the ambiguity of co-ordinates & goes under the name of gauge freedom.

Another way of thinking about this is that the definition of a perturbation depends on how we foliate (& thread) space with surfaces. As an example we can get rid of a perturbation by simply choosing a surface of constant density.

[For more see Baumann's notes]

Lecture 14

General Relativity is invariant under $\underbrace{\text{diffeomorphisms}}$

an invertible function that maps a smooth manifold to another such that the function and its inverse are smooth

If the universe is represented by a manifold M with metric $g_{\mu\nu}$ and matter fields ψ, and

$$\phi : M \longrightarrow M$$

(active coordinate transformation)

is a diffeomorphism, then the sets

$$(M, g_{\mu\nu}, \psi) \qquad \text{and} \qquad (M, \phi^* g_{\mu\nu}, \phi^* \psi)$$

represent the same physical situation

There is no preferred coordinate system for spacetime

When we do perturbation theory, diffeomorphism invariance still applies, but now, not every diffeomorphism will map a small perturbation into a small perturbation. As it turns out (see Carroll 7.1), the "safe" diffeomorphisms have the form

$$\boxed{x^\mu \rightarrow x^\mu + \epsilon \xi^\mu}$$

for which one can check that, for example, the linear change induced in the Riemann tensor is $\delta R_{\mu\nu\rho\sigma} = 0$.

How do perturbations change upon a gauge transformation?

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(\eta, x) \quad , \quad \xi^0 = T \quad , \quad \xi^i = L^i = \partial^i L + \underbrace{\hat{L}^i}_{\text{divergenceless}}$$

$$ds^2 = a^2(\eta)\left[d\eta^2 - \delta_{ij}\, dx^i dx^j\right] \quad \longleftarrow \text{ Unperturbed}$$

Invariance of the space-time interval:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}(\tilde{x})\, d\tilde{x}^\alpha d\tilde{x}^\beta \Rightarrow \boxed{\tilde{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}}$$

The original perturbed spacetime:

$$ds^2 = a^2(\eta)\left[(1+2A)\, d\eta^2 - 2B_i\, dx^i d\eta - (\delta_{ij} + h_{ij})\, dx^i dx^j\right]$$

The gauge-transformed perturbed spacetime

$$ds^2 = a^2(\tilde{\eta})\left[(1+2\tilde{A})\, d\tilde{\eta}^2 - 2\tilde{B}_i\, d\tilde{x}^i d\tilde{\eta} - (\delta_{ij} + \tilde{h}_{ij})\, d\tilde{x}^i d\tilde{x}^j\right]$$

Calculate:

$$g_{00} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial x^0} \frac{\partial \tilde{x}^\beta}{\partial x^0} = \left(\frac{\partial \tilde{\eta}}{\partial \eta}\right)^2 \tilde{g}_{00} + \left(\frac{\partial \tilde{\eta}}{\partial \eta}\right)\left(\frac{\partial \tilde{x}^i}{\partial \eta}\right) \tilde{g}_{0i} + \left(\frac{\partial \tilde{x}^i}{\partial \eta}\right)\left(\frac{\partial \tilde{x}^j}{\partial \eta}\right) \tilde{g}_{ij}$$

$$= (1+T')^2 \tilde{g}_{00} + \underbrace{(1+T')\, L^{i\prime} a^2(-2B_i) + L^{i\prime} L^{j\prime} \tilde{g}_{ij}}_{\text{second order}}$$

$$\approx (1+T')^2 \tilde{g}_{00}$$

$$\Rightarrow \quad a(\eta)^2(1+2A) = (1+T')^2 a(\tilde{\eta})^2(1+2\tilde{A})$$

$$= (1+2T'+\cdots)(a(\eta)+a'(\eta)T+\cdots)^2(1+2\tilde{A})$$

$$= a^2(\eta)(1+\mathcal{H}T+\cdots)^2(1+2\tilde{A}+2T'+\cdots)$$

$$= a^2(\eta)(1+2T'+2\mathcal{H}T+2\tilde{A}+\cdots)$$

$$\Rightarrow \boxed{\tilde{A} = A - T' - \mathcal{H}T}$$

Do one more,

$$g_{0i} = \tilde{g}_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial\eta}\frac{\partial\tilde{x}^\beta}{\partial x^i} = \left(\frac{\partial\tilde{\eta}}{\partial\eta}\right)\left(\frac{\partial\tilde{\eta}}{\partial x^i}\right)\tilde{g}_{00} + \left(\frac{\partial\tilde{\eta}}{\partial\eta}\right)\left(\frac{\partial\tilde{x}^j}{\partial x^i}\right)\tilde{g}_{0j}$$

$$+\left(\frac{\partial\tilde{x}^j}{\partial\eta}\right)\left(\frac{\partial\tilde{\eta}}{\partial x^i}\right)\tilde{g}_{0j} + \left(\frac{\partial\tilde{x}^j}{\partial\eta}\right)\left(\frac{\partial\tilde{x}^k}{\partial x^i}\right)\tilde{g}_{jk}$$

$$= (1+T')\partial_i T\,\tilde{g}_{00} + \left[(1+T')(\delta_i^j+\partial_i L^j) + L^{j\prime}\partial_i T\right]\tilde{g}_{0j} + L^{j\prime}(\delta_i^k+\partial_i L^k)\tilde{g}_{jk}$$

$$\simeq \partial_i T\,\tilde{g}_{00} + a^2(\delta_i^j(1+T')+\partial_i L^j)(-\tilde{B}_j) - \delta_i^k L^{j\prime}(\delta_{jk}+\tilde{h}_{jk})a^2$$

$$\simeq a^2(\tilde{\eta})(1+2\tilde{A})\partial_i T - a^2(\tilde{\eta})\tilde{B}_i - a^2(\tilde{\eta})\delta_{ij}L^{j\prime}$$

$$\simeq a^2(\eta)\left[\partial_i T - \tilde{B}_i - L_i'\right]$$

$$= -a^2(\eta)B_i$$

$$\Rightarrow \boxed{\tilde{B}_i = B_i + \partial_i T - L_i'}$$

Just one remaining, so might as well do it

$$g_{ij} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial x^i} \frac{\partial \tilde{x}^\beta}{\partial x_j}$$

$$= \left(\frac{\partial \tilde{\eta}}{\partial x^i}\right)\left(\frac{\partial \tilde{\eta}}{\partial x^j}\right) \tilde{g}_{00} + \left(\frac{\partial \tilde{\eta}}{\partial x^i}\frac{\partial \tilde{x}^k}{\partial x^j} + \frac{\partial \tilde{\eta}}{\partial x^j}\frac{\partial \tilde{x}^k}{\partial x^i}\right)\tilde{g}_{0k} + \left(\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}\right)\tilde{g}_{kl}$$

$$= \partial_i T \partial_j T \tilde{g}_{00} + \left[\partial_i T (\delta_j^k + \partial_j L^k) + \partial_j T (\delta_i^k + \partial_i L^k)\right] \tilde{g}_{0k}$$

$$+ (\delta_i^k + \partial_i L^k)(\delta_j^l + \partial_j L^l)\tilde{g}_{kl}$$

$$\simeq (\partial_i T \delta_j^k + \partial_j T \delta_i^k)\tilde{g}_{0k} + (\delta_i^k \delta_j^l + \delta_i^k \partial_j L^l + \delta_j^l \partial_i L^k)\tilde{g}_{kl}$$

$$\simeq \tilde{g}_{ij} + \tilde{g}_{il}\partial_j L^l + \tilde{g}_{kj}\partial_i L^k$$

$$= -a(\tilde{\eta})^2 \left[\delta_{ij} + \tilde{h}_{ij} + \delta_{il}\partial_j L^l + \delta_{kj}\partial_i L^k\right]$$

$$= -a(\eta)^2 (1 + 2\mathcal{H}T)(\delta_{ij} + \tilde{h}_{ij} + \partial_j L_i + \partial_i L_j)$$

$$\simeq -a(\eta)^2 (\delta_{ij} + \tilde{h}_{ij} + 2\mathcal{H}T\delta_{ij} + \partial_j L_i + \partial_i L_j)$$

$$= -a(\eta)^2 (\delta_{ij} + h_{ij})$$

$$\Rightarrow \boxed{\tilde{h}_{ij} = h_{ij} - 2\partial_{(i}L_{j)} - 2\mathcal{H}T\delta_{ij}}$$

In terms of the SVT decomposition,

$$\partial_i \tilde{B} + \hat{\tilde{B}}_i = \partial_i B + \hat{B}_i + \partial_i T - \partial_i L' - \hat{L}_i'$$

$$2\tilde{C}\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}\tilde{E} + 2\partial_{(i}\hat{\tilde{E}}_{j)} + 2\hat{\tilde{E}}_{ij}$$
$$= 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{E}_{ij} - 2\mathcal{H}T\delta_{ij} - 2\partial_i\partial_j L - 2\partial_{(i}\hat{L}_{j)}$$

$-2\partial_i\partial_j L =$ some steps ....

$$A \to A + T' - \mathcal{H}T$$

$$B \to B + T - L'$$

$$\hat{B}_i \to \hat{B}_i - \hat{L}_i'$$

$$\Rightarrow \quad C \to C - \mathcal{H}T - \tfrac{1}{3}\nabla^2 L$$

$$E \to E - L$$

$$\hat{E} \to \hat{E}_i - \hat{L}_i$$

$$\hat{E}_{ij} \to \hat{E}_{ij}$$

Gauge-invariant Bardeen variables:

$$\Psi = A + \mathcal{H}(B - E') + (B - E')'$$

$$\Phi = -C - \mathcal{H}(B - E') + \tfrac{1}{3}\nabla^2 E$$

$$\hat{\Phi}_i = \hat{E}_i' - \hat{B}_i$$

$$\hat{E}_{ij}$$

An alternative solution to the gauge problem is to fix the gauge and keep track of all perturbations.

Newtonian gauge: Use T and L to set $B = E = 0$

$$\Rightarrow \Psi = A, \quad \Phi = -C$$

Scalars $\Rightarrow$ $ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right]$

In weak field limit, Ψ plays the role of potential

It is simple because hypersurfaces of constant time are orthogonal to worldlines of observers at rest. Also, induced geometry in constant-time surfaces is isotropic.

No anisotropic stress $\Rightarrow$ $\Psi = \Phi$

Spatially flat gauge: Convenient (?) for inflationary perturbations

$$C = E = 0$$

$\Rightarrow$ all scalar perturbations are $\delta\phi$

Synchronous gauge: $\Phi = B = 0$

Popular for numerical methods

Uniform density gauge: Useful for description of perturbations in super-horizon scales

$$\delta\rho = 0, \quad E = 0, \quad -\Psi \equiv R$$

Comoving gauge: $\delta q = 0, \quad E = 0.$

Matter perturbations

Matter in a homogeneous and isotropic universe,

$$\bar{T}^\mu{}_\nu = (\bar{\rho}+\bar{P})\,\bar{u}^\mu \bar{u}_\nu - \bar{P}\,\delta^\mu{}_\nu$$

For a comoving observer

$$\bar{u}_\mu = a\,\delta^0_\mu \quad ; \quad \bar{u}^\mu = \bar{a}^1\,\delta^\mu_0$$

In a perturbed universe, ρ, P and u_μ can be functions of position. Also, there can be an anisotropic stress contribution.

$$\Rightarrow \boxed{\delta T^\mu{}_\nu = (\delta\rho+\delta P)\,\bar{u}^\mu\bar{u}_\nu + (\bar{\rho}+\bar{P})(\delta u^\mu \bar{u}_\nu + \bar{u}^\mu \delta u_\nu) - \delta\bar{P}\,\delta^\mu{}_\nu - \Pi^\mu{}_\nu}$$

traceless, as it can be absorbed in P

also $u^\mu \Pi_{\mu\nu} = 0$ $\quad \}\ \Pi^0_0 = \Pi^0_i = 0$

As

$$g_{\mu\nu}u^\mu u^\nu = \bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu + \delta g_{\mu\nu}\bar{u}^\mu\bar{u}^\nu + 2\bar{u}_\mu\delta u^\mu = 1$$

$$\Rightarrow \delta g_{\mu\nu}\bar{u}^\mu\bar{u}^\nu + 2\bar{u}_\mu\,\delta u^\mu = 0$$

$$\Rightarrow 2\bar{u}_\mu\delta u^\mu = 2a\,\delta^0_\mu\,\delta u^\mu = 2a\,\delta u^0 = -\bar{a}^2\,\delta g_{00} = -2A \Rightarrow \boxed{\delta u^0 = -A/a}$$

and with

$$\boxed{\delta u^i \equiv v^i/a}$$

coordinate velocity, $v^i = dx^i/d\tau$

$$\Rightarrow u^\mu = \bar{a}^1(1-A,\, v^i)$$

$$\Rightarrow u_0 = g_{0\mu}u^\mu = g_{00}u^0 + g_{0i}u^i \simeq g_{00}u^0 = a^2(1+2A)\,\bar{a}^1(1-A) \simeq a\,(1+A),$$

$$u_i = g_{i0}u^0 + g_{ij}u^j = -a^2B_i \cdot a^{-1}(1-A) - a^2(\delta_{ij}+h_{ij})a^{-1}v^j$$

$$\approx -aB_i - av_i \quad = -a(B_i+v_i)$$

$$\Rightarrow u_\mu \approx a\left(1+A, -(v_i+B_i)\right)$$

$$\Rightarrow \delta T^0_0 = (\delta\rho+\delta P)\bar{u}^0\bar{u}_0 + (\bar{\rho}+\bar{P})(\delta u^0\bar{u}_0 + \bar{u}^0\delta u_0) - \delta P$$

$$= \delta\rho + \delta P + (\bar{\rho}+\bar{P})(-a^{-1}Aa + a^{-1}aA) - \delta P \quad = \delta\rho$$

$$\delta T^i_0 = (\bar{\rho}+\bar{P})(\delta u^i\bar{u}_0 + \bar{u}^i\delta u_0) = (\bar{\rho}+\bar{P})(a^{-1}v^i a) = (\bar{\rho}+\bar{P})v^i$$

$$\delta T^i_j = -\delta P\,\delta^i_j - \Pi^i_j$$

$$\Rightarrow \quad \begin{aligned} T^0_0 &= \bar{\rho}(\eta) + \delta\rho \equiv \bar{\rho}(1+\delta) \qquad \leftarrow \ \delta: \text{density contrast} \\ T^i_0 &= (\bar{\rho}+\bar{P})\,v^i \equiv q^i \qquad \leftarrow \ q^i: \text{momentum density} \\ T^i_j &= -(\bar{P}+\delta P)\delta^i_j - \Pi^i_j \end{aligned}$$

Also, this implies that upon the existence of several species,

$$\boxed{\delta\rho = \sum_a \delta\rho_a \ , \quad \delta P = \sum_a \delta P_a \ , \quad q^i = \sum_a q^i_{(a)} \ , \quad \Pi^i{}_j = \sum_a \Pi^i{}_{j(a)}}$$

↑ velocities don't add

The SVT decomposition also applies:

$$q_i = \partial_i q + \hat{q}_i$$

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\Pi_{j)} + \hat{\Pi}_{ij}$$

Under a gauge transformation,

$$T^{\mu}{}_{\nu}(x) = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} = \tilde{T}^{\alpha}{}_{\beta}(\tilde{x})$$

Repeating what was done before, you get

$$\delta\rho \rightarrow \delta\rho - T\bar{\rho}'$$

$$\delta P \rightarrow \delta P - T\bar{P}'$$

$$q_i \rightarrow q_i + (\bar{\rho} + \bar{P}) L_i'$$

$$v_i \rightarrow v_i + L_i'$$

$$\Pi_{ij} \rightarrow \Pi_{ij}$$

One particular gauge-invariant combination:

$$\underbrace{\bar{\rho}\Delta}_{\text{comoving-gauge density perturbation}} \equiv \delta\rho + \rho'(v+B) \qquad , \qquad v_i = \partial_i v$$

Lecture 15

<u>Plan</u>: Perturbed Einstein Equations in Newtonian Gauge.

<u>Review</u>:

General Perturbed metric (scalar perturbations)

$$ds^2 = a^2\left[(1+2\Psi)d\eta^2 - 2\,\partial_i B\, dx^i d\eta - \left(\delta_{ij}(1-2\Phi) + 2\partial_{\langle i}\partial_{j\rangle} h\right)dx^i dx^j\right]$$

There is additional gauge freedom (co-ordinate choice). Restrict to transformations that map small perturbations to small perturbations.

There are two ways of dealing with this

(1) Take combinations of perturbations that are gauge invariant = do not depend of foliation/threading or choice of co-ordinates (at linear order).

(2) Specify co-ordinate choice explicitly.

Focus of a specific choice of co-ordinates & limiting to <u>scalar perturbations</u>

$$\left.\begin{aligned} \eta &\rightarrow \eta + T(\eta, \vec{x}) \\ x^i &\rightarrow x^i + \partial^i L(\eta, \vec{x}) \end{aligned}\right\} \text{ scalar co-ordinate transformation}$$

We can use T & L to get rid of two of our 4 scalar perturbations. We can always choose T and L such that $E = B = 0$. This is called the <u>Newtonian gauge</u>. In this gauge the perturbed metric takes the following form:

$$\boxed{ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - (1-2\Phi)dx^i dx^j\right]}$$

Φ, Ψ are similar to the usual newtonian gravitational potential. Also remember that here $\Phi, \Psi \ll 1$.

Also note:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = \underbrace{a^2\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}}_{\bar{g}_{\mu\nu}} + \underbrace{a^2\begin{pmatrix} 2\Psi & & & 0 \\ & 2\Phi & & \\ & & 2\Phi & \\ 0 & & & 2\Phi \end{pmatrix}}_{\delta g_{\mu\nu}}$$

MISC

Cosmic time to conformal time

$$\frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} \quad ; \quad \text{"}\cdot\text{"} = \frac{d}{dt} \quad , \quad \text{"}\,'\,\text{"} = \frac{d}{d\eta}$$

$$\Rightarrow \quad H = \frac{\dot{a}}{a} \quad \& \quad \mathcal{H} = \frac{a'}{a}$$

$$H = \frac{\mathcal{H}}{a}.$$

Perturbations up to linear order

① Let $A = \bar{A} + \delta A$, $B = \bar{B} + \delta B$

$$\therefore \quad AB = \bar{A}\bar{B} + \bar{A}\,\delta B + \bar{B}\,\delta A + \underbrace{(\text{higher order in } \delta A \,\&\, \delta B)}_{\text{ignore in linear perturbation theory.}}$$

② $\bar{A} + \delta A = \bar{B} + \delta B$

$\Rightarrow \quad \bar{A} = \bar{B}$

$\delta A = \delta B.$

③ $$\frac{1}{1+\delta A} \approx 1 - \delta A$$

Once we have a form for the metric, it is easy to calculate

$$g_{\mu\nu} + \delta g_{\mu\nu} \longrightarrow \Gamma^{\mu}_{\alpha\beta} + \delta\Gamma^{\mu}_{\alpha\beta} \longrightarrow G^{\mu}_{\nu} + \delta G^{\mu}_{\nu}$$

Christoffels

$$\Gamma^{0}_{00} = \mathcal{H} + \Psi'$$

$$\Gamma^{0}_{i0} = \partial_i \Psi$$

$$\Gamma^{i}_{00} = \delta^{ij}\partial_j \Psi$$

$$\Gamma^{0}_{ij} = \mathcal{H}\delta_{ij} - [\Phi' + 2\mathcal{H}(\Phi + \Psi)]\delta_{ij}$$

$$\Gamma^{i}_{j0} = [\mathcal{H} - \Phi']\delta^{i}_{j}$$

$$\Gamma^{i}_{jk} = -2\,\delta^{i}_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l \Phi$$

Einstein Tensor

$$G^{0}_{0} = \frac{3\mathcal{H}^2}{a^2} + \frac{2\nabla^2\Phi}{a^2} - \frac{6\mathcal{H}}{a^2}(\Phi' + \mathcal{H}\Psi) \qquad (\nabla^2 \equiv \delta^{ij}\partial_i\partial_j)$$

$$G^{0}_{i} = -\frac{2}{a^2}\mathcal{H}\partial_i\Psi + \frac{2}{a^2}\partial_i\Phi'$$

$$G^{i}_{j} = \frac{1}{a^2}(\mathcal{H}^2 + 2\mathcal{H}')\delta^{i}_{j} - \frac{1}{a^2}\Big[\nabla^2(\Psi - \Phi) + 2\Phi'' + 2(\mathcal{H}^2 + 2\mathcal{H}')\Psi$$
$$+ 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\Big]\delta^{i}_{j}$$
$$- \frac{1}{a^2}\delta^{ik}\partial_k\partial_j(\Phi - \Psi)$$

Sample calculation

$$\Gamma^0_{00} = \frac{1}{2} g^{0\lambda}\left(2\partial_0 g_{\lambda 0} - \partial_\lambda g_{00}\right)$$

$$= \frac{1}{2}\bar{g}^{0\lambda}\left(2\partial_0 \bar{g}_{\lambda 0} - \partial_\lambda \bar{g}_{00}\right) + \frac{1}{2}\delta g^{0\lambda}\left(2\partial_0 \bar{g}_{\lambda 0} - \partial_\lambda \bar{g}_{00}\right)$$

$$+ \frac{1}{2}\bar{g}^{0\lambda}\left(2\partial_0 \delta g_{\lambda 0} - \partial_\lambda \delta g_{00}\right)$$

$$= \frac{1}{2}\bar{g}^{00}\partial_0 \bar{g}_{00} + \frac{1}{2}\delta g^{00}\partial_0 \bar{g}_{00} + \frac{1}{2}\bar{g}^{00}\partial_0 \delta g_{00}$$

$$= \frac{1}{2a^2} 2aa' + \frac{-4}{a^2} 2aa' + \frac{1}{2}\frac{1}{a^2}\partial_0(a^2 2\Psi)$$

$$= \mathcal{H} \cancel{-2\mathcal{H}\Psi} \cancel{+2\mathcal{H}\Psi} + \Psi'$$

$$= \underbrace{\mathcal{H}}_{\bar{\Gamma}^0_{00}} + \underbrace{\Psi'}_{\delta\Gamma^0_{00}}$$

In the above calculation, we repeatedly used

$\bar{g}_{00} = a^2 \quad \bar{g}_{0i} = 0 \quad \bar{g}_{ij} = -a^2\delta_{ij}$

$\bar{g}^{00} = \frac{1}{a^2} \quad \bar{g}^{0i} = 0 \quad \bar{g}^{ij} = -\frac{1}{a^2}\delta^{ij}$

$\delta g_{00} = 2a^2\Psi \quad \delta g_{ij} = a^2 2\Phi\delta_{ij}$

$\delta g_{0i} = 0$

$\delta g^{00} = -\frac{2}{a^2}\Psi \quad \delta g^{ij} = -\frac{2\Phi\delta^{ij}}{a^2}$

$\delta g^{0i} = 0$

Getting to the Einstein tensor is a bit more involved, but nothing different from the previous calculation.

I recommend that write your own mathematica notebook to calculate these components or learn how to use an existing one.
An example is provided on the CANVAS site for this course.

Perturbed Energy Momentum tensor.

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu$$

where $\bar{T}^\mu{}_\nu = (\bar{\rho}+\bar{p})\bar{u}^\mu \bar{u}_\nu - \bar{p}\,\delta^\mu_\nu$

$$\delta T^\mu_\nu = (\delta\rho + \delta p)\bar{u}^\mu \bar{u}_\nu + (\bar{\rho}+\bar{p})(\delta u^\mu \bar{u}_\nu + \bar{u}^\mu \delta u_\nu) - \delta p\, \delta^\mu_\nu - \underbrace{\Pi^\mu{}_\nu}_{\text{anisotropic stress}} \to 0$$

ignore.

(relevant for neutrinos)

where

$u^\mu = \bar{u}^\mu + \delta u^\mu$ $\qquad$ * $\delta u^\mu \ll \bar{u}^\mu$

$\rho = \bar{\rho} + \delta\rho \equiv \bar{\rho}(1+\delta)$ $\qquad$ * $\frac{\delta\rho}{\bar{\rho}} \equiv \delta \ll 1$.

$P = \bar{P} + \delta P$ $\qquad$ * $\delta p \ll \bar{P}$

It is useful to understand δu^μ.

Recall that $\bar{g}_{\mu\nu}\bar{u}^\mu \bar{u}^\nu = 1 \Rightarrow \bar{u}^0 = \frac{1}{a}$ (assuming

$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu$

$= a^2(\eta)[d\eta^2 - \delta_{ij}dx^i dx^j]$)

$\therefore \quad \bar{u}^\mu = \frac{1}{a}(1, 0)$

Similarly $\quad g_{\mu\nu} u^\mu u^\nu = 1$

$$\Rightarrow \bar{g}_{\mu\nu}\bar{u}^\mu \bar{u}^\nu + \bar{g}_{\mu\nu}(\delta u^\mu \bar{u}^\nu + \bar{u}^\mu \delta u^\nu) + \delta g_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = 1$$

$$\Rightarrow \cancel{1} + 2\bar{g}_{\mu\nu}\delta u^\mu \bar{u}^\nu + \delta g_{00}(\bar{u}^0)^2 = \cancel{1}$$

$$\Rightarrow 2a^2(\delta u^0)\left(\frac{1}{a}\right) + \cancel{a^2}\, 2\Psi \frac{1}{\cancel{a^2}} = 0 \Rightarrow \delta u^0 = -\frac{\Psi}{a}$$

Thus $\quad u^0 = \frac{1}{a}(1-\Psi)$.

We define $\quad u^i = \frac{v^i}{a} \quad$ such that $\quad \boxed{u^\mu = \frac{1}{a}(1-\Psi,\ v^i)}$

Thus $\quad \bar{u}^\mu = \frac{1}{a}(1, 0) \qquad \delta u^\mu = \frac{1}{a}(-\Psi, v^i)$

The components of the energy momentum tensor.

$$T^0{}_0 = \bar{\rho}(1+\delta) \qquad : \delta = \text{density contrast}.$$

$$T^i{}_0 = (\bar{\rho} + \bar{P})v^i$$

$$T^i{}_j = -(\bar{P} + \delta P)\delta^i_j + \cancel{\Pi^i{}_j}^{\,0}$$

Note that for i different species.

$$T^\mu{}_\nu = \sum_I T^{\mu(I)}_\nu$$

Lecture 16

Plan: Perturbed Einstein & conservation equations.

Review:

$$ds^2 = a^2\left[(1+2\Psi)d\eta^2 - \delta_{ij}(1-2\Phi)dx^i dx^j\right]$$
$$= (\bar{g}_{\mu\nu} + \delta g_{\mu\nu})\, dx^\mu dx^\nu$$

$$\underbrace{\bar{g}_{\mu\nu} + \delta g_{\mu\nu}}_{\text{perturbed metric}} \longmapsto \bar{\Gamma}^\alpha_{\mu\nu} + \delta\Gamma^\alpha_{\mu\nu} \longmapsto \underbrace{\bar{G}^\mu{}_\nu + \delta G^\mu{}_\nu}_{\text{Einstein tensor}}$$

$$\underbrace{\bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu}_{\text{Perturbed energy momentum tensor}} = (\bar{\rho}+\bar{p})\bar{u}^\mu \bar{u}_\nu - \bar{p}\,\delta^\mu_\nu$$
$$+ (\delta\rho + \delta p)\bar{u}^\mu \bar{u}_\nu + (\bar{\rho}+\bar{p})(\bar{u}^\mu \delta u_\nu + \delta u^\mu \bar{u}_\nu)$$
$$- \delta p\, \delta^\mu_\nu + \cancel{\Pi^\mu{}_\nu}^{\,0}$$

$$u^\mu = \bar{u}^\mu + \delta u^\mu \qquad \delta u^\mu = \frac{1}{a}(1-\Psi, v^i)$$

$$T^0{}_0 = \bar{\rho}(1+\delta)$$
$$T^i{}_0 = (\bar{\rho}+\bar{p})v^i$$
$$T^i{}_j = -(\bar{p}+\delta p)\delta^i_j + \cancel{\Pi^i{}_j}^{\,0}$$

$$T^\mu{}_\nu = \sum_I T^{\mu\,(I)}_\nu$$

$$\delta\rho = \sum_I \delta\rho_I$$ total density perturbation

$$\delta P = \sum_I \delta P_I$$ total pressure perturbation

$$(\bar{\rho}+\bar{P})v^i = \sum_I (\bar{\rho}_I+\bar{P}_I)\, v^i_I$$ total momentum density perturbation (velocities themselves do not add!).

Q: How do Φ, Ψ, S, δP, q^i etc evolve, & how are they related?

Ans Use

$$\underbrace{G^\mu{}_\nu = 8\pi G\, T^\mu{}_\nu}_{\text{Einstein Eq}} \quad \&/\text{OR}. \quad \underbrace{\nabla_\mu T^\mu{}_\nu = 0}_{\text{conservation equations.}}$$

Some of the equations in the combined set above are redundant. Moreover some are evolution equations, whereas others are constraint equations.

First we will write down the Einstein equations.

Einstein Equations

$$G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}$$

$$\bar{G}^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu} = 8\pi G \left(\bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu} \right)$$

$$\Rightarrow \quad \bar{G}^{\mu}{}_{\nu} = 8\pi G \bar{T}^{\mu}{}_{\nu} \quad \& \quad \delta G^{\mu}{}_{\nu} = 8\pi G \, \delta T^{\mu}{}_{\nu}$$

Consider the 0-0 component.

$$\bar{G}^{0}{}_{0} = 8\pi G \bar{T}^{0}{}_{0} \quad \& \quad \delta G^{0}{}_{0} = 8\pi G \, \delta T^{0}{}_{0}$$

$$\Rightarrow \quad \frac{3\mathcal{H}^2}{a^2} = 8\pi G \, \bar{\rho} \quad \& \quad \frac{2\nabla^2\Phi}{a^2} - \frac{6\mathcal{H}}{a^2}(\Phi' + \mathcal{H}\Psi) = 8\pi G \, \bar{\rho}\delta$$

$$\Rightarrow \quad \mathcal{H}^2 = \frac{8\pi G}{3} a^2 \bar{\rho} \quad \& \quad \nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 \bar{\rho}\delta$$

Similarly, one can get the other Einstein equations for the perturbations.

For $\Pi^{\mu}{}_{\nu} = 0$ (no anisotropic stress), the $i \neq j$ Einstein equation yields $\Phi = \Psi$, which allows us to simplify our equations further

Putting all the Einstein equations together, we have. (with $\Phi = \Psi$ from $i \neq j$ eq).

0-0 $\quad \nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho} \delta$

0-i $\quad \Phi' + \mathcal{H}\Phi = -4\pi G a^2 (\bar{\rho} + \bar{P}) v \qquad v^i = \delta^{ij}\partial_j v$

scalar parts

i=j $\quad \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P$

along with $\quad \mathcal{H}^2 = \dfrac{8\pi G}{3} a^2 \bar{\rho}$

The above Einstein equations for the perturbations will be supplemented by the conservation equations, which we turn to next.

Conservation equations $\quad \nabla_\mu T^\mu{}_\nu = 0$

Two ways to proceed with $\nabla_\mu T^\mu{}_\nu = 0$.

① $\nabla_\mu T^\mu{}_\nu = 0 \Rightarrow \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha = 0$

Use the expressions for $T^\mu{}_\nu$ & $\Gamma^\mu_{\alpha\beta}$ to first order in the perturbations and get to the continuity & Euler equations for the fluid variables.

② Recall that we derived (for a fluid without anisotropic stress) previously. $\nabla_\mu T^\mu{}_\nu = 0$

$$\Rightarrow \boxed{\begin{aligned} &u^\mu \nabla_\mu \rho + (\rho + P) \nabla_\mu u^\mu = 0 \quad \leftarrow \text{Continuity eq} \\ &(\rho + p) u^\mu \nabla_\mu u^\nu = (g^{\mu\nu} - u^\mu u^\nu) \nabla_\mu P \leftarrow \text{Euler eq.} \end{aligned}}$$

We can use these to find the eq. of motion.

We will proceed with ② because it is easier algebraicly.

Let us start with the continuity equation.

$$u^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu u^\mu = 0$$

Now, using $\rho = \bar{\rho}(1+\delta)$, $P = \bar{P} + \delta P$

& $u^\mu = \bar{u}^\mu + \delta u^\mu = \frac{1}{a}(1-\Psi, v^i)$

(upto linear order in pert)

as well as the Christoffel symbols for $\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\lambda} u^\lambda$ we arrive at.

$$\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{P}) = 0$$

← Continuity Eq. Background Level.

$$\delta' + \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)\left(\partial_i v^i - 3\Phi'\right) + 3\mathcal{H}\left(\frac{\delta P}{\bar{\rho}} - \frac{\bar{P}}{\bar{\rho}}\delta\right) = 0$$

Continuity eq. Perturbed level.

Let us do this in a bit more detail.

$$u^\mu \nabla_\mu \rho = u^\mu \partial_\mu \rho \qquad \because \rho \text{ is a scalar}$$

$$= (\bar{u}^\mu + \delta u^\mu)\, \partial_\mu\left(\bar{\rho}(1+\delta)\right)$$

$$= \bar{u}^\mu \partial_\mu \bar{\rho} + \delta u^\mu \partial_\mu \bar{\rho} + \bar{u}^\mu \partial_\mu (\bar{\rho}\delta)$$

$$= \frac{\bar{\rho}'}{a} - \frac{\Psi}{a}\bar{\rho}' + \frac{1}{a}\bar{\rho}'\delta + \frac{\bar{\rho}}{a}\delta'$$

$$= \underbrace{\frac{\bar{\rho}'}{a}}_{0^{th}\text{ order}} + \underbrace{\frac{\bar{\rho}'}{a}(\delta - \Psi) + \frac{\bar{\rho}}{a}\delta'}_{1^{st}\text{ order}}.$$

Now

$$\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\lambda} u^\lambda$$

$$= \partial_\mu \bar{u}^\mu + \bar{\Gamma}^\mu_{\mu\lambda}\bar{u}^\lambda + \partial_\mu \delta u^\mu + \delta\Gamma^\mu_{\mu\lambda}\bar{u}^\lambda$$

$$= (\bar{u}^0)' + \bar{\Gamma}^\mu_{\mu 0}(\bar{u}^0) + \partial_\mu \delta u^\mu + \delta\Gamma^\mu_{\mu 0}(\bar{u}^0)$$

$$= \frac{3\mathcal{H}}{a} + \frac{1}{a}\left[\partial_i v^i - 3(\Phi + 3\mathcal{H}\Psi')\right]$$

$$\Rightarrow (\rho + P)\nabla_\mu u^\mu = \left(\bar{\rho}(1+\delta) + \bar{P} + \delta P\right)(\nabla_\mu u^\mu)$$

$$= \frac{3\mathcal{H}}{a}(\bar{\rho} + \bar{P}) + \frac{3\mathcal{H}}{a}(\bar{\rho}\delta + \delta P) + \frac{\bar{\rho} + \bar{P}}{a}\left[\partial_i v^i - 3(\Phi' + 3\mathcal{H}\Psi)\right]$$

Putting it all together

$$u^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu u^\mu = 0$$

$$\Rightarrow \quad \bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{P}) = 0$$

$$\delta' + \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)\left(\partial_i v^i - 3\Phi'\right) + 3\mathcal{H}\left(\frac{\delta P}{\bar{\rho}} - \frac{\bar{P}\delta}{\bar{\rho}}\right) \quad 0$$

Similarly, we can use $(\rho + P)u^\mu \nabla_\mu u^\nu = (g^{\mu\nu} - u^\mu u^\nu)\nabla_\mu P$ to get

$$v' + \left(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho} + \bar{P}}\right)v = -\frac{\delta P}{\bar{\rho} + \bar{P}} - \Psi$$

where we used $v^i = \delta^{ij}\partial_j v$

Thus for a perfect fluid, the conservation equation $\nabla_\mu T^\mu{}_\nu = 0$ yields.

$$\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{P}) = 0$$

$$\boxed{\begin{aligned}
&\delta' + \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)\left(\nabla^2 v - 3\Phi'\right) + 3\mathcal{H}\left(\frac{\delta P}{\rho} - \frac{\bar{P}}{\bar{\rho}}\delta\right) = 0 \\
&v' + \left(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho} + \bar{P}}\right)v = -\frac{\delta P}{\bar{\rho} + \bar{P}} - \Psi
\end{aligned}}$$

Summary: Scalar perturbations, Newtonian gauge.

$$ds^2 = a^2\left[(1+2\Psi)\,d\eta^2 - (1-2\Phi)\,\delta_{ij}\,dx^i dx^j\right]$$

Einstein Eqns.

$$\Phi = \Psi \qquad i\neq j$$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho}\,\delta \qquad 0\text{-}0$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho}+\bar{P})v \qquad 0\text{-}i$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\,\delta P \qquad i\text{-}i$$

Conservation Equations

$$\delta' + \left(1+\frac{\bar{P}}{\bar{\rho}}\right)\left(\nabla^2 v - 3\Phi'\right) + 3\mathcal{H}\left(\frac{\delta P}{\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta = 0 \qquad \text{cont.}$$

$$v' + \left(\mathcal{H} + \frac{3\bar{P}'}{\bar{\rho}+\bar{P}}\right)v = -\frac{\delta P}{\bar{\rho}+\bar{P}} - \Phi \qquad \text{Euler.}$$

Examples :

(1) Non-relativistic matter only universe : $\rho_{tot} = \rho_m$

Einstein Equations :

$$\mathcal{H}^2 = \frac{8\pi G}{3} \bar{\rho}_m a^2 \qquad e1$$

$$\Phi = \Psi \qquad e2$$

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho}_m \delta_m \qquad e3$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 \bar{\rho}_m v_m \qquad e4$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi \approx 0 \qquad e5$$

Conservation equations :

$$\bar{\rho}_m' = -3\mathcal{H}\bar{\rho}_m \qquad c1$$

$$\delta_m' = -\left(\nabla^2 v_m - 3\Phi'\right) \qquad c2$$

$$v_m' + \mathcal{H} v_m = -\Phi \qquad c3$$

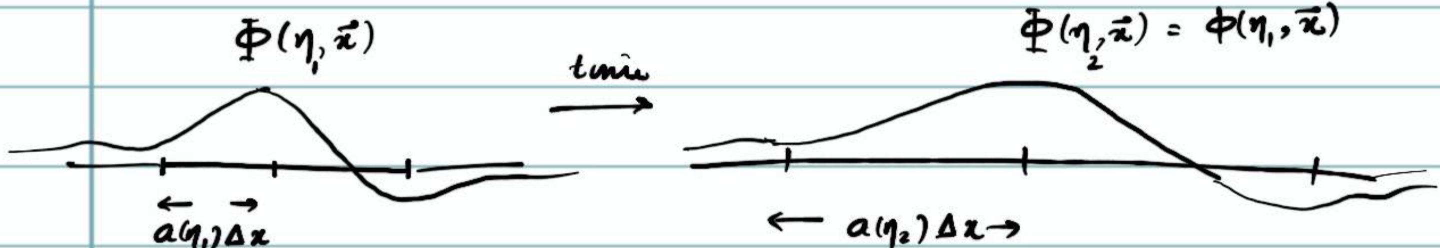
From e1 & c1, we get $\bar{\rho}_m \propto a^{-3}$, $\mathcal{H} = \frac{2}{\eta}$ & $a \propto \eta^2$

Note that $\mathcal{H} = \frac{2}{\eta} \Rightarrow 2\mathcal{H}' + \mathcal{H}^2 = 0$

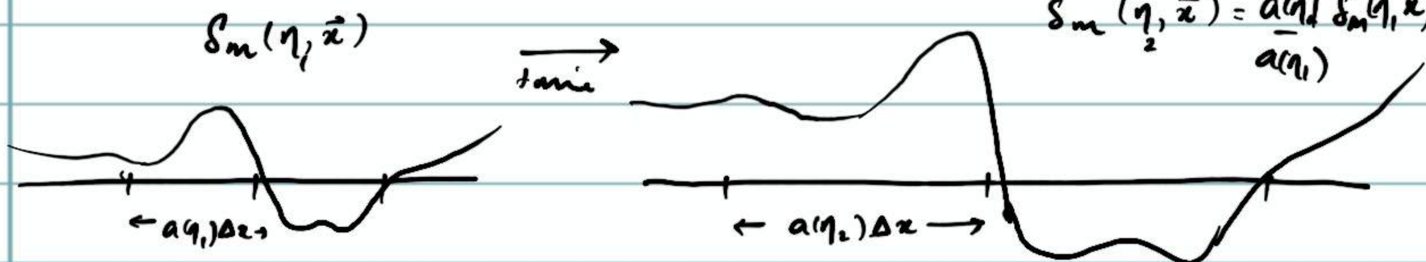
Hence e5 $\Rightarrow \quad \Phi'' + 3\mathcal{H}\Phi' = 0$

$\Rightarrow \quad \Phi = A + \frac{B}{\eta^5} = \text{const} + (\text{decaying mode})$

$\therefore$ $\Phi(\eta, \vec{x}) = \text{const}$ in a matter dominated universe !



Using e3, $a^3 \bar{\rho}_m \delta_m \approx \text{const} \Rightarrow \delta_m(\eta, \vec{x}) \propto a(\eta) \propto \eta^2$



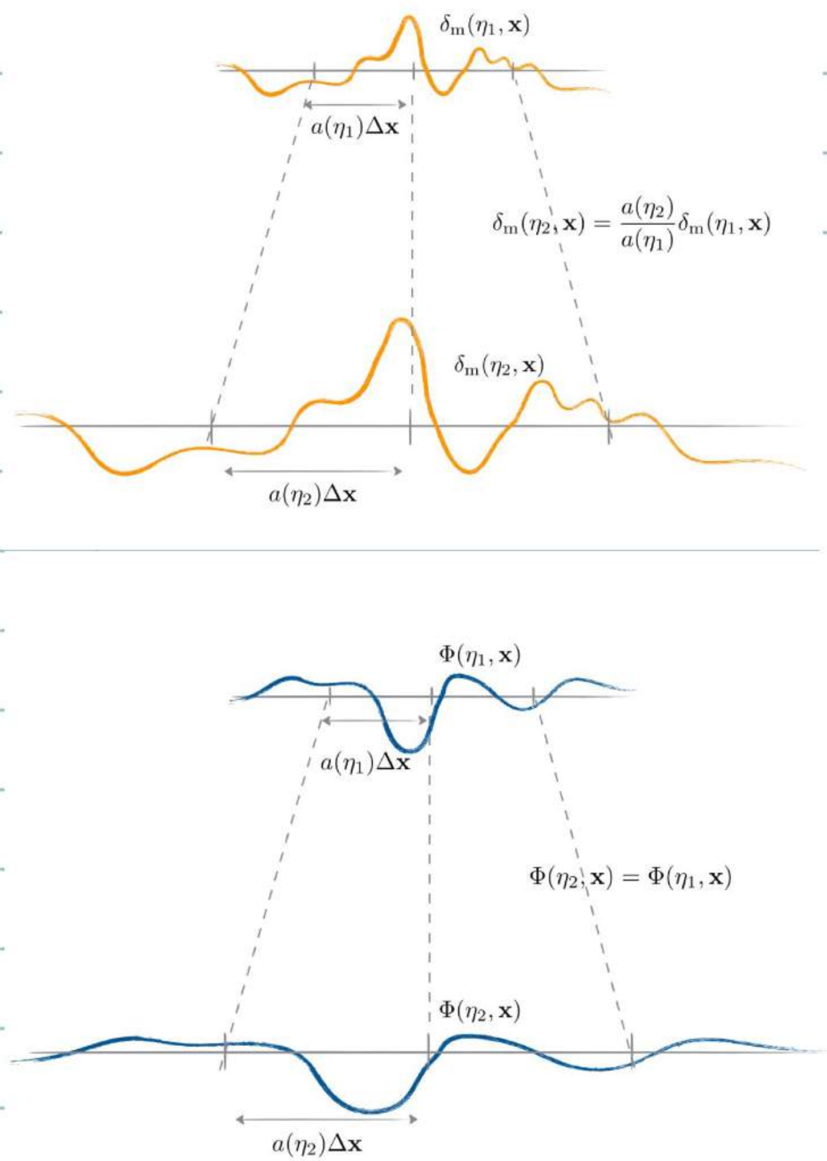
We could also combine c2 & c3 to get

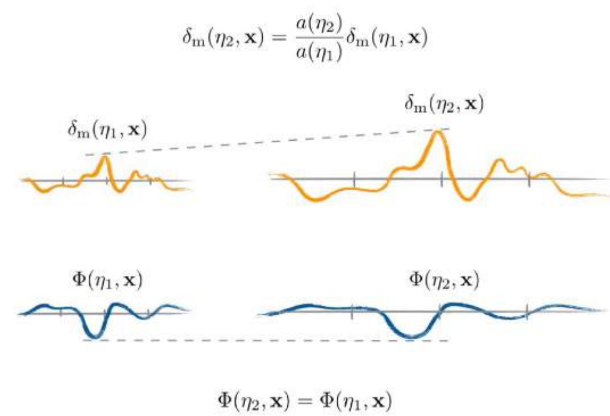
$$\delta_m'' + 2\mathcal{H}\delta_m' + \frac{4\pi G}{3} a^2 \bar{\rho}_m \delta_m = 0$$

$\Rightarrow \quad \delta_m \propto \eta^2 \propto a(\eta)$ as before.

& $\quad v_m \propto \eta \propto a^{1/2}(\eta)$

Same physics, better plots ...





Lecture 17

Plan: Perturbed Einstein & conservation equations.

Summary: Scalar perturbations, Newtonian gauge.

$$ds^2 = a^2\left[(1+2\Psi)\,d\eta^2 - (1-2\Phi)\,\delta_{ij}\,dx^i dx^j\right]$$

Einstein Eqns.

$\Phi = \Psi$ $\qquad i \neq j$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho}\,\delta \qquad \text{0-0}$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho}+\bar{P})\,v \qquad \text{0-i}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\,\delta P \qquad \text{i-i}$$

Conservation Equations

$$\delta' + 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\nabla^2 v - 3\Phi') \qquad \text{cont.}$$

$$v' + 3\mathcal{H}\left(\frac{1}{3} + \frac{\bar{P}'}{\bar{\rho}'}\right)v = -\frac{\delta P}{\bar{\rho}+\bar{P}} - \Phi \qquad \text{Euler.}$$

For a matter only universe $\rho_{tot} = \rho_m$, we use the $i=j$ Einstein equation to get

$$\Phi'' + \frac{6}{\eta}\Phi' = 0$$

$$\Rightarrow \Phi(\eta, \vec{x}) = \text{const.} = f(\vec{x}) .$$ [ignoring the decaying mode]

Initial conditions?

We can combine the two conservation equations

$$\delta_m' = -\nabla^2 v_m + 3\Phi'$$

$$v_m' + \mathcal{H} v_m = -\Phi$$

to get $\delta_m'' + \mathcal{H}\delta_m' = \nabla^2\Phi + 3(\Phi'' + \mathcal{H}\Phi')$

For $\Phi'(\eta, \vec{x}) = 0 \Rightarrow \delta_m'' + \mathcal{H}\delta_m' = \nabla^2\Phi$

$$= 3\mathcal{H}^2\Phi + 4\pi G a^2 \bar{\rho}_m \delta_m$$

Since $\mathcal{H}^2\Phi \sim \dfrac{\Phi}{\eta^2} \longrightarrow 0$ at late times.

$$\therefore \quad \delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2 \bar{\rho}_m \delta_m = 0$$

Ignoring the decaying mode.

$$\delta_m \propto \eta^2 \propto a(\eta).$$

ie $\delta_m(\eta, \vec{x}) \propto a(\eta) g(\vec{x}) \leftarrow$ initial conditions?

Since all the equations are linear, it is useful to move to Fourier space.

$$f_{\vec{k}}(\eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} f(\eta, \vec{x}) \quad \Big| \quad f(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} f_{\vec{k}}(\eta)$$

$\left(k = \frac{2\pi}{\lambda}\right)$ where $\vec{k}, \vec{x}$ are co-moving $\left\{\begin{array}{l} \vec{r}_{phys} = a\vec{x} \\ \vec{k}_{phys} = \frac{\vec{k}}{a} \end{array}\right\}$ The equations in Fourier space are obtained by replacing $\nabla^2 \rightarrow -k^2$. To reduce clutter, I write $f_{\vec{k}}$ as f. (The appearance of k in the equations is an indicator of fourier space)

Fourier Space Equations

$\phi = \psi$

$$-k^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho}\delta$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho} + \bar{P})v$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P$$

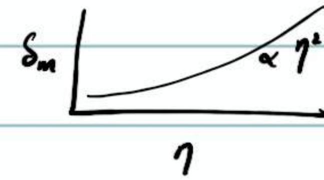
$$\delta' + 3\mathcal{H}\left(\frac{\delta P}{\delta \rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta = \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(k^2 + 3\phi')$$

$$v' + 3\mathcal{H}\left(\frac{1}{3} + \frac{\bar{P}'}{\bar{\rho}'}\right)v = -\frac{\delta P}{\bar{\rho} + \bar{P}} - \Phi$$

For a matter only universe $\rho \approx \rho_{tot}$.

$$\Phi_k(\eta) = \text{const.}$$

$$\delta_{m_k}(\eta) \propto a(\eta)$$



Let us now consider a radiation only universe : $P_r = \frac{1}{3}\rho_r \qquad \frac{\delta P_r}{\delta \rho_r} = \frac{1}{3} \qquad \frac{P_r'}{\rho_r'} = \frac{1}{3}$

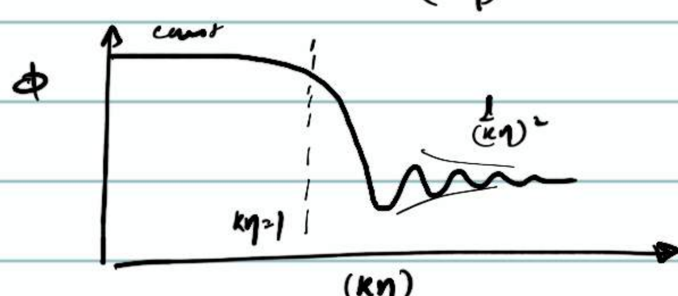
$$a(\eta) \propto \eta \Rightarrow \mathcal{H} = \frac{1}{\eta} \Rightarrow 2\mathcal{H}' + \mathcal{H}^2 = -\mathcal{H}^2 = -\frac{1}{\eta^2}$$

From the 0-0 & i=j Einstein equations, we get

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{1}{3}k^2\Phi = 0$$

$k\eta \ll 1 \Rightarrow \Phi \approx \text{const.}$

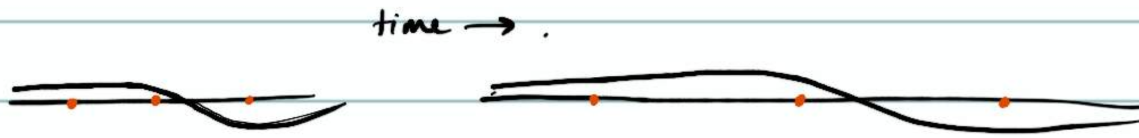
$k\eta \gg 1 \Rightarrow \Phi \propto \dfrac{\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{(k\eta)^2}$



Note the simple behavior for $k\eta \ll 1$

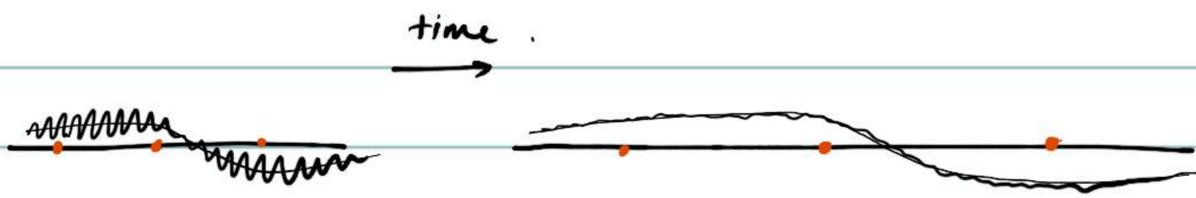
Real space picture

For $\lambda_{phys} \ll H^{-1}$ (ie $k\eta \gg 1$)



For $\lambda_{phys} \gg H^{-1}$ (ie $k\eta \ll 1$)

Combine



"Superhorizon" fluctuations do not evolve.

"Subhorizon" fluctuations decay away

Let us talk about scales a bit more carefully.

Some nomenclature (k = comoving wavenumber).

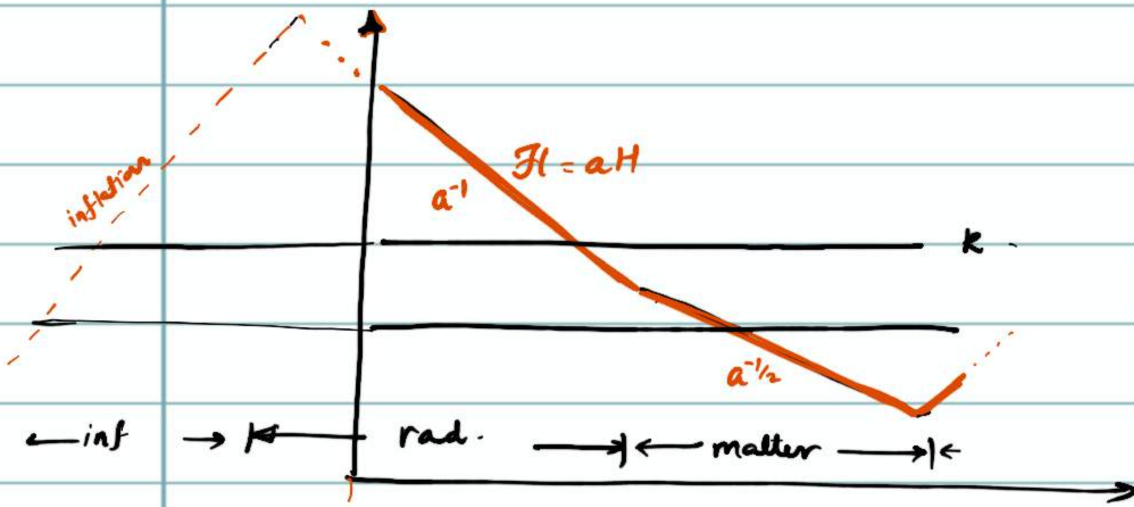
$\frac{k}{\mathcal{H}} > 1$ inside Horizon (subhorizon)

$\frac{k}{\mathcal{H}} < 1$ outside Horizon (superhorizon)

Note $k \gtrless \mathcal{H} \Leftrightarrow \lambda \lessgtr \mathcal{H}^{-1}$ (k, λ comoving).

In terms of physical wavelengths $\lambda_{phys} = \lambda a$

$$\lambda \lessgtr \mathcal{H}^{-1} \Leftrightarrow \lambda_{phys} \lessgtr H^{-1}$$



Note: if $\frac{k}{\mathcal{H}} > 1$ at some time $\Rightarrow \frac{k}{\mathcal{H}} < 1$ in the past. (at least without inflation).

So all perturbations seem to have wavelengths that were outside the horizon or "super horizon" in the past.

On these superhorizon scales, the dynamics tends to be pretty simple.
For example, as we saw earlier $\Phi \approx$ const for $k\eta \sim \frac{k}{\mathcal{H}} \ll 1$.

in a radiation dominated universe.

We can generalize to the case with many different species readily:

Let I = dm, γ, ν, b etc
dark matter ... baryons

$\phi = \psi$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \sum_I \bar{\rho}_I \delta_I$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 \sum_I (\bar{\rho}_I + \bar{P}_I) v_I$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \sum_I \delta P_I$$

$$\delta_I' + 3\mathcal{H}\left(\frac{\delta P_I}{\delta \rho_I} - \frac{\bar{P}_I}{\bar{\rho}_I}\right)\delta_I = -\left(1 + \frac{\bar{P}_I}{\bar{\rho}_I}\right)(\nabla^2 v_I - 3\phi')$$

$$v_I' + 3\mathcal{H}\left(\frac{1}{3} + \frac{\bar{P}_I'}{\bar{\rho}_I'}\right)v_I = -\frac{\delta P_I}{\bar{\rho}_I + \bar{P}_I} - \Phi$$

Note: In the Einstein eq Φ, ψ are sourced by all the perturbations

In the conservation equation each species I responds to the total Φ & ψ.

It is commen to parametize $\delta P_I = c_{s_I}^2 \delta\rho_I$

sound speed

and $\frac{\bar{P}_I}{\bar{\rho}_I} = w_I$ = equation of state parameter.

With these parameters, the Einstein and conservation equations become.

$\phi = \psi$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \sum_I \bar{\rho}_I \delta_I$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 \sum_I (1+w_I)\bar{\rho}_I v_I$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \sum_I c_{s_I}^2 \bar{\rho}_I \delta_I$$

$$\delta_I' + (1+w_I)(\nabla^2 v_I - 3\Phi') + 3\mathcal{H}(c_{s_I}^2\delta_I - w_I)\delta_I = 0$$

$$v_I' + \left[\mathcal{H}(1-3w_I) + \frac{w_I'}{1+w_I}\right] v_I = -\frac{c_{s_I}^2}{1+w_I}\delta_I - \Phi$$

Perturbation variables : δ_I, v_I, ϕ, ψ

Backgroud variables : $\mathcal{H}$, $\bar{\rho}_I$, $\underbrace{w_I, c_{s_I}^2}_{\text{given}}$ $\left[\begin{array}{l} \mathcal{H}^2 = \frac{8\pi G}{3}\sum \bar{\rho}_I \\ \bar{\rho}_I' = -3\mathcal{H}(\bar{\rho}_I + \bar{P}_I). \end{array} \right.$

Solve! (Some eq are redundant).

Example : (non-rel matter + radiation)

* **Matter** : $w_m \approx 0$; $c_{s_m}^2 \approx 0$

$$\Rightarrow \quad \delta_m' = -\left(\nabla^2 v_m - 3\Phi'\right)$$

$$v_m' + \mathcal{H} v_m = -\Phi$$

* **Radiation** : $w_r = \frac{1}{3}$, $c_{s_r}^2 = \frac{1}{3}$

$$\Rightarrow \quad \delta_r' = -\frac{4}{3}\left(\nabla^2 v_r - 3\Phi'\right)$$

$$v_r' = -\frac{1}{3}\delta_r - \Phi$$

conservation equations.

$$\Phi = \Psi$$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2\left(\bar{\rho}_m \delta_m + \bar{\rho}_r \delta_r\right)$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2\left(\rho_m v_m + \frac{4}{3}\rho_r v_r\right)$$

$$\Phi'' - 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi = \frac{4\pi G a^2}{3}\bar{\rho}_r \delta_r .$$

Einstein Equations

Thus it makes sense to set initial conditions for $\lambda_{phys} > H^{-1}$ or equivalently $\frac{k}{\mathcal{H}} \ll 1$. This is what we do next.

Lecture 18

Plan : (1) Initial Conditions for perturbations
(2) Conserved curvature perturbation

Review:

$$ds^2 = a^2\left[(1+2\Psi)d\eta^2 - (1-2\Psi)\delta_{ij}\,dx^i dx^j\right]$$

Metric : scalar perturbations, Newtonian Gauge.

Einstein Equations & Conservation equations
(Fourier Space).

$w_I = \frac{\bar{P}_I}{\bar{\rho}_I}$, $c_{s_I}^2 = \frac{\delta P_I}{\delta\rho_I}$

$$\Phi = \Psi$$

$$k^2\Phi + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 \sum_I \bar{\rho}_I \delta_I$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 \sum_I (1+w_I)\bar{\rho}_I v_I$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \sum_I c_{s_I}^2 \bar{\rho}_I \delta_I$$

$H = \frac{\mathcal{H}}{a}$.

$a \sim \frac{1}{\eta}$

$\therefore \mathcal{H} \propto \frac{1}{\eta}$.

$$\delta_I' - (1+w_I)(k^2 v_I + 3\Phi') + 3\mathcal{H}(c_{s_I}^2 - w_I)\delta_I = 0$$

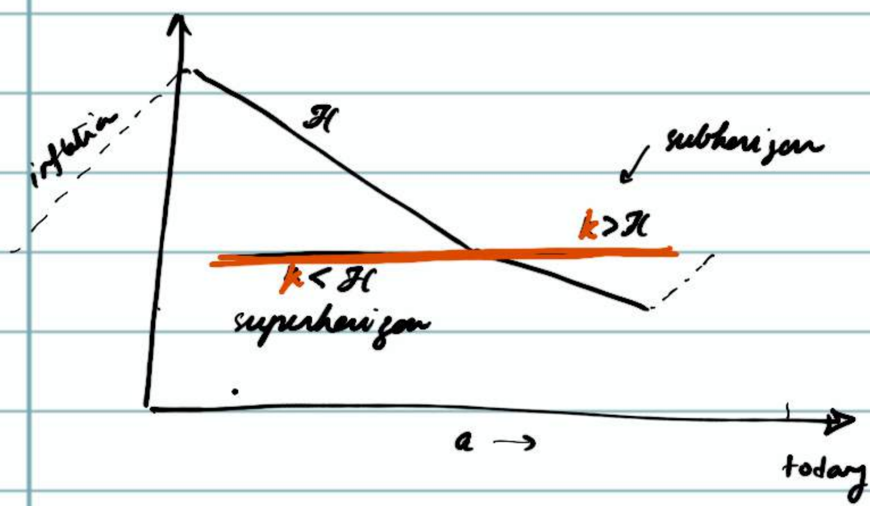
$$v_I' + \left[\mathcal{H}(1-3w_I) + \frac{w_I'}{1+w_I}\right]v_I = -\frac{c_{s_I}^2}{1+w_I}\delta_I - \Phi$$

Perturbation variables : δ_I, v_I, Φ, Ψ

Background variables : $\mathcal{H}$, $\bar{\rho}_I$, $\underbrace{w_I, c_{s_I}^2}_{\text{given}}$

$$\left[\begin{array}{l} \mathcal{H}^2 = \frac{8\pi G}{3}\sum \rho_I \\ \bar{\rho}_I' = -3\mathcal{H}(\bar{\rho}_I + \bar{P}_I). \end{array}\right.$$

Also recall how a given subhorizon mode was likely superhorizon in the past.



To solve the differential equations (Einstein & Conservation) for the perturbations, we need to specify initial conditions. As we can see from the above plot, at early times, many of the wavelengths of the perturbations were superhorizon. So it makes sense to set up initial conditions on superhorizon scales. Moreover, the dynamics of the perturbations are exceptionally simple on superhorizon scales.

Let us focus on Superhorizon Scales for the moment.

There is another extremely useful simplification that arises when we consider "Adiabatic fluctuations" which satisfy the following property.

$$\frac{\delta_I}{1+w_I} = \frac{\delta_J}{1+w_J}$$

For example: $\delta_r = \frac{4}{3}\delta_m$.

This correlation between fluctuations in different species arises if the source of the perturbation is the same streching / squishing of spacetime. The different $\delta_I \neq \delta_J$ arise due to the different responses of the energy densities based on their equation of state.

[For more, see Baumann's Notes].

Note that for adiabatic perturbations

$\sum \bar{\rho}_I \delta_I$ is dominated by the one with the largest $\bar{\rho}_I$.

Thus during radiation domination, from the 0-0 einstein equation (on superhorizon scales),

$$2\Phi_{RD} \approx -\delta_r = -\frac{4}{3}\delta_m .$$

[Recall that we had shown earlier $\Phi = \text{const}$ on $k \ll \mathcal{H}$ during radiation domination].

Thus knowledge about Φ in the early universe is sufficient to yield the initial conditions on δ_r & δ_m etc on superhorizon scales.

*. What sets Φ_{RD}? Ans: Quantum Fluctuations during inflation!

* During matter domination, on superhorizon scales:

$$2\Phi_{MD} \approx -\delta_m$$

What is the relationship between Φ_{MD} & Φ_{RD} on superhorizon scales.

For this consider the i=j Einstein equation

$c_{s_r}^2 = \frac{1}{3}$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{4\pi G a^2}{3} \bar{\rho}_r \delta_r$$

$$= \frac{4\pi G}{3} a^2 \bar{\rho}_r \frac{4}{3} \delta_m \quad \leftarrow \text{adiabatic}$$

From the conservation equation for matter

$$\delta_m' = 3\Phi'$$

$k^2 \to 0$
superhorizon
$w_m = c_{s_m}^2 = 0$

$$\delta_m = 3\Phi + \text{const}$$

Since $\delta_m \to -2\Phi_{MD}$ during matter domination

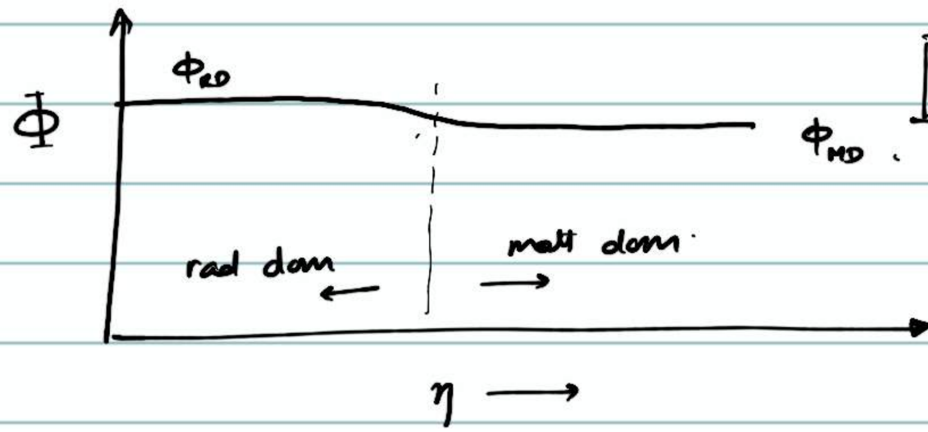
$$\delta_m = 3\Phi - 5\Phi_{MD}$$

$$\therefore \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{4\pi G a^2}{3} \bar{\rho}_r \frac{4}{3}(3\Phi - 5\Phi_{MD})$$

$$\Rightarrow \quad -\mathcal{H}^2 \Phi_{RD} = \frac{4\pi G a^2}{3} \bar{\rho}_r \frac{4}{3}(3\Phi_{RD} - 5\Phi_{MD})$$

$$\Rightarrow \quad \Phi_{RD} = -\frac{2}{3}(\Phi_{RD} - 5\Phi_{MD}) \Rightarrow \boxed{\Phi_{MD} = \frac{9}{10}\Phi_{RD}}$$

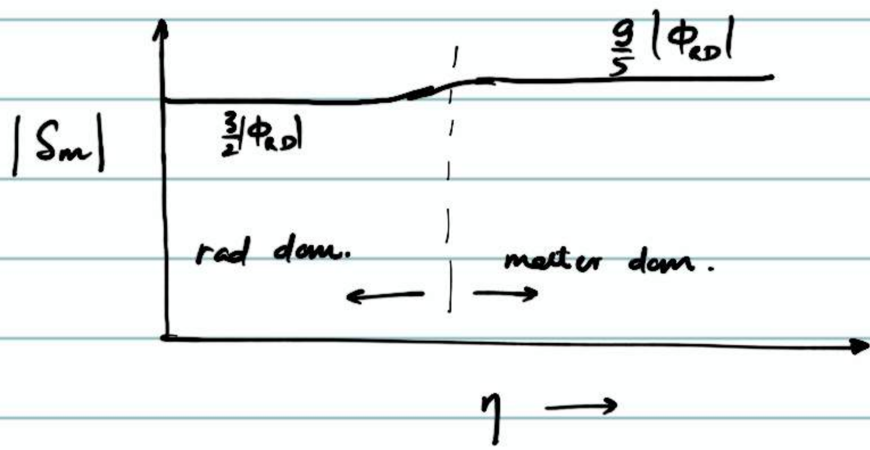
Thus on superhorizon scales

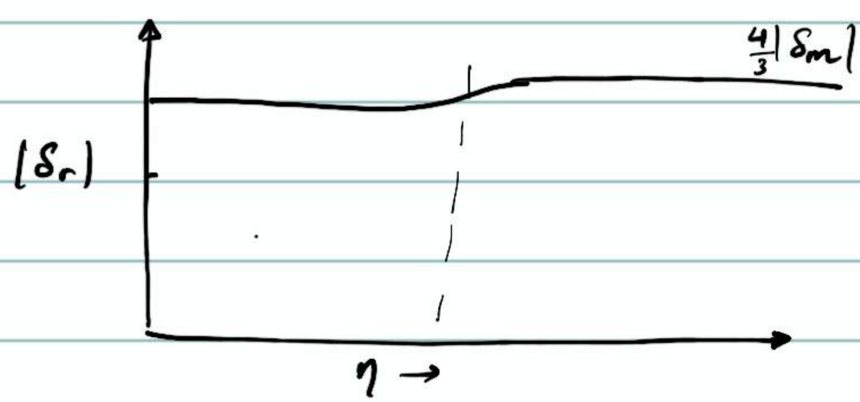


$$\boxed{\Phi_{MD} = \frac{9}{10}\Phi_{RD}.}$$

What about δ_m on Super horizon scales?

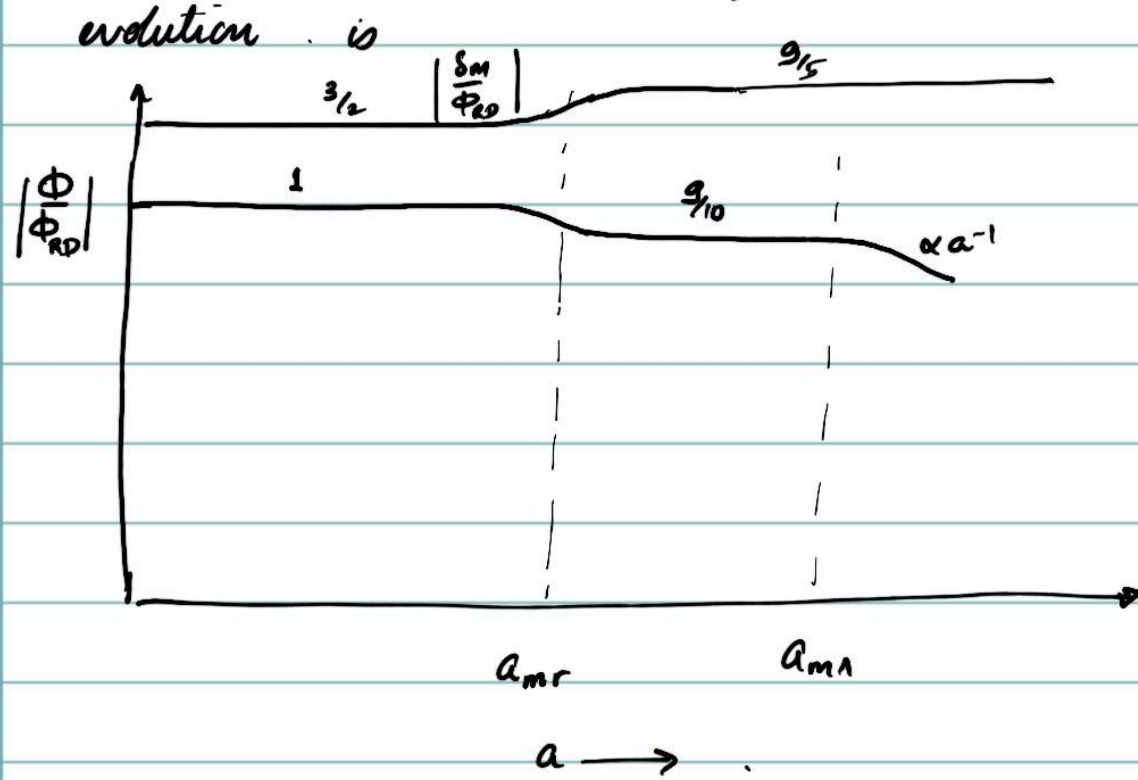
$$\delta_m = 3\Phi - 5\Phi_{MD}.$$





To complete this superhorizon evolution picture, let us also consider dark energy domination. There are no perturbation in DE. Hence simply using the $i=j$ Einstein eq with RHS $=0$, and $\mathcal{H} \propto a$, we get $\Phi \propto a^{-1}$ as a solution

Thus the final result for superhorizon scale evolution is



superhorizon evolution.

$k > \mathcal{H}$.

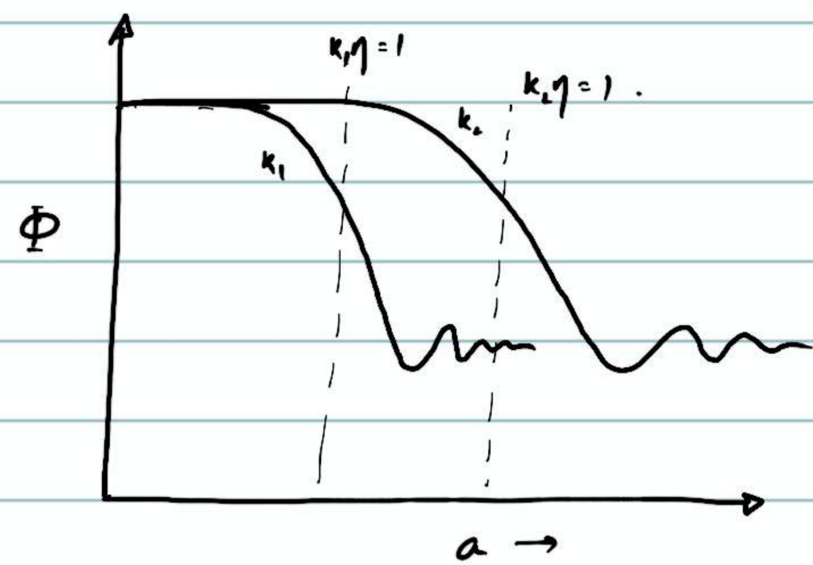
Let us consider subhorizon behaviour now, i.e. after the modes of interest enter the horizon.

Let us first review the behavior of Φ on all scales, i.e. all eras.

Recall that during radiation domination, the 0-0, 0-i Einstein equations can be combined to yield

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{1}{3}k^2\Phi = 0 \Rightarrow \Phi = \Phi_{RD}\left(\frac{\sin x - x\cos x}{x^3}\right) \quad x = \frac{k\eta}{\sqrt{3}}$$

↑ valid on super & subhorizon scales



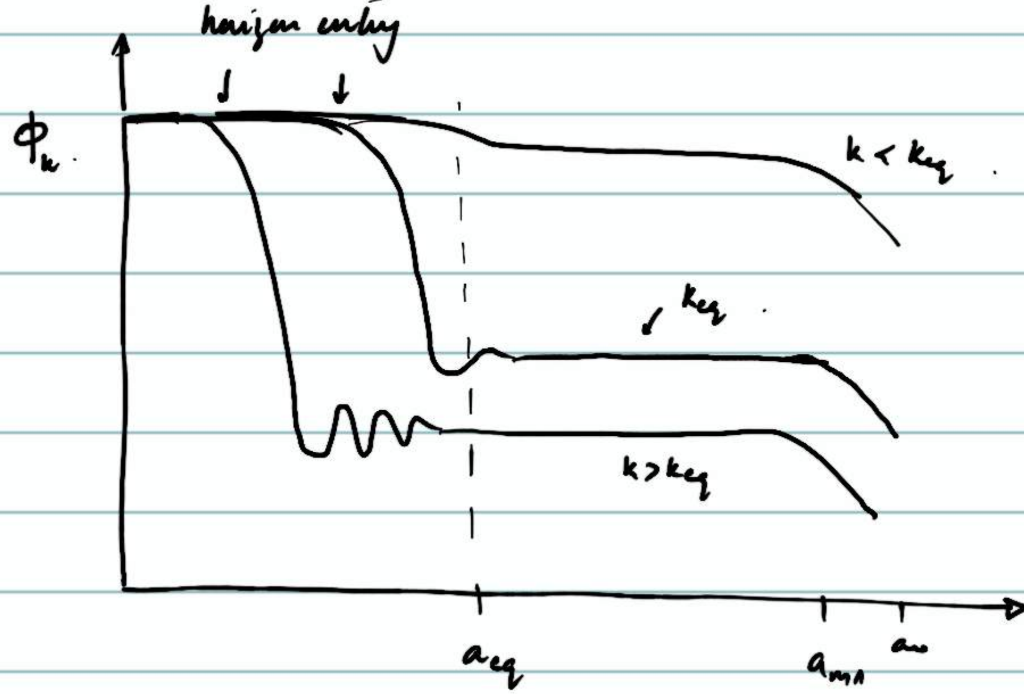
$a \propto \eta$.

During matter domination: Φ = const. (all scales).

During Λ domination $\Phi \propto a^{-1}$

Thus the general behavior of the gravitational potential on super & sub-horizon scales

is



$k_{eq} = \mathcal{H}_{eq}$.

Dark matter perturbations δ_m

From the conservation equations.

$$\delta_m' = k^2 v_m + 3\Phi'$$
$$v_m' + \mathcal{H} v_m = -\Phi$$

These can be combined to yield

$$\delta_m'' + \mathcal{H}\delta_m' = 3\Phi'' + 3\mathcal{H}\Phi' - k^2\Phi .$$

We have already solved for the behavior of the potential earlier, so let us use that knowledge to understand the behavior of δ_m

a) Radiation domination (subhorizon scales)

During RD & on sub-horizon scales. $(k\tau \gg 1)$

$$\Phi \sim \frac{\cos x}{x^2} \qquad x = \frac{k\eta}{\sqrt{3}}$$

$\Rightarrow$ The RHS of the δ_m'' eq then yields. $(x \gg 1)$

$$3\Phi'' \sim \frac{\cos x}{x^2} \qquad 3\mathcal{H}\Phi' \sim \frac{\sin x}{x^2} \qquad \Phi \sim \frac{\cos x}{x^2}$$

The contribution is oscillatory & decaying.

Ignoring this contribution [really there is a small ~~constant~~ piece from matter Φ]

$$\delta_m'' + \mathcal{H}\delta_m' \approx 0 \qquad \text{with } \mathcal{H} \sim \frac{1}{\eta} \text{ for RD}$$

$$\delta_m \approx \text{const} + \ln \tau$$
$$\approx \text{const} + \ln a \qquad a \propto \eta \text{ during RD}$$

Thus, during RD, $\boxed{\delta_m \propto \ln a}$ $k\tau \gg 1$.

b) matter domination

Here Φ = const

$$\Rightarrow \quad \delta_m'' + \mathcal{H}\delta_m' = -k^2\Phi$$

We also know from the 00 einstein eq Φ During MD,

$$-k^2\Phi - 3\mathcal{H}^2\Phi = 4\pi G a^2 \bar{\rho}_m \delta_m$$

$$\Rightarrow \quad \Phi = -\frac{4\pi G a^2 \bar{\rho}_m \delta_m}{\mathcal{H}^2\left(\frac{k^2}{\mathcal{H}^2}+3\right)} \simeq -\frac{3}{2}\frac{\delta_m}{\left(\frac{k^2}{\mathcal{H}^2}+3\right)}$$

For subhorizon scales $k^2\Phi \approx -\frac{3}{2}\mathcal{H}^2\delta_m$

$$\therefore \quad \delta_m'' + \mathcal{H}\delta_m' = \frac{3}{2}\mathcal{H}^2\delta_m$$

For MD, $\mathcal{H} \sim \frac{2}{\eta}$

$$\therefore \quad \delta_m'' + \frac{2}{\eta}\delta_m' = \frac{3}{2}\left(\frac{4}{\eta^2}\right)\delta_m$$

$$\therefore \quad \delta_m \approx c_1\eta^2 + \frac{c_2}{\eta^3}$$

since $a \sim \eta^2$ during MD

$$\propto a$$

Thus $\boxed{\delta_m \propto a}$ during MD on subhorizon scales $k\eta \gg 1$.

* Number

2) <u>Radiation perturbations</u> δ_r (really the fluid is radiation + baryons)

a) (Radiation domination).

Using the 00 einstein equation

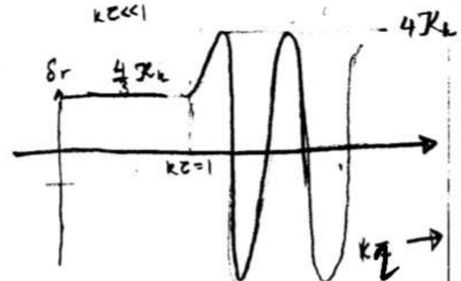
$$\delta_r = -\frac{2}{3}(k\tau)^2\Phi_k - 2\tau\Phi_k' - 2\Phi_k .$$

Using our solution for the potential Φ derived earlier.

$$\delta_r \longrightarrow -2\Phi = \frac{4}{3}\mathcal{R}_k \qquad k\eta \ll 1$$

$$\longrightarrow -\frac{2}{3}(k\eta)^2\Phi_k \qquad k\eta \gg 1$$

$$= 4R_k\cos\left(\frac{k\eta}{\sqrt{3}}\right) \quad \leftarrow$$



[Another way of arriving at this same conclusion is by using the continuity & euler equations.

For radiation perturbations in a radiation dominated universe

$$\delta_r' = \frac{4}{3}\left(k^2 v_r + 3\Phi_k'\right) \Rightarrow \delta_r'' = \frac{4}{3}\left(k^2 v_r' + 3\Phi_k''\right)$$

$$v_r' = -\frac{1}{4}\delta_r - \Phi_k$$

$$= \frac{4}{3}\left[k^2\left(-\frac{1}{4}\delta_r - \Phi_k\right) + 3\Phi_k''\right]$$

$$= -\frac{k^2}{3}\delta_r - \frac{4}{3}k^2\Phi + 4\Phi_k''$$

$$\therefore \delta_r'' + \frac{k^2}{3}\delta_r = -\frac{4}{3}k^2\Phi_k + 4\Phi_k''$$

One can easily see from here that, when Φ_k has decayed (deep inside the Horizon $k\eta \gg 1$,

$$\delta_r'' + \frac{k^2}{3}\delta_r \approx 0 \quad \Rightarrow \delta_r \propto \cos\left(\frac{k\eta}{\sqrt{3}}\right) \quad \text{as we found earlier.}$$]

b) Radiation perturbations during matter domination.

From the continuity eq for

$$\delta_r'' + \frac{k^2}{3}\delta_r = -\frac{4}{3}k^2\Phi + 4\Phi_k''$$

[Note the continuity equation & euler eq for δ_r during radiation and matter domination is the same]

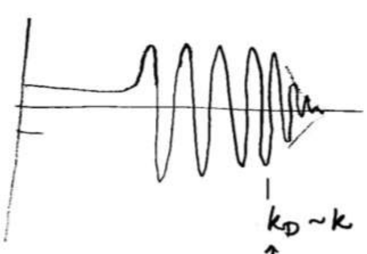
During matter domination $\Phi_k = \text{const}$. Hence $\Phi_k'' \to 0$.

$$\delta_r'' + \frac{k^2}{3}\delta_r = -\frac{4}{3}k^2\Phi_k$$

which has a solution $\delta_r = -4\Phi_k + \text{oscillations in} \left(\frac{k\tau}{\sqrt{3}}\right)$

[In reality we cannot treat radiation as a fluid after decoupling. Hence this is only valid untill decoupling]

[Additional complication: Even before decoupling, the coupling between photons and baryons is never perfect. At small enough scales, δ_r decays exponentially because the photons "diffuse out" of the perturbation. This is called silk damping. 15



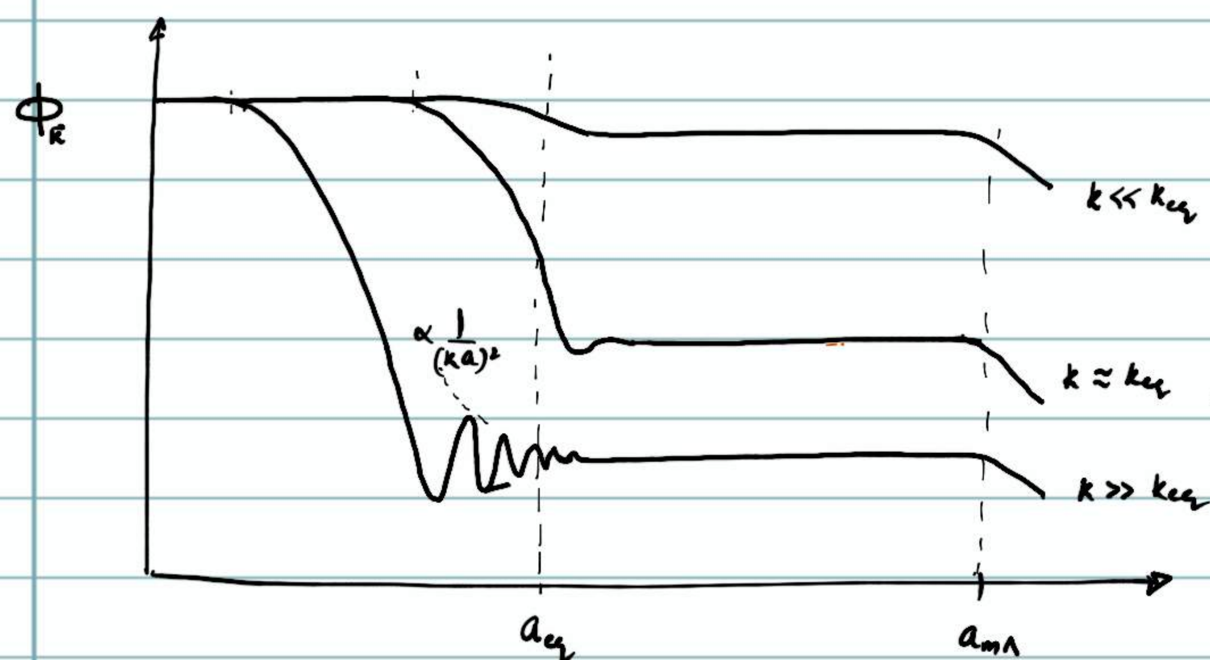
silk damping scale, function of time]

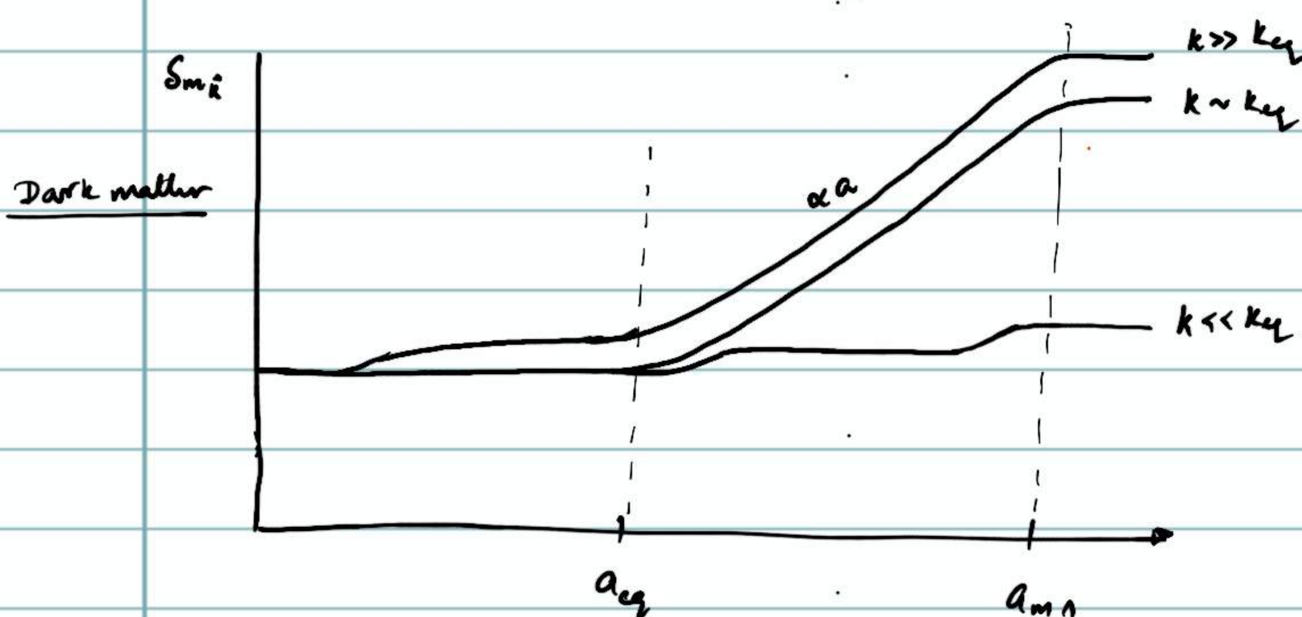
Lecture 19

Plan : 1) Statistics – 2 pt correlation function

2) Matter Power spectrum .

Review :





Photons and Baryons

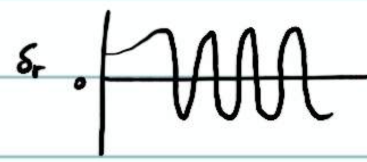
For photons

$$\delta_r'' + \frac{k^2}{3}\delta_r = -\frac{4}{3}k^2\Phi + 4\Phi''$$

subhorizon.

During radiation domination $\quad \Phi \sim \dfrac{e^{ik\eta/\sqrt{3}}}{(k\eta)^2}$ for $k\eta \gg 1$

$$\Rightarrow \quad \delta_r'' + \frac{k^2}{3}\delta_r \approx 0 \qquad \Rightarrow$$



$$\Rightarrow \quad \delta_r \sim e^{ik\eta/\sqrt{3}}$$

During matter domination $\quad \Phi \simeq$ constant.

$$\Rightarrow \quad \delta_r'' + \frac{k^2}{3}\delta_r = -\frac{4}{3}k^2\Phi \simeq \text{const}$$

$\Rightarrow$ δ_r oscillates around a constant different from zero.
$= -4\Phi_0$

What about baryons?

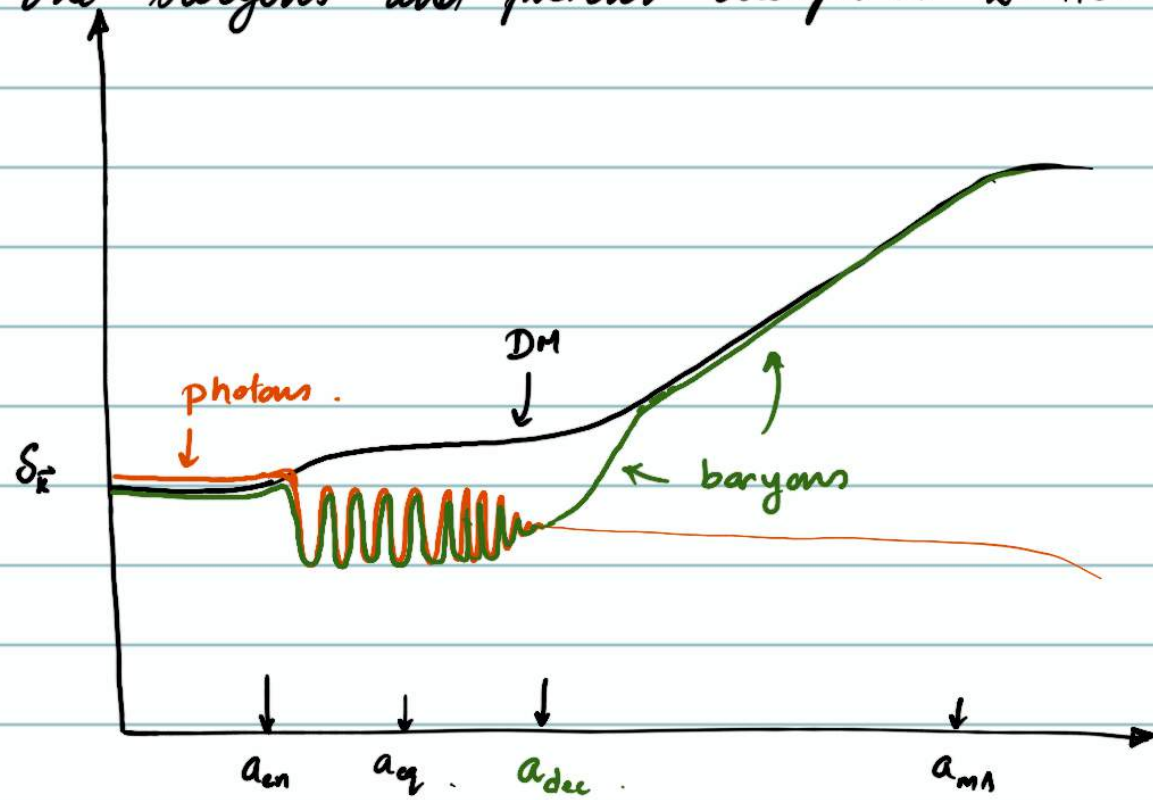
- coupled to the photons. They modify the photon equation.

$$\delta_r'' + \frac{R}{1+R}\delta_r' + c_s^2 k^2 \delta_r = -\frac{4}{3}k^2\Phi + 4\Phi'' + \frac{4R'}{1+R}\Phi.$$

where $\quad R = \dfrac{3}{4}\dfrac{\bar{\rho}_b}{\bar{\rho}_r} \qquad c_s^2 = \dfrac{1}{3(1+R)}.$

δ_r oscillates around $-4(1+R)\Phi_0$ and has damping included in it. Oscillations have a frequency $\frac{k}{3(1+R)}$.

The baryons and photons decouple at $z \approx 1100$.

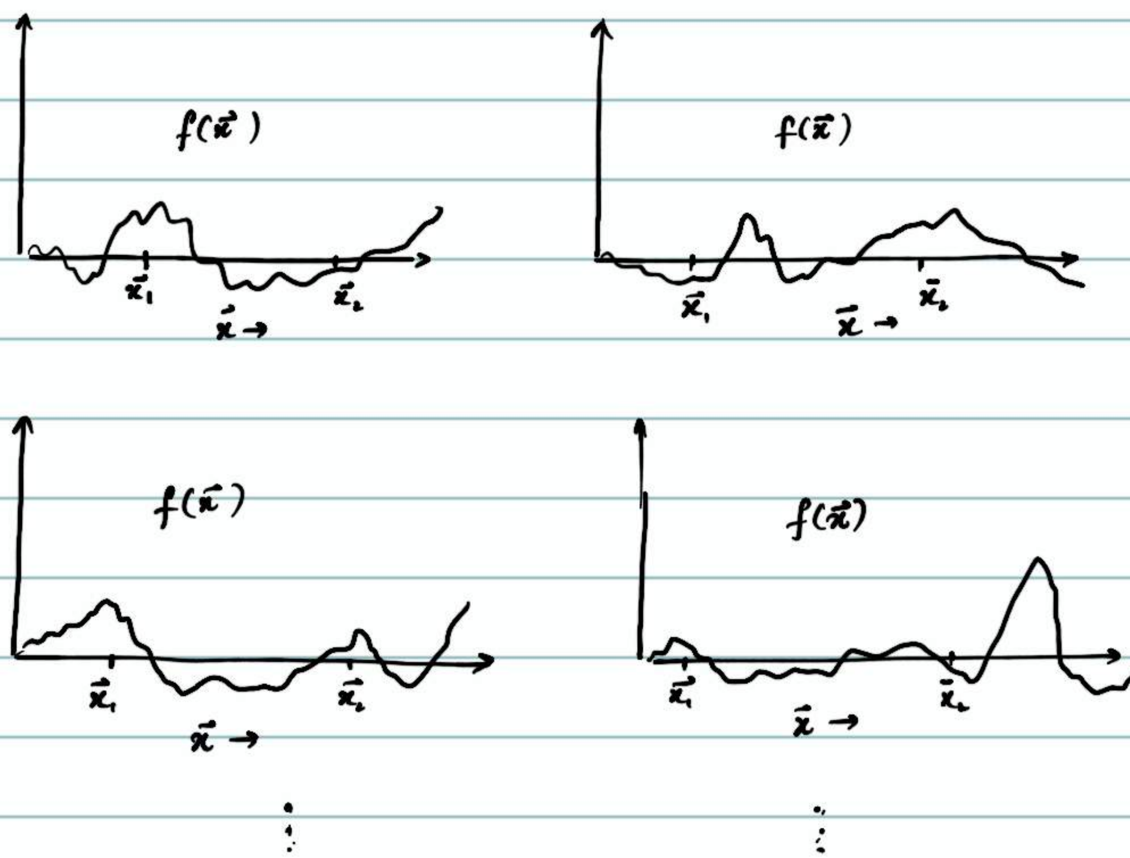


Before a_{dec}: baryon & photon together

After a_{dec}: baryons follow DM.

Statistical Description of Fields.

Let $f(\vec{x})$ be drawn from an ensemble.



The two-point auto-correlation function

$$\xi_f(\vec{x}_1, \vec{x}_2) \equiv \langle f(\vec{x}_1) f(\vec{x}_2) \rangle$$ average over the ensemble.

If the ensemble is generated by a process that is statistically homogeneous & isotropic.

Statistical Homogeneity & isotropy

$\Rightarrow \quad \xi_f(\vec{x}_1, \vec{x}_2) = \xi_f(|\vec{x}_1 - \vec{x}_2|) = \xi_f(r)$

(We will restrict ourselves to such cases) where $\vec{r} = \vec{x}_1 - \vec{x}_2$

The Power spectrum of f : $P_f(k)$ is defined as the Fourier transform of the 2-point correlation function

$$P_f(k) \equiv \int d^3r \, e^{i\vec{k}\cdot\vec{r}} \xi_f(r) \quad \& \quad \xi_f(r) = \int \frac{d^3k}{(2\pi)^3} P_f(k) e^{-i\vec{k}\cdot\vec{r}}$$

Power spectrum

Note that for $f_{\vec{k}} = \int d^3x \, e^{i\vec{k}\cdot x} f(\vec{x})$

$$\langle f_{\vec{k}_1} f_{\vec{k}_2} \rangle = (2\pi)^3 \delta(\vec{k}_1 - \vec{k}_2) P_f(k) .$$

Result of $\langle f(\vec{x}) f(\vec{y}) \rangle = \xi_f(|\vec{x} - \vec{y}|)$

Statistical Hom. & Isotropy.

Note that $P_f(k) \propto |f_{\vec{k}}|^2$

Note that

$$\xi_f(0) = \langle f^2(\vec{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} \, P_f(k)$$

$$= \int d\ln k \left[\frac{k^3}{2\pi^2} P_f(k) \right]$$

Thus $\frac{k^3 P_f(k)}{2\pi^2}$ is the contribution to the variance of a field at a point per logarithmic interval in wavenumber.

$\frac{k^3 P_f(k)}{2\pi^2} = \text{const} \Rightarrow$ equal contribution per log k interval.

$\equiv$ scale invariant power spectrum.

[Note: inflation typically generates an almost scale-invariant spectrum]

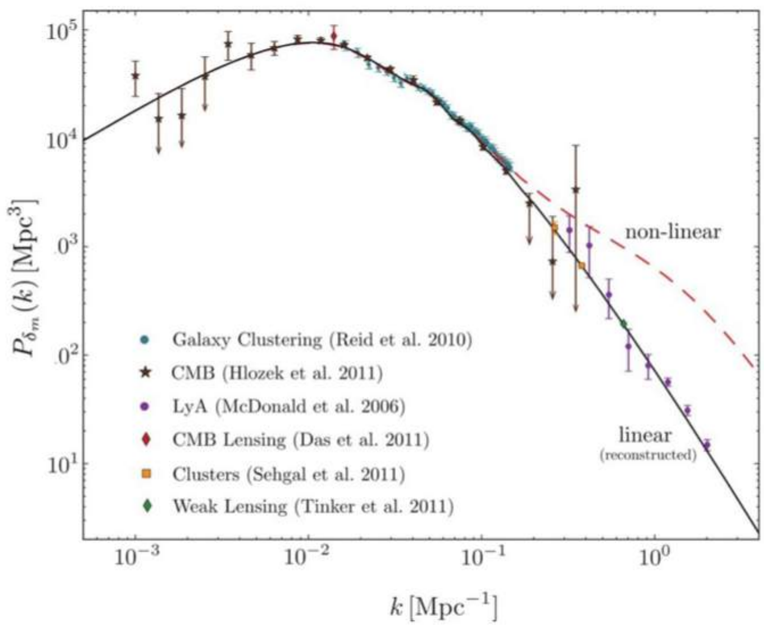
<u>Question</u>: Why statistical description?

<u>Ans</u>: We do not anticipate being able to predict the exact functional form of the perturbations. We expect to predict the statistical properties of the perturbations and compare those to the observed ones!

<u>Question</u>: Theoretical Calculations based on an ensemble, but observationally we only measure one member of the ensemble. How can we relate the two?

<u>Ans</u>: In some cases, we can divide the universe into smaller subsystems and treat each one as a member of the ensemble. Thus ensemble avg. can be compared to avg. of subsystems.

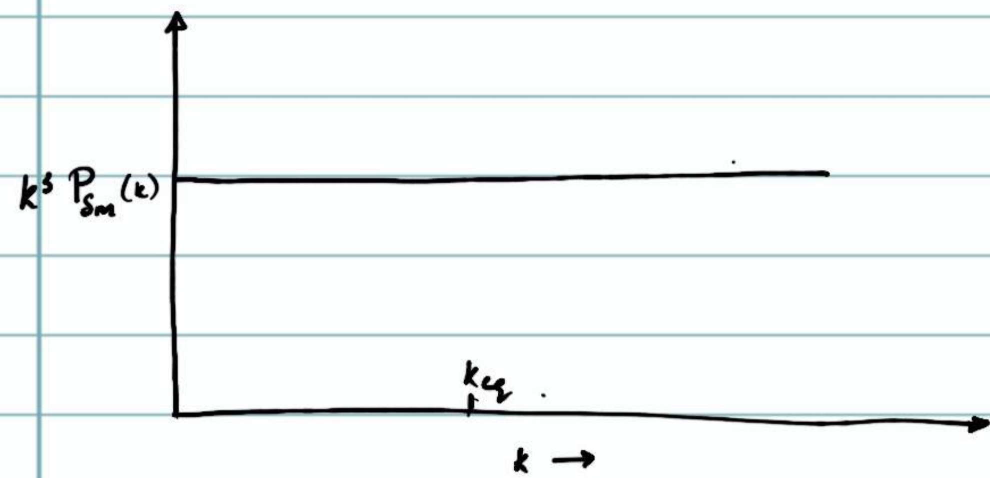
Let us understand the following power spectrum : $P_{\delta_m}(k)$. [ie. $f = \delta_m$]



We will explain the above shape using our understanding of $\delta_{m_k}(a)$ and scale invariant initial conditions.

Dark matter power spectrum.

1. At $t = t_1$, $k \ll \mathcal{H}_1$ for all modes on the plot.

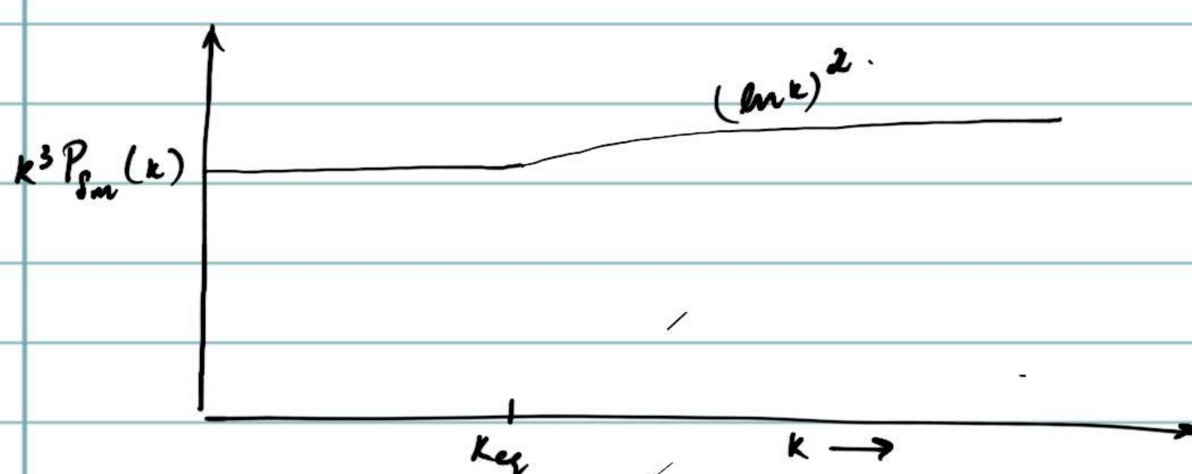


2. At $t = t_{eq}$, $k > \mathcal{H}_{eq}$ modes have been growing as $\propto \ln\left(\frac{a_{eq}}{a_{en}(k)}\right)^2$ where $k = \mathcal{H}_{en} = \mathcal{H}(a_{en}) \propto a_{en}^{-1}$

$$\Rightarrow a_{en} \propto k^{-1}.$$

Hence $k^3 P_{\delta m}(k) \propto (\ln k)^2$ for $k > k_{eq}$.

For $k < k_{eq}$, there is not much evolution (apart from a small constant shift).



(ignore Λ).

3. For $t = t_0 > t_{eq}$, all subhorizon modes grow $\propto a$. The amount by which modes will have grown depends on when they became subhorizon.

For $k < k_{eq}$ $\quad k^{3/2}\delta_m \propto \left(\dfrac{a_0}{a_{en}}\right) \propto k^2$

$$\because \quad k = \mathcal{H}(a_{en}) \propto a_{en}^{-1/2}$$

$$\Rightarrow \quad a_{en} \propto k^{-2}.$$

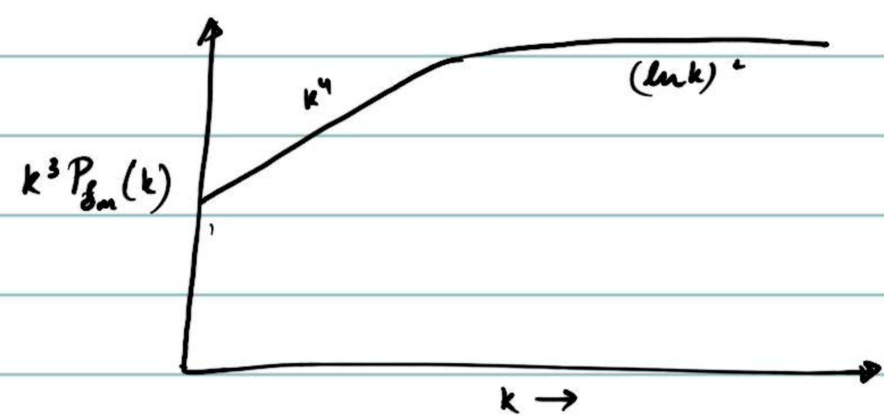
$aH = \frac{1}{a}$.

For $k > k_{eq}$ $\quad k^{3/2}\delta_m \propto \left(\dfrac{a_0}{a_{eq}}\right) \cdot \ln\left(\dfrac{a_{eq}}{a_{en}}\right) \propto \ln k$.

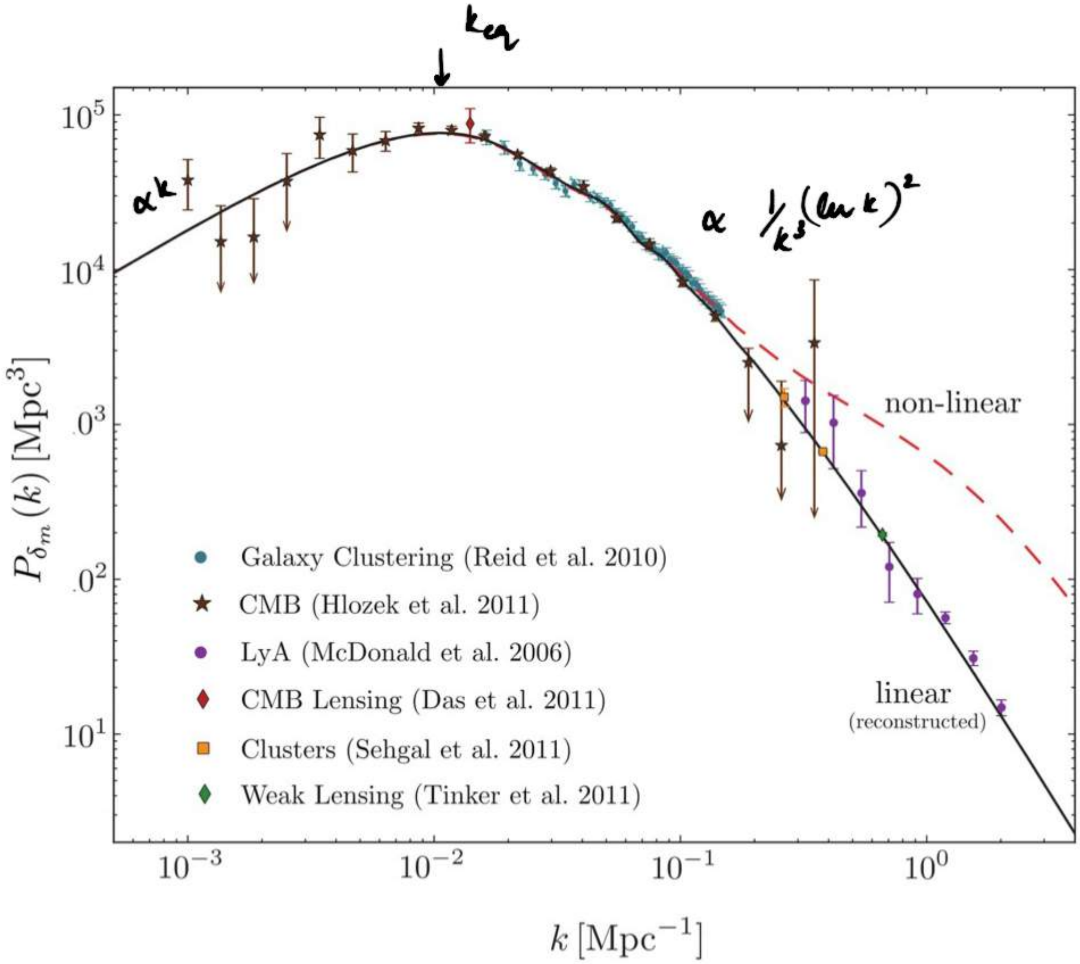
$$\because \quad k = \mathcal{H}(a_{en}) \propto a_{en}^{-1}$$

$$\Rightarrow \quad a_{en} \propto k^{-1}$$

Thus $\quad \dfrac{k^3 P_{\delta_m}(k)}{2\pi^2} \propto \begin{cases} k^4 & k < k_{eq} \\ (\ln k)^2 & k > k_{eq} \end{cases}$



This explains the shape of the power spectrum

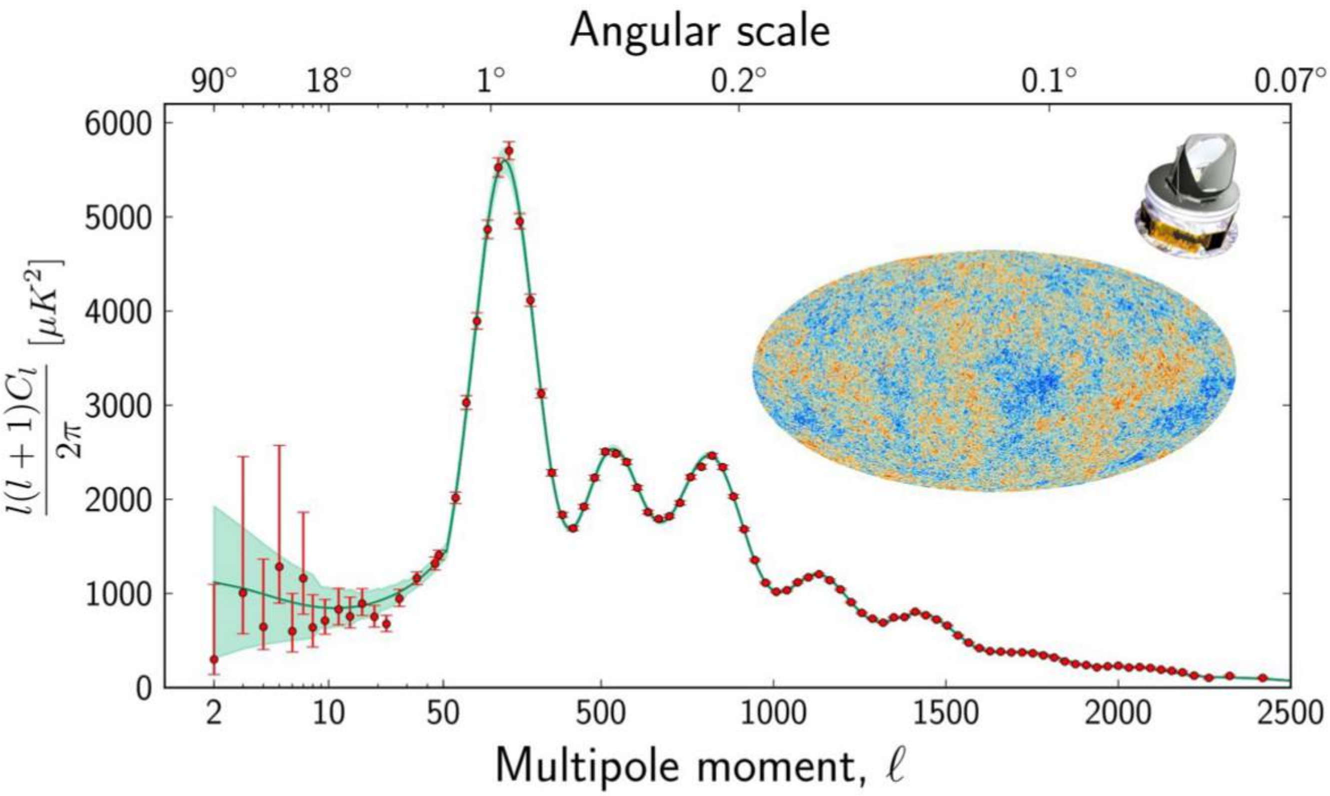


For nonlinear evolution, more work needed.
For simulations, see for example.

www.slac.stanford.edu/~kaehler/homepage/visualizations/
dark-matter.html.

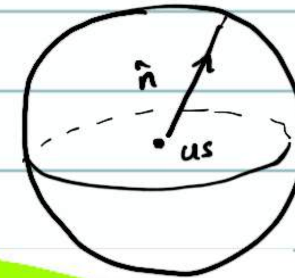
Lecture 20

Another well known power spectrum.



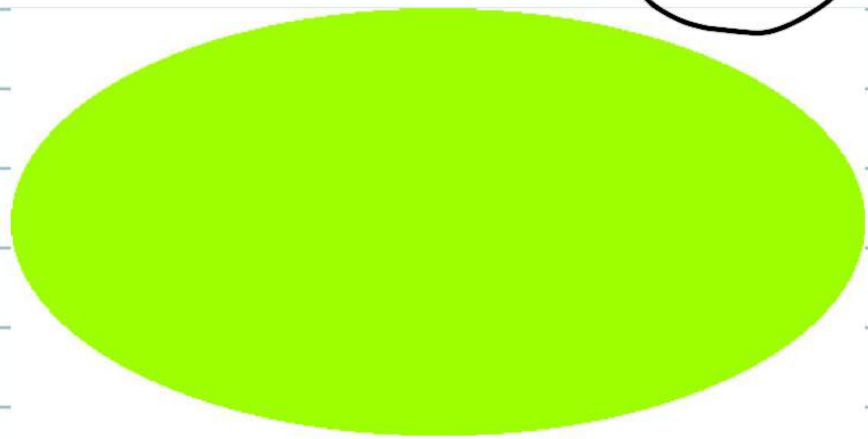
Can we understand this?

The Cosmic Microwave Background.



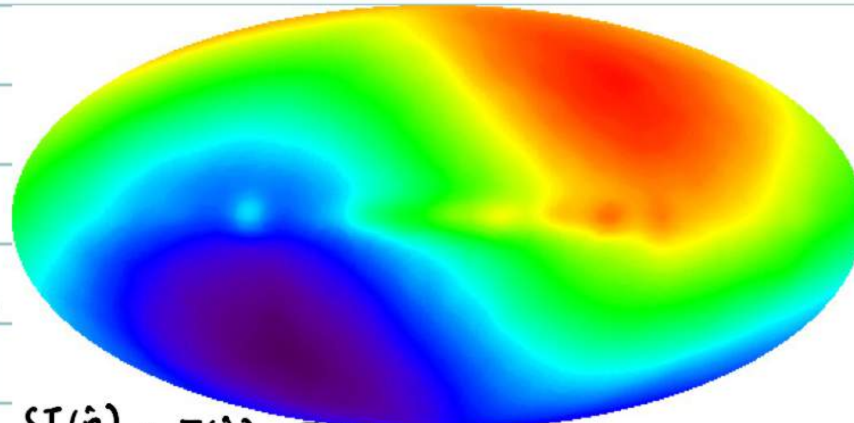
(0) At zeroth order the temperature of the CMB is uniform.

$\bar{T} = 2.73$ K



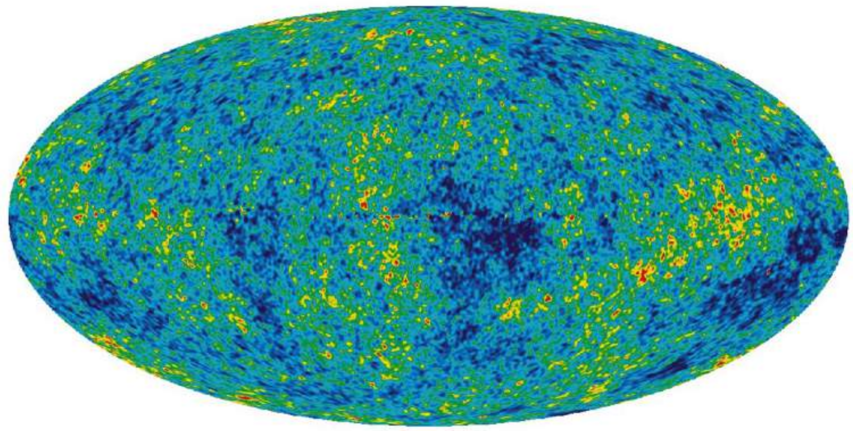
Dipole. Solar system motion leads to a doppler shift in the photon momentum which yields a yin-yang pattern of temperature anisotropy

$v \approx 370$ km s^{-1}



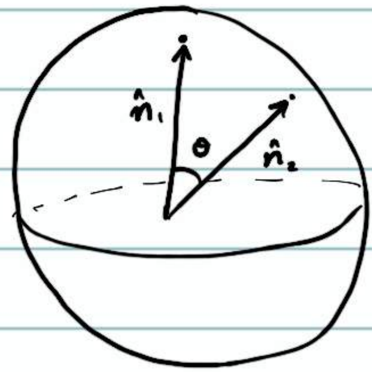
$$\frac{\delta T(\hat{n})}{T} = \frac{T(\hat{n}) - \bar{T}}{T} = \hat{n}\cdot\vec{v}$$

Primordial: After removing the dipole, the following is left on the sky.



(galaxy also removed).

Statistical description of fields on a sphere.



2 pt angular correlation function.

$$\langle f(\hat{n}_1) f(\hat{n}_2) \rangle \equiv C(\hat{n}_1, \hat{n}_2)$$

$$= C(\hat{n}_1 \cdot \hat{n}_2) \quad \leftarrow \text{statistical isotropy.}$$

$$= C(\theta)$$

C_ℓ = "Fourier transform" of $C(\theta)$ in angular space.

$$C(\theta) = \sum_\ell \frac{(2\ell+1)}{4\pi} C_\ell \, P_\ell(\cos\theta) \qquad \hat{n}_1 \cdot \hat{n}_2 = \cos\theta.$$

Legendre polynomials.

$$C_\ell = 2\pi \int_{-1}^{1} d(\cos\theta) \, P_\ell(\cos\theta) \, C(\theta).$$

extra

Note

$$f(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\hat{n})$$

$$f_{\ell m} = \int d\hat{n}\, f(\hat{n})\, Y^*_{\ell m}(\hat{n})$$

$$\langle f(\hat{n}_1) f(\hat{n}_2) \rangle \underset{\substack{\uparrow \\ \text{isotropy}}}{=} C(\theta) \qquad \Rightarrow \langle f_{\ell m} f_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell .$$

$\hat{n}_1 \cdot \hat{n}_2 = \cos\theta$.

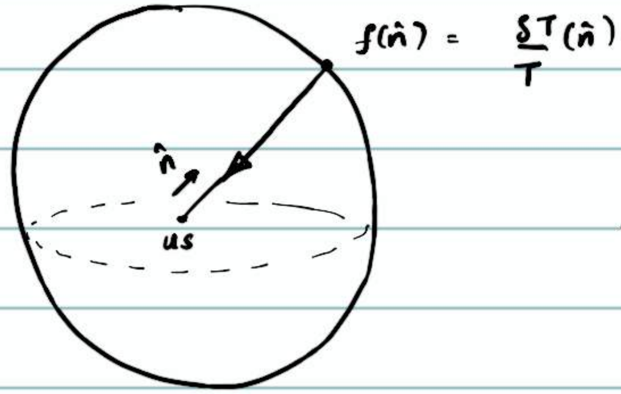
[Compare with $\langle f(\vec{x}_1) f(\vec{x}_2) \rangle \underset{\substack{\uparrow \\ \text{Hom + isotropy}}}{=} \xi_f(|\vec{x}_1 - \vec{x}_2|) \Rightarrow \langle f_{\vec{k}_1} f^*_{\vec{k}_2} \rangle$

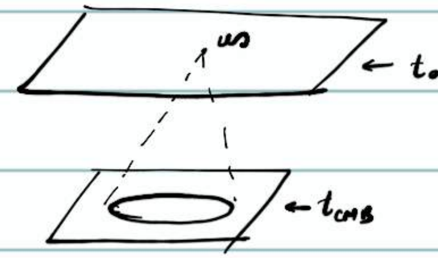
$= (2\pi)^3 \delta(\vec{k}_1 - \vec{k}_2) P_f(\vec{k}_1)$]

$$C(0) = \langle f^2(\hat{n}) \rangle = \sum_\ell \frac{(2\ell+1)}{4\pi} C_\ell \approx \int d\ln\ell \underbrace{\frac{\ell(\ell+1)}{2\pi} C_\ell}_{\substack{\uparrow \\ \text{common to plot this .}}}$$

↑ variance at given angular position .

Primordial CMB :





$$f(\hat n) = \frac{\delta T}{T}(\hat n)$$

Recall that $ds^2 = a^2(\eta)\left[(1+2\Phi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right]$.

& geodesic equation

$$\frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta$$

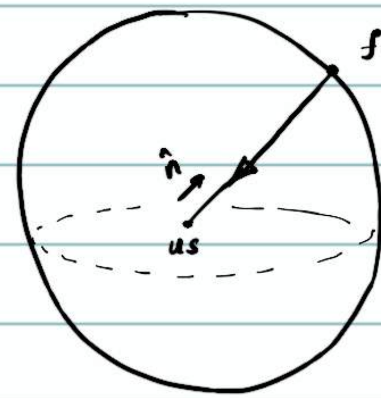
Together, we get $\frac{1}{P}\frac{dp}{d\eta} = -\frac{1}{a}\frac{da}{d\eta} - \hat p^i\partial_i\Phi + \partial_\eta\Phi$

* Using $ap \propto aT = a\bar T\left(1+\frac{\delta T}{\bar T}\right)$ and solving the above equation from rec. to today we get.

$$\frac{\delta T}{T}(\hat n) = \left(\frac{1}{4}\delta_\gamma + \Phi\right) + \hat n\cdot\bar v_e + 2\int_{rec}^{today} d\eta\,\partial_\eta\Phi$$

added for motion of electrons at last scattering surface.

(This assumes instantaneous recombination)



$$f(\hat{n}) = \frac{\delta T}{T}(\hat{n})$$

Instantaneous recombination approximation:

$$\frac{\delta T}{T}(\hat{n}) \approx \underbrace{\overset{(a)}{\left(\frac{\delta_\gamma}{4} + \Phi\right)\Bigg|_{LSS}} - \overset{(b)}{\left(\hat{n}\cdot\vec{v}_e\right)\Bigg|_{rec}}}_{\text{at recombination}} + \underbrace{2\int_{rec}^{today} d\eta\, \partial_\eta \Phi}_{\text{effects of potential evolution along line of sight}}$$

(c)

(a) $\left(\frac{\delta_\gamma}{4} + \Phi\right)_{rec}$ = Sachs-Wolfe Term.

= contribution from photon density perturbation and the gravitational potential well from which photons have to climb out.

Note: For large angular scales, $\delta_m = -2\Phi$ (dec. is during MD)

& $\delta_\gamma = \frac{4}{3}\delta_m \Rightarrow \left(\frac{\delta_\gamma}{4} + \Phi\right)_{rec} = \frac{-1}{8}\delta_\gamma\Big|_{rec}$

Thus $\quad \delta_\gamma > 0 \Rightarrow \frac{\delta T}{T} < 0 \quad$ & $\delta_\gamma < 0 \Rightarrow \frac{\delta T}{T} > 0$

hot spot at t_{rec} appears as a cold spot to us! (large angular scales). The effect is due to photons having to climb out of potential wells.

(b) $-(\hat{n} \cdot \vec{v}_e)$ = Doppler term:
This is the effect on the photon energy due to the scattering off moving electrons.

(c) $\int_{rec}^{today} d\eta \, \partial_\eta \Phi$ = Integrated Sachs Wolfe Effect. (ISW)

This the effect on the energy of photons due to time evolving gravitational potentials. During matter domination, Φ = const. At recombination, although we are in MD, there is still enough radiation to make Φ evolve. Moreover Φ also evolves during ΛD. Both of these contribute to the ISW effect.

The Calculation of C_ℓ proceeds as follows.

$\frac{k^3 P_f(k)}{2\pi^2} \approx \text{const}$; $f = \Phi, \delta_\gamma, \delta_b$ etc from inflation.
& superhorizon evolution.

$\downarrow$

Evolution of $\Phi, \delta_\gamma, \delta_b$ etc

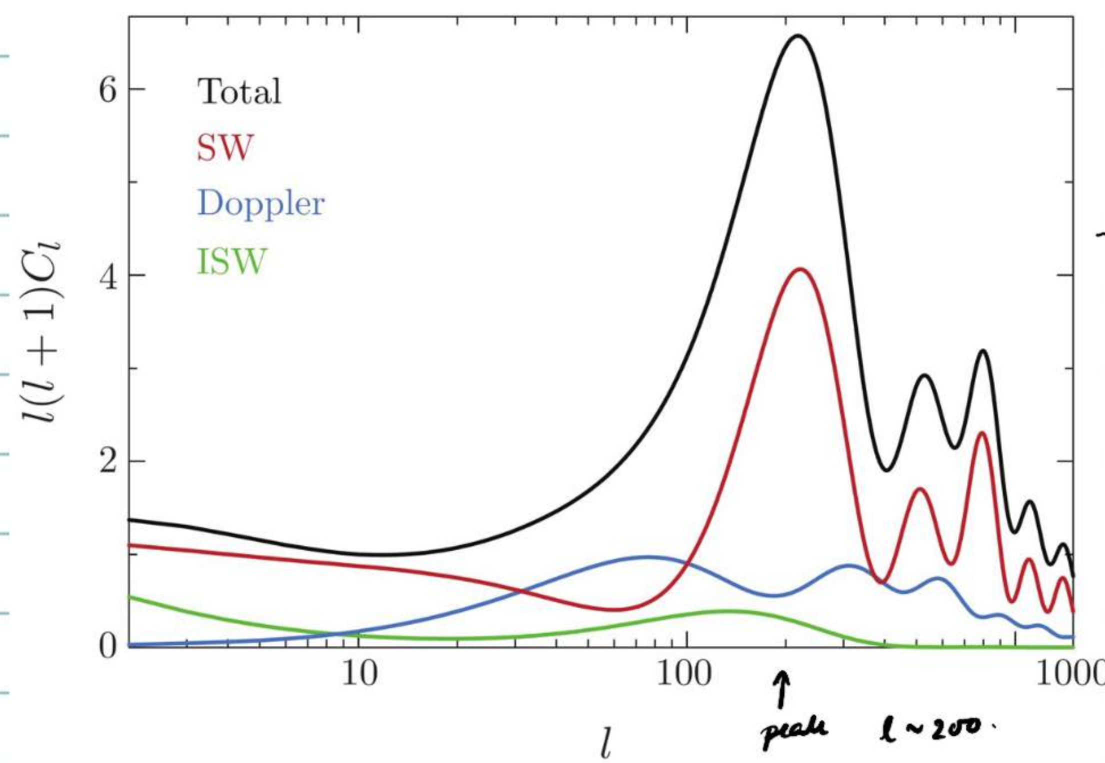
$\downarrow$

Projection on the sky

$\downarrow$

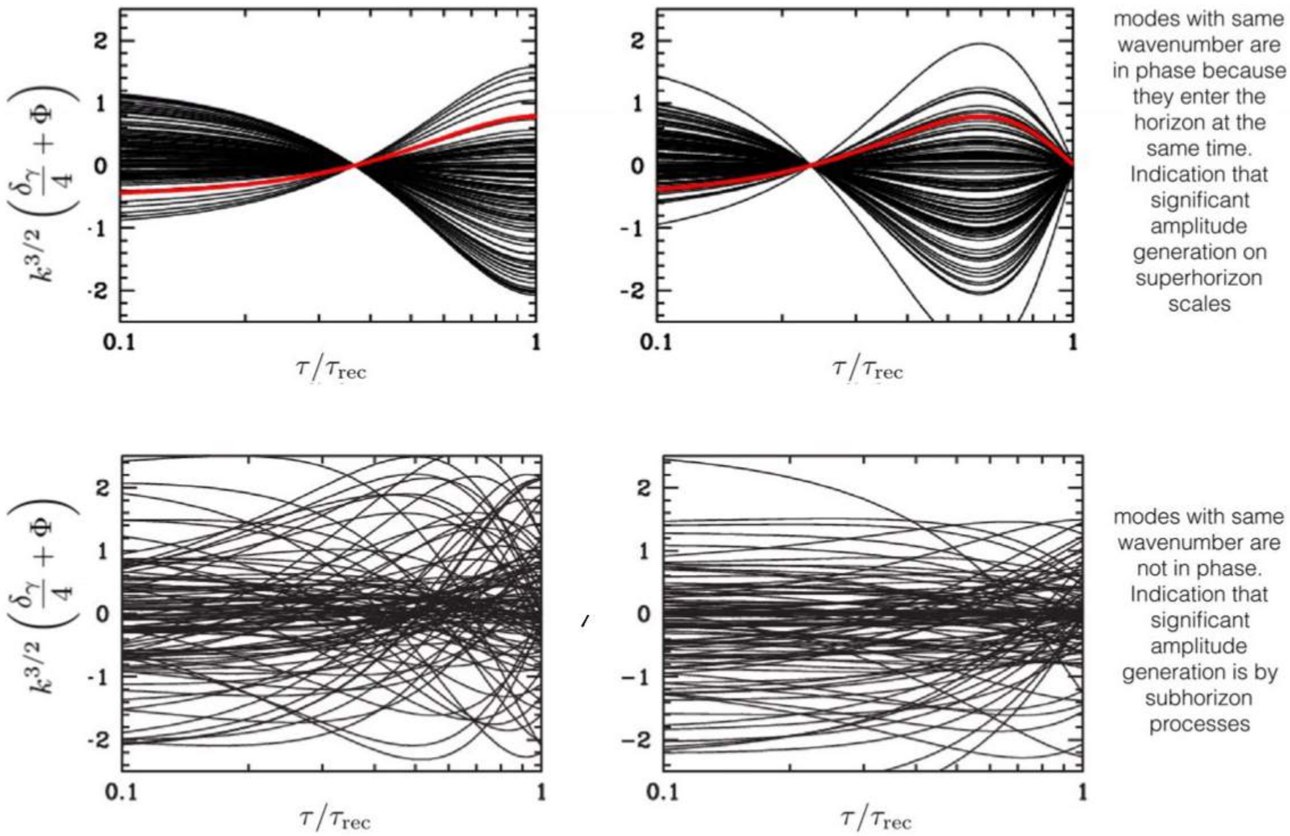
C_ℓ.

$\theta \sim 1°$.



peak $\ell \sim 200$.

Why peaks and troughs?



Peaks & troughs related to the "phase" of the oscillating δ_γ, Φ etc.

All modes with same $|\vec{k}|$, but different $\vec{k}$ entered horizon at the same time $k = \mathcal{H}$.
Hence at $t = t_{dec} \approx t_{rec}$, they will all be in phase!
For some k_1, all $\vec{k}_1$ modes will be at a max.
For some k_2, all $\vec{k}_2$ modes will be at a min.

The fact that we see peaks and troughs means that amplitudes must be significant on superhorizon scales ← Inflation!

If all the modes were being stirred up from subhorizon scales, they would not be in phase, and we would lose the peaks and valleys.

* The C_ℓ's first peak is at $\theta \sim 1°$, or $\ell \sim 200$..

$\theta \sim \frac{1}{\ell}$ & $\theta \sim \frac{\text{distance travelled by sound wave}}{\text{Angular diameter distance}}$.

(small angles)

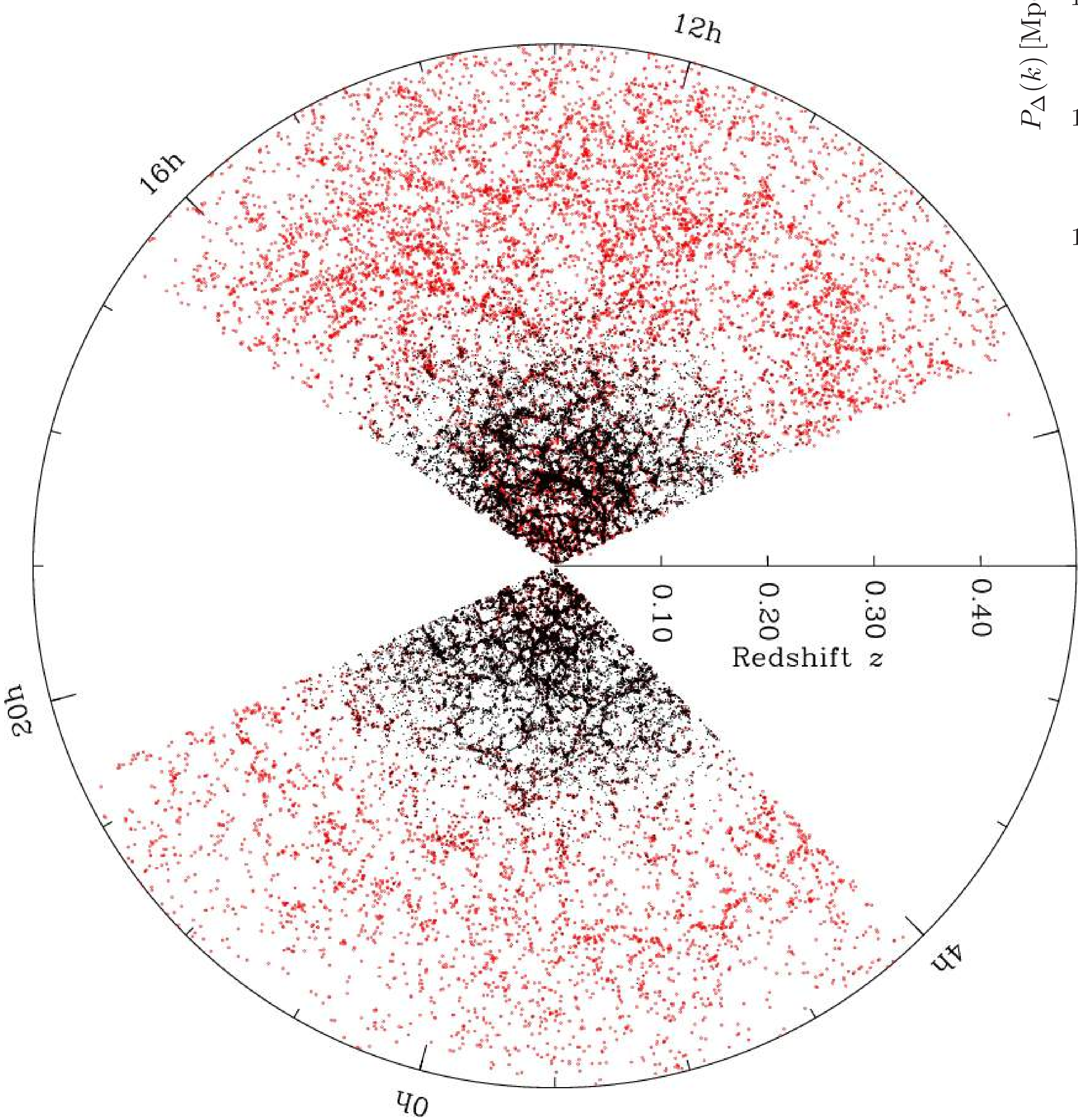
Location depends on sound speed c_s as well as geometry! Its location is a strong indication that the universe is spatially flat.

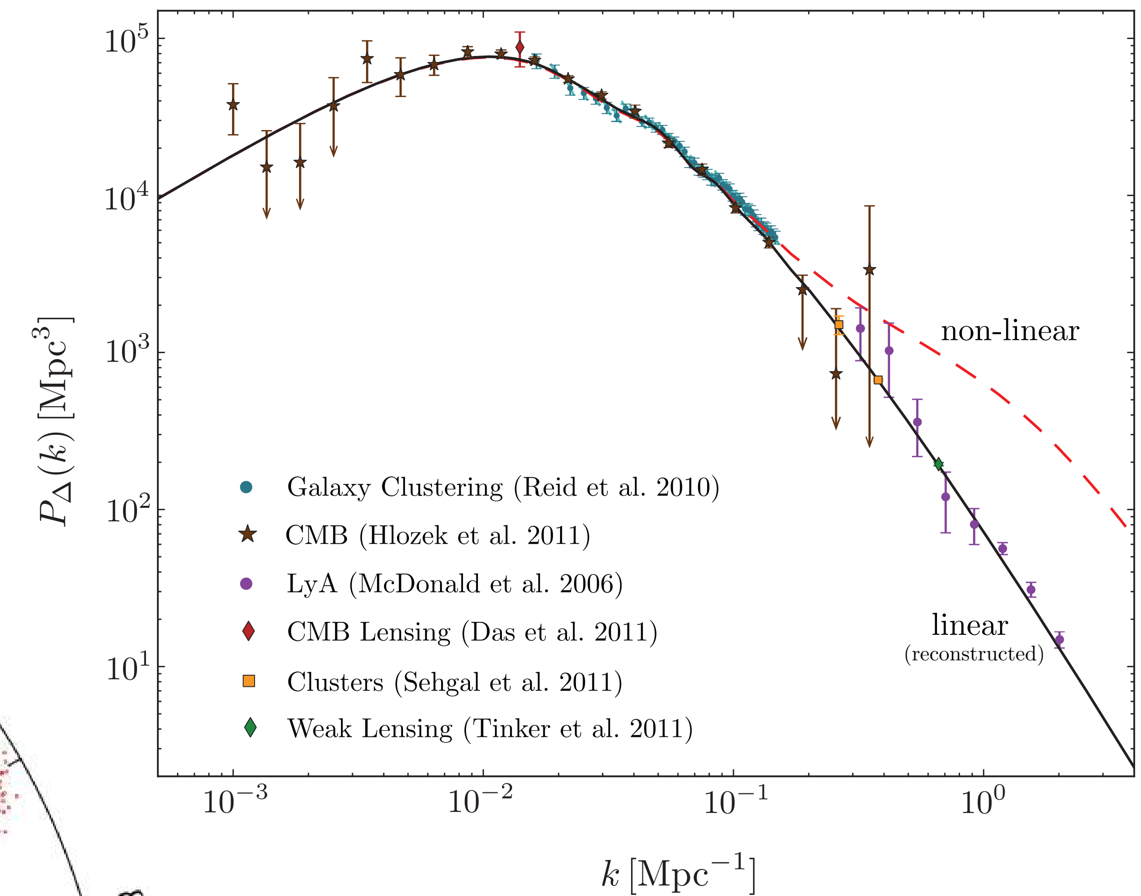
* Lots of additional information in the CMB. Ω_m, Ω_b, H_0, Ω_r, Ω_Λ ....

See background.uchicago.edu/~whu/metaanim.html.

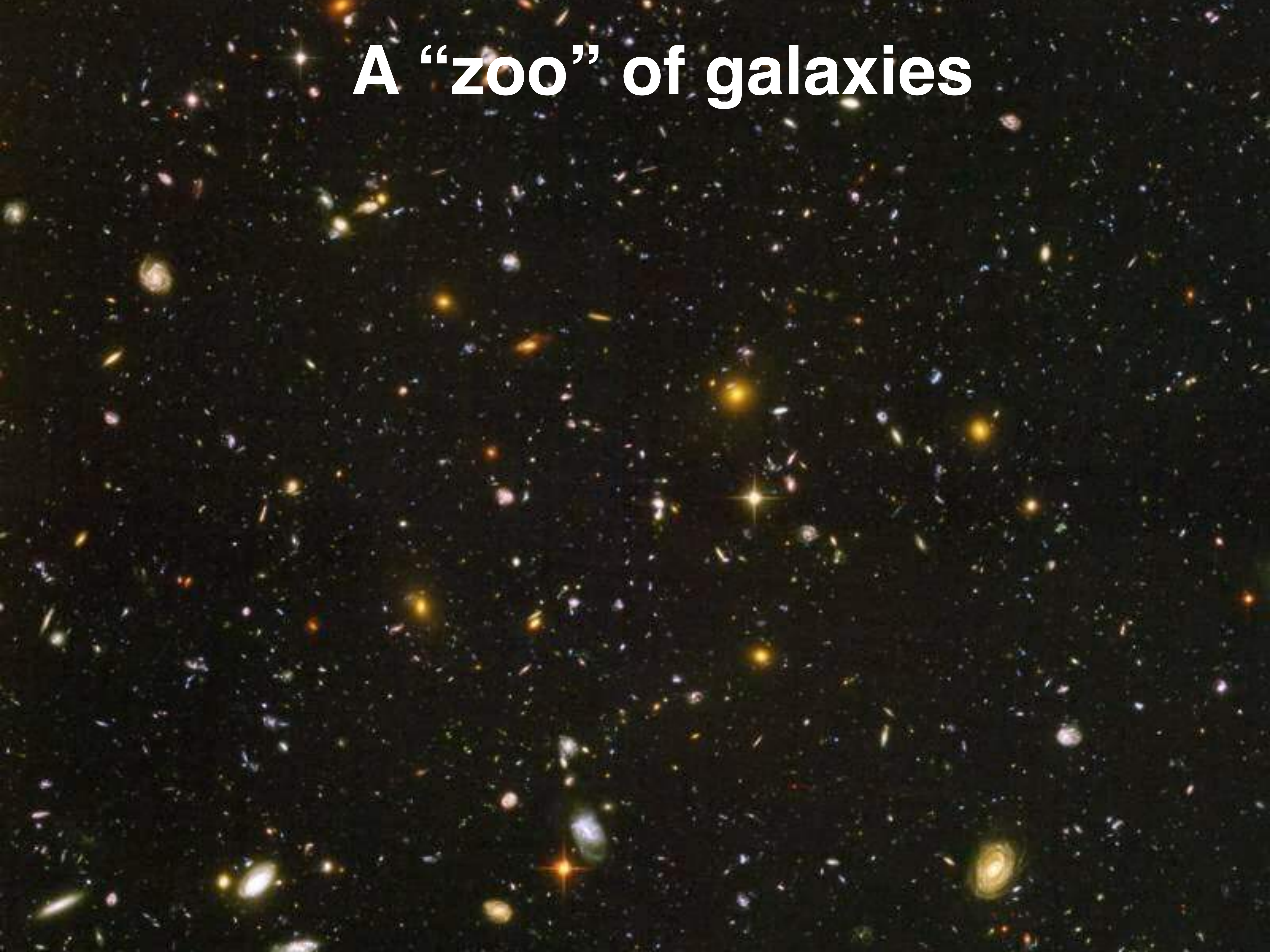
Lecture 22

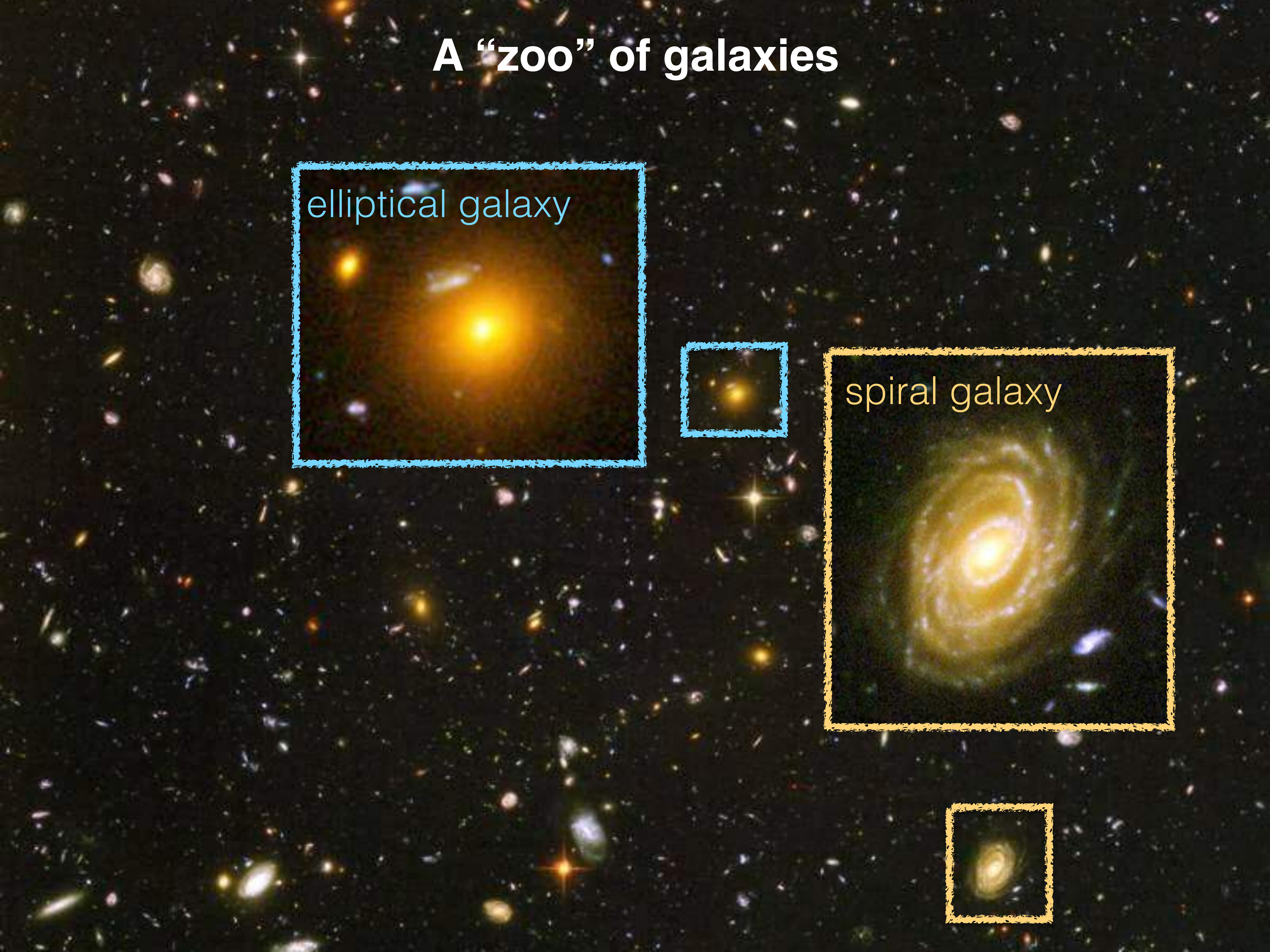
GALAXIES

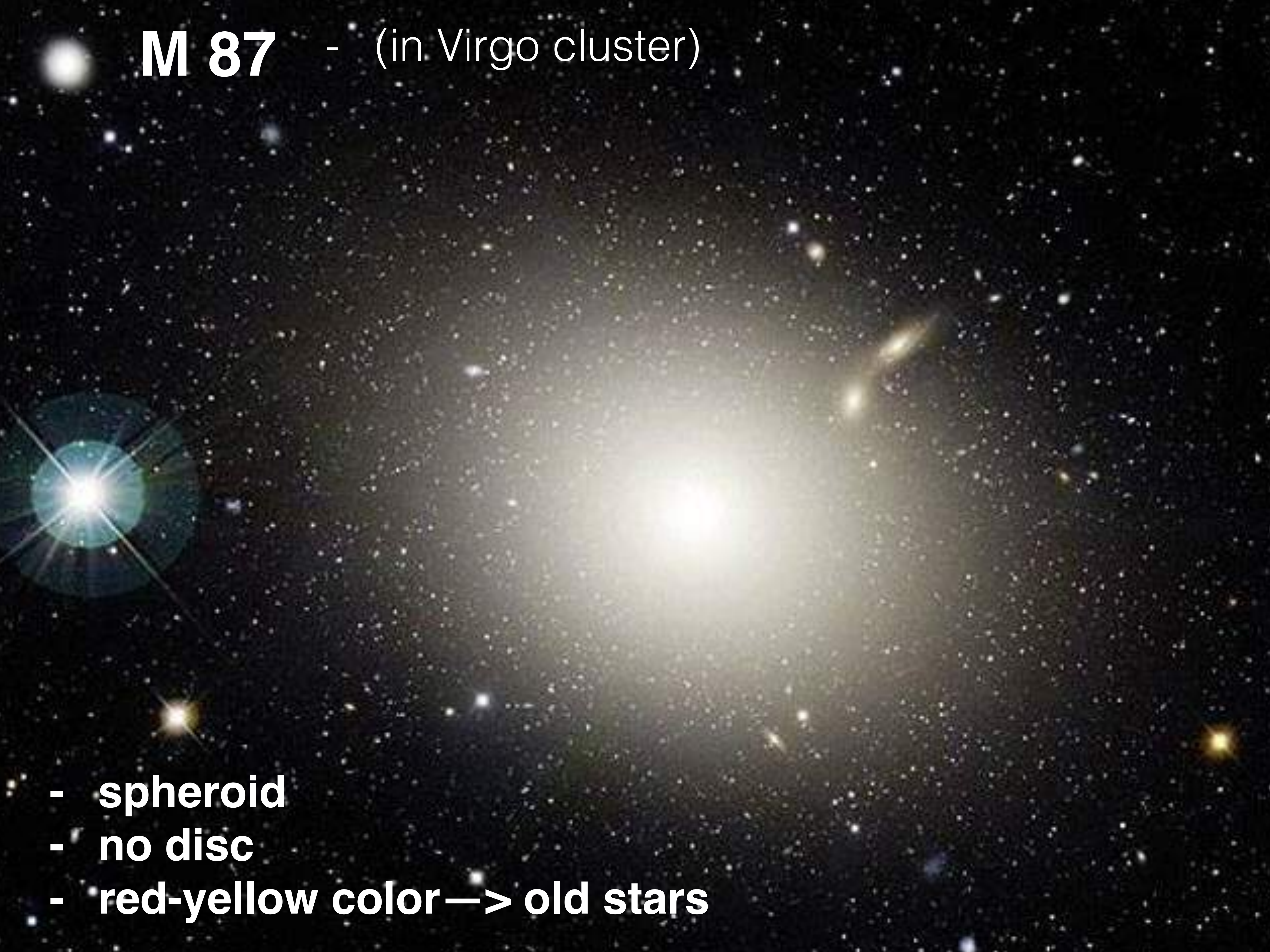

12h
16h
8h
20h
4h
0h
0.10
0.20
0.30
0.40
Redshift z


$P_\Delta(k)$ [Mpc^3]
k [Mpc^{-1}]
non-linear
linear
(reconstructed)
Galaxy Clustering (Reid et al. 2010)
CMB (Hlozek et al. 2011)
LyA (McDonald et al. 2006)
CMB Lensing (Das et al. 2011)
Clusters (Sehgal et al. 2011)
Weak Lensing (Tinker et al. 2011)

A “zoo” of galaxies

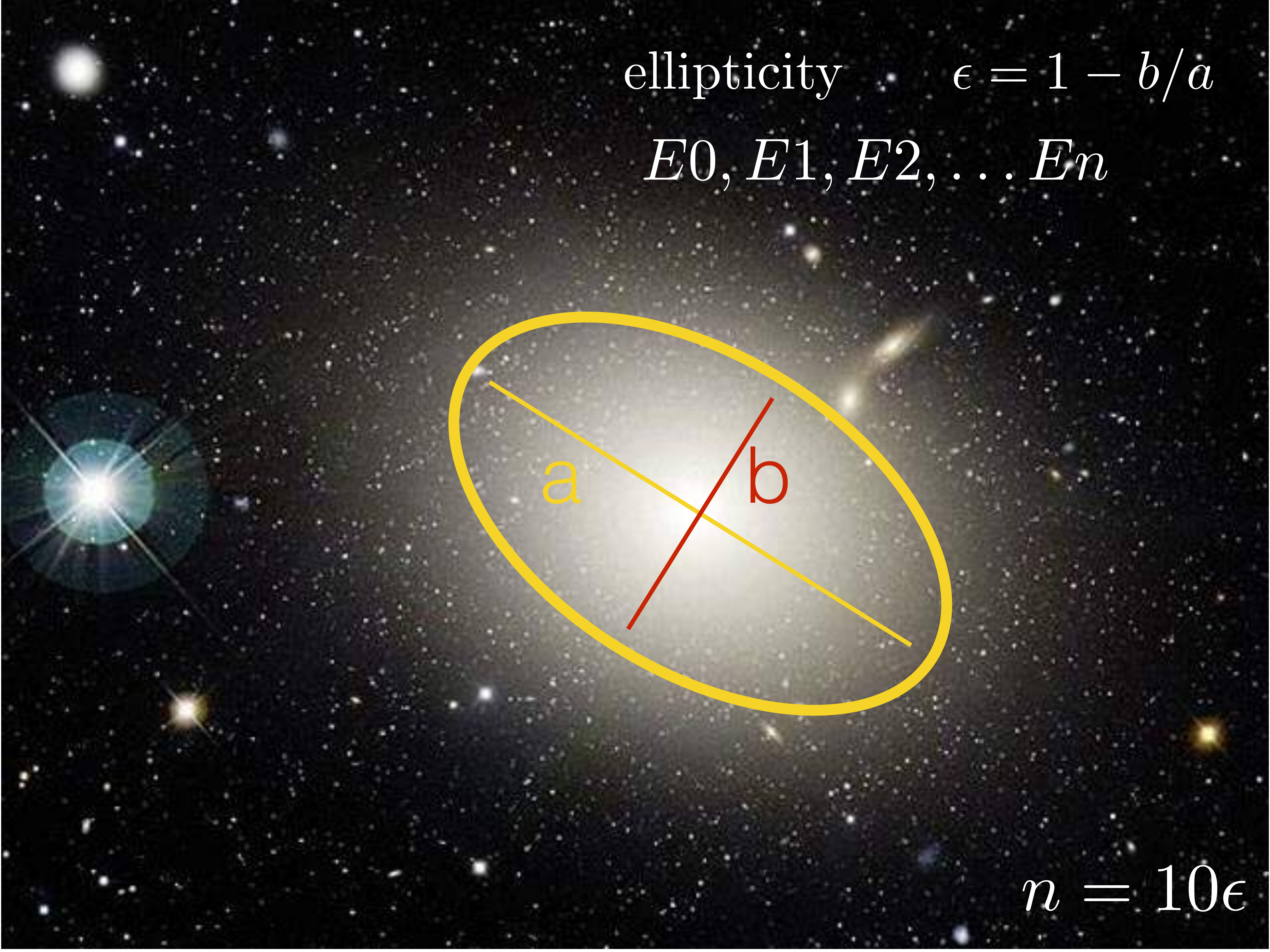



A "zoo" of galaxies
elliptical galaxy
spiral galaxy



M 87 - (in Virgo cluster)

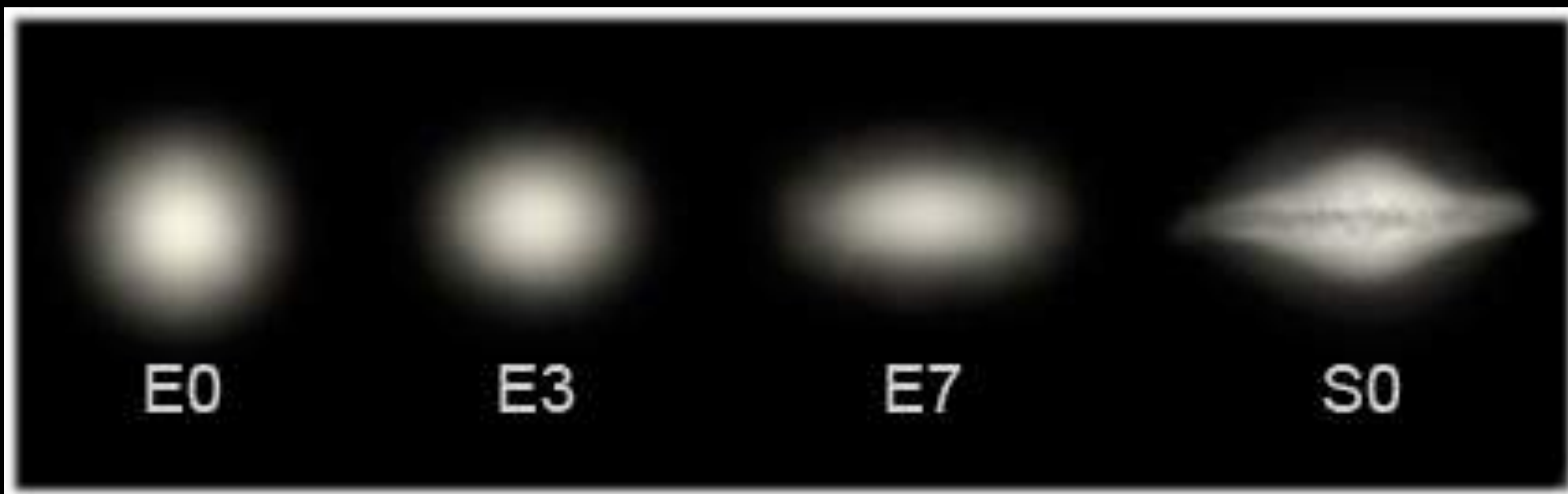
- **spheroid**
- **no disc**
- **red-yellow color —> old stars**


ellipticity $\epsilon = 1 - b/a$
$E0, E1, E2, \ldots En$
a
b
$n = 10\epsilon$

Classification of Ellipticals

ellipticity $\epsilon = 1 - b/a$

$E0, E1, E2, \ldots En$

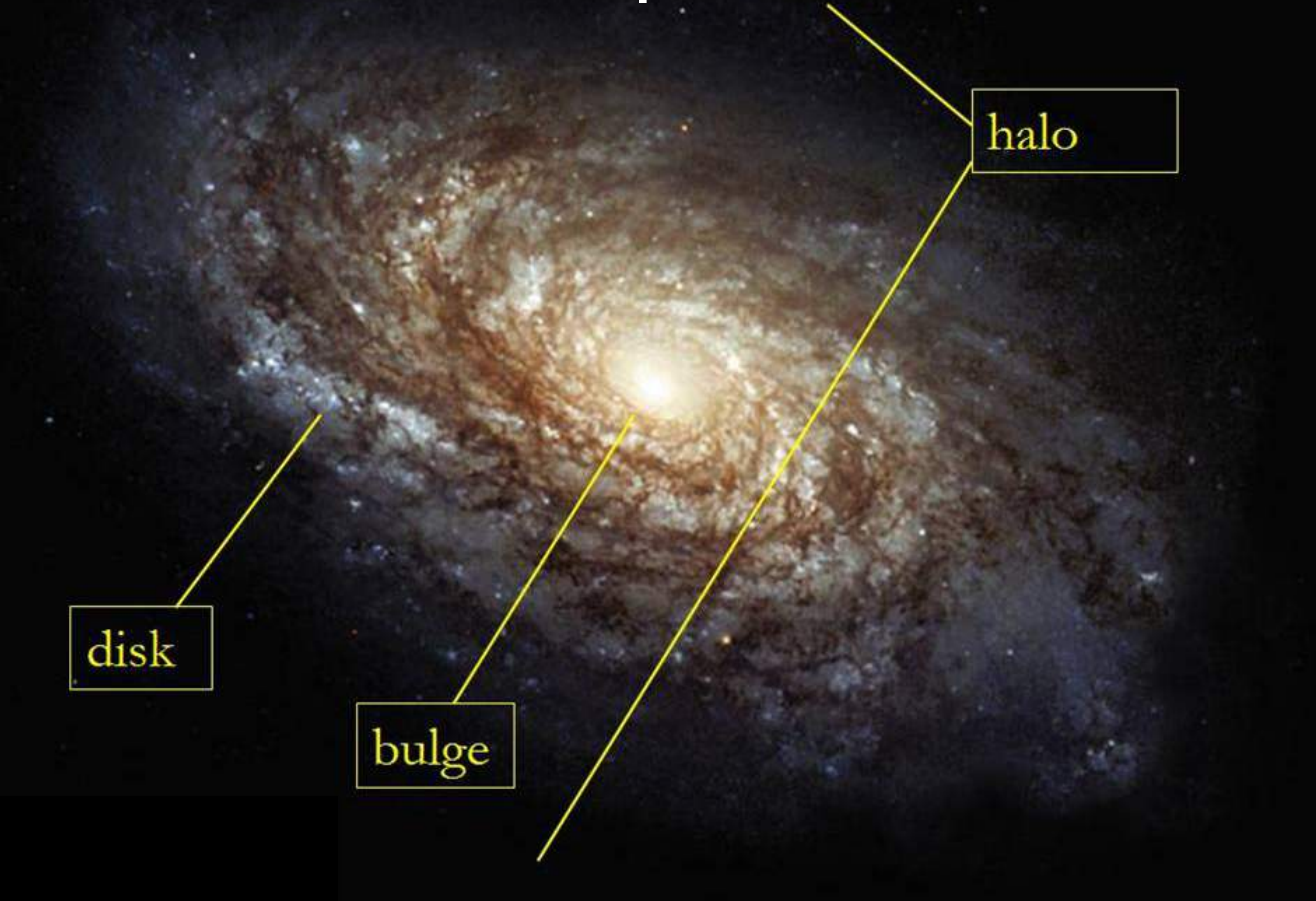


$n = 10\epsilon$

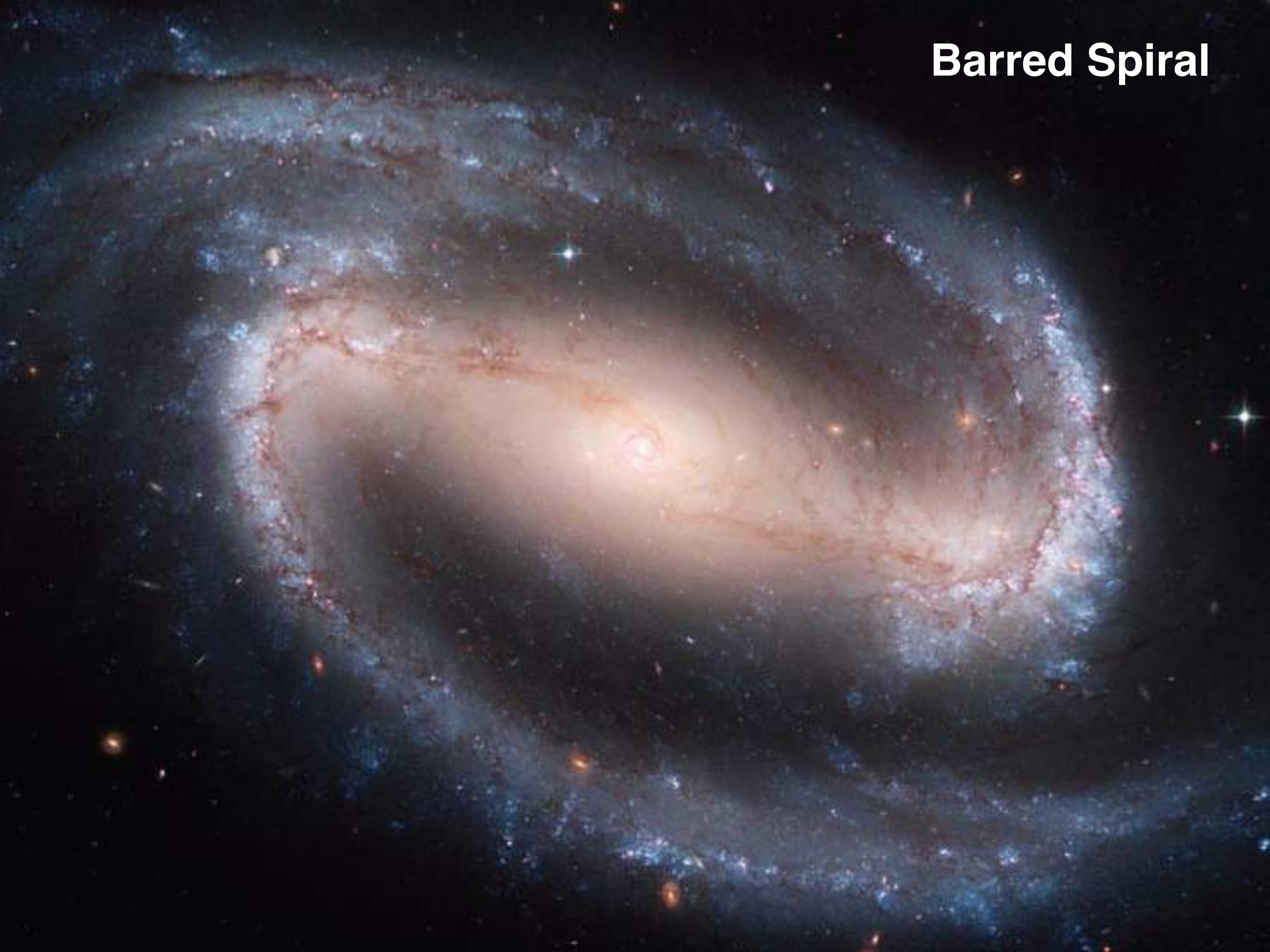
Lenticular Galaxy

- intermediate between spirals and ellipticals
- dusty stellar disc
- less dusty spheroid

Spirals



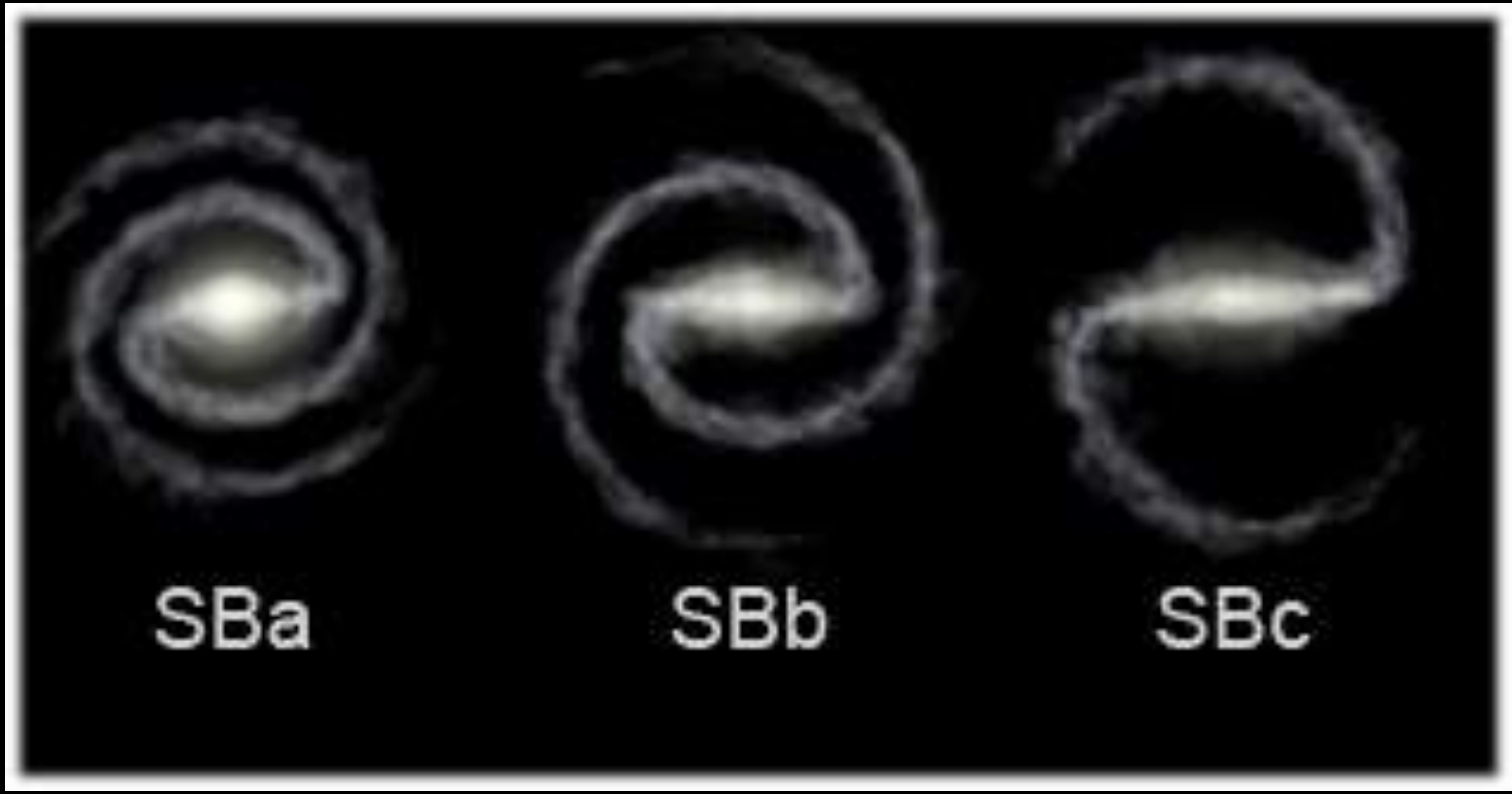
Grand Design Spirals


Barred Spiral

Classification of Spirals

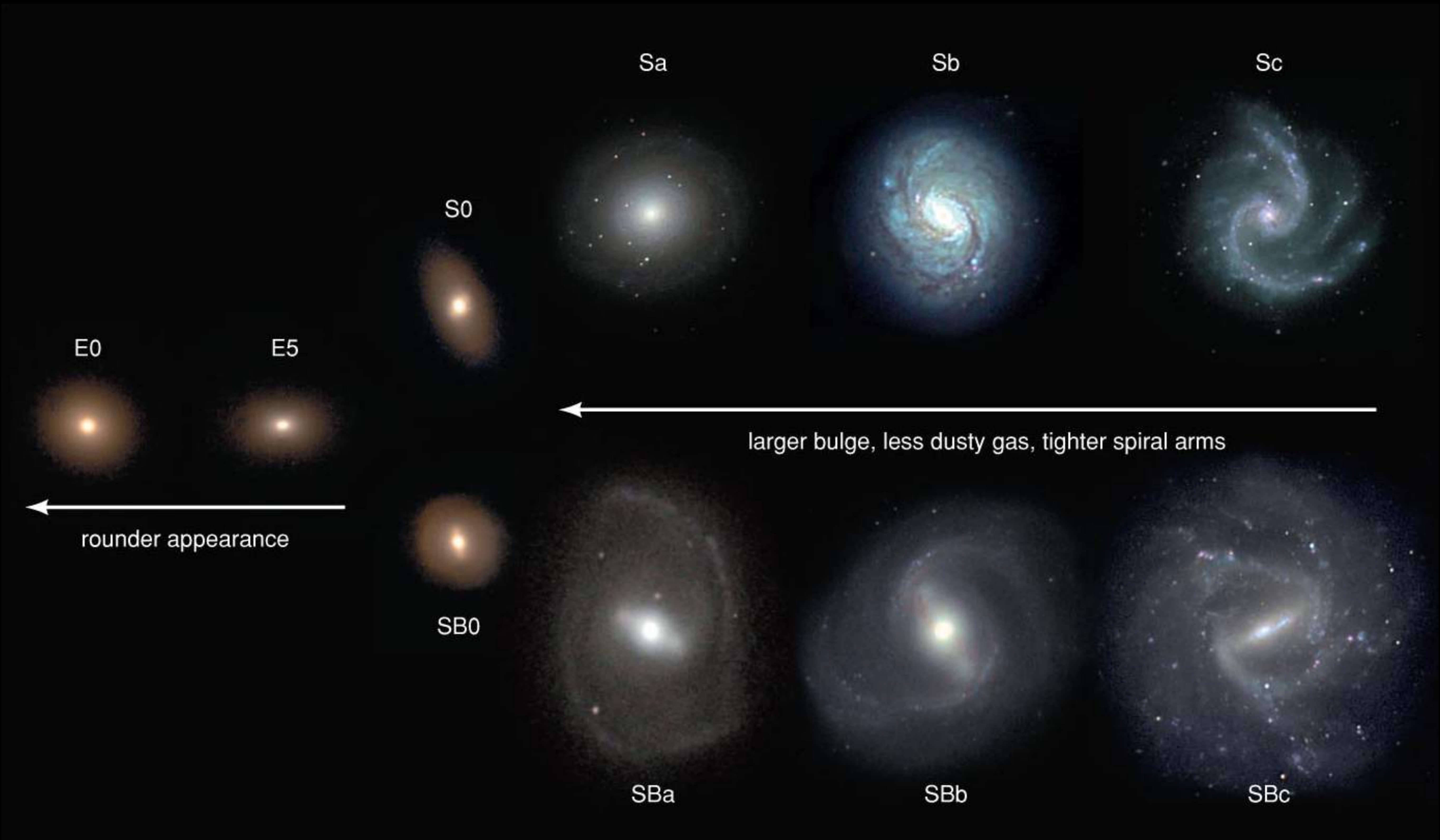
disk/bulge

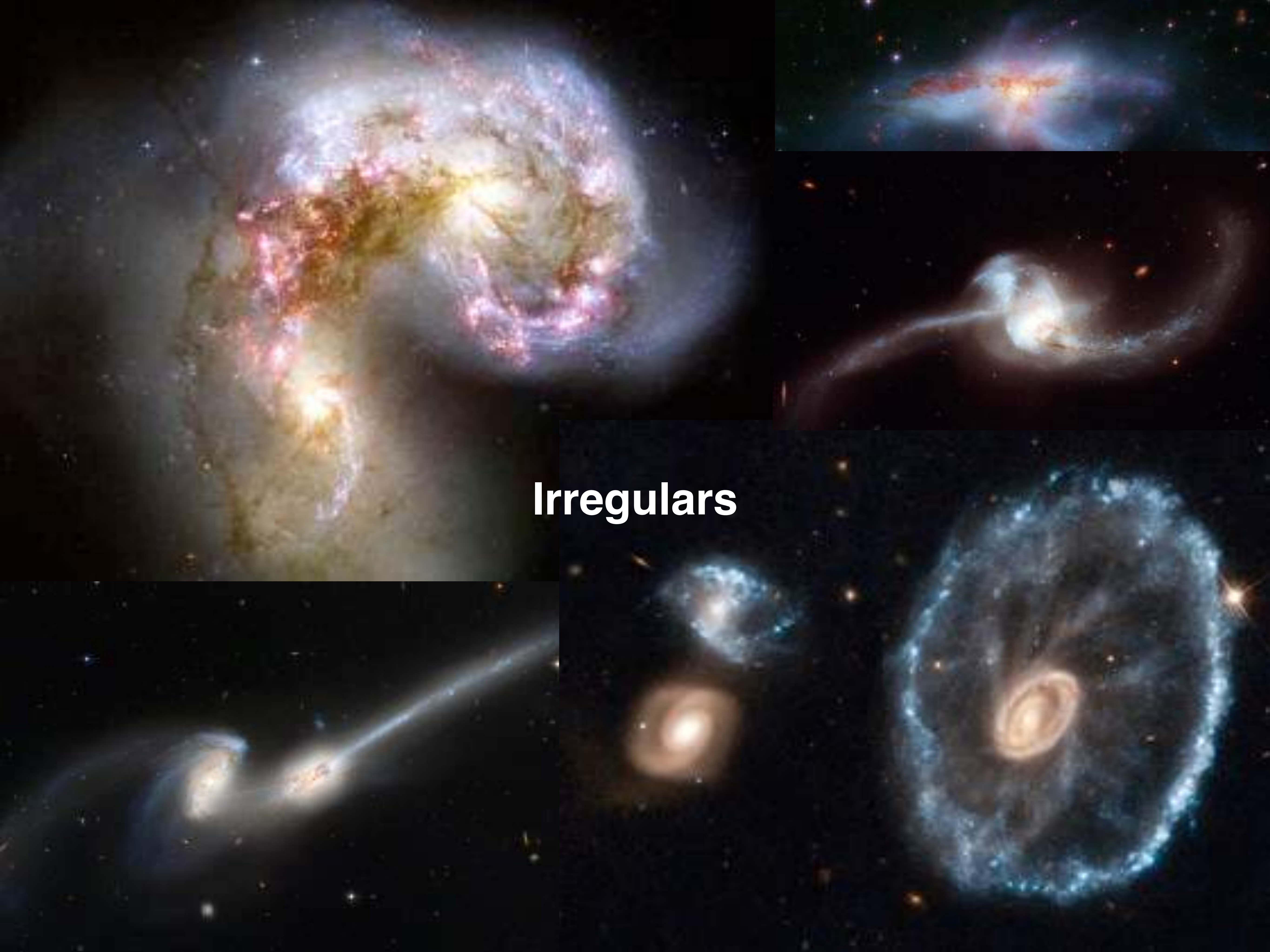
pitch angle

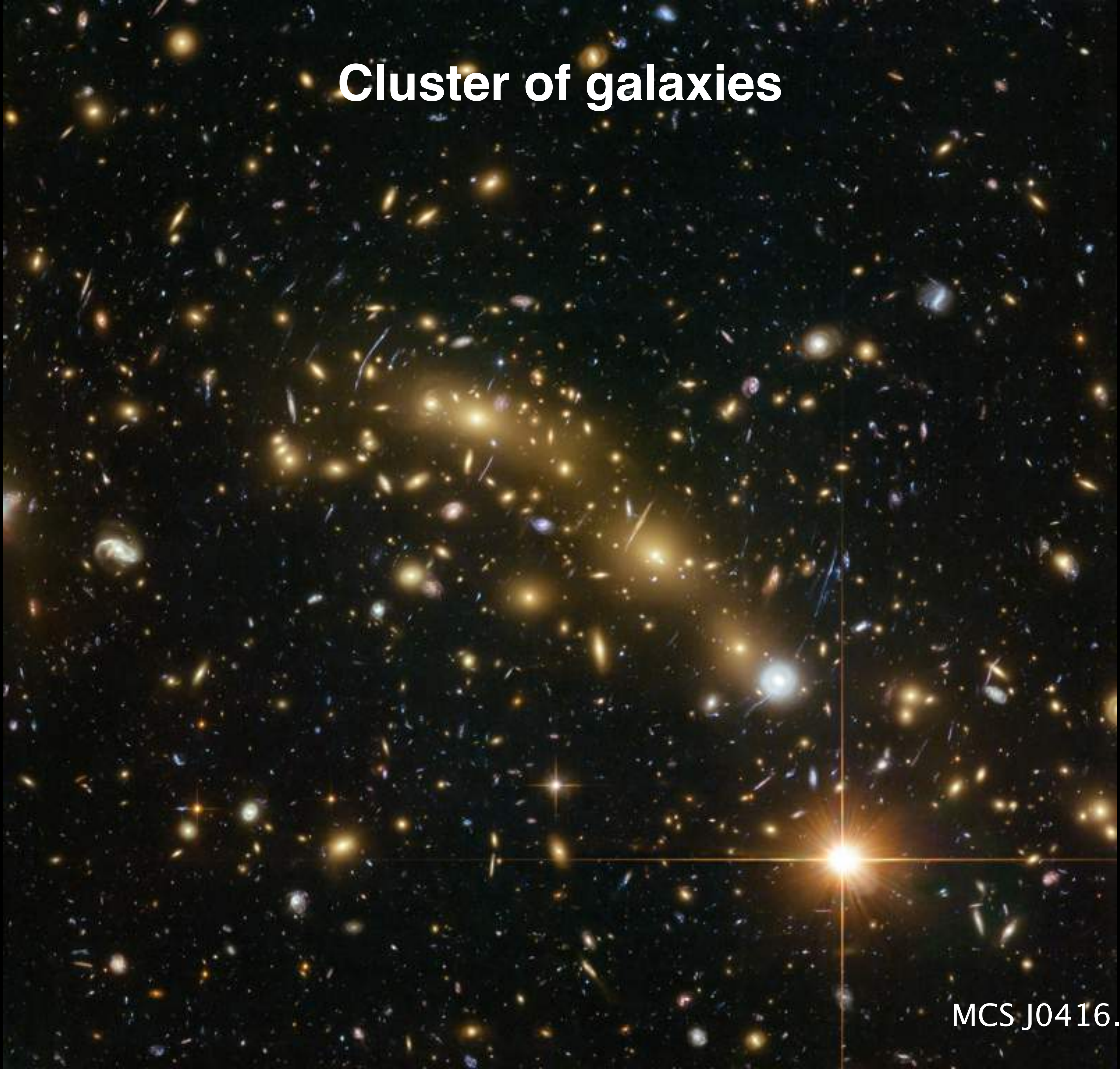


disc/bar

Hubble Sequence (not evolutionary!)




Irregulars


Cluster of galaxies
HST
MCS J0416.1–2403

Gravitational Dynamics

1. Virial theorem
2. Time scales for relaxation

see notes

Gravitational Dynamics. (collisionless).

1. Virial Theorem
2. Relaxation time scales.

Virial Theorem:

Consider a collection of N point particles.
(For the moment we are considering a collection of stars).
Their gravitational dynamics is controlled by Newton's Laws.

can also be applied to DM particles or galaxies as particles

$$\frac{d^2\vec{r}_i}{dt^2} = -\sum_{j\neq i}^{N} \frac{G\, m_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j)$$

The Kinetic & Potential Energy of this system is given by

$$KE = \frac{1}{2}\sum_{i=1}^{N} m_i \left|\frac{d\vec{r}_i}{dt}\right|^2 \qquad PE = \frac{G}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

$$\frac{1}{2}\sum_i \frac{d}{dt}\left(m_i|\vec{r}_i|^2\right) = \sum_i m_i \; \vec{r}_i \cdot \vec{v}_i$$

$$\frac{1}{2}\sum_i \frac{d^2}{dt^2}\left(m_i|\vec{r}_i|^2\right) = \sum_i m_i|\vec{v}_i|^2 - \sum_i m_i \vec{r}_i \cdot \sum_{j\neq i} \frac{G m_j (\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3}$$

$$= \sum_i m_i|\vec{v}_i|^2 - \frac{1}{2}\left(\sum_j m_j \vec{r}_j \cdot \sum_{i\neq j} \frac{G m_i (\vec{r}_j-\vec{r}_i)}{|\vec{r}_i-\vec{r}_j|^3} + \sum_i m_i \vec{r}_i \cdot \sum_{\substack{\cancel{i\neq j}\\ j\neq i}} \frac{G m_j (\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3}\right)$$

$$= \sum_i m_i|\vec{v}_i|^2 - \frac{1}{2}\left(\sum_{\substack{ij\\ i\neq j}} \frac{G m_i m_j}{|\vec{r}_i-\vec{r}_j|}\right)$$

Estimate the mean free time between star collision in a spherical galaxy $R \sim 10$ kpc.

$N \sim 10^{11}$.

Let $I \equiv \frac{1}{2} \sum_i m_i |\vec{r}_i|^2$: moment of inertia

Using Newton's law, we can show that

$$\frac{d^2 I}{dt^2} = \frac{1}{2} \frac{d^2}{dt^2} \underbrace{\sum_i^N \left(m_i |\vec{r}_i|^2\right)}_{I} = 2KE - |PE| .$$

moment of inertia

total Kinetic & potential energies of the system

Let us take the time average of both sides over some long time T.

$$\frac{\frac{dI(T)}{dt} - \frac{dI(0)}{dt}}{T} = 2\langle KE \rangle_T - \langle |PE| \rangle_T .$$

$\frac{dI}{dt}$ is bounded $\Rightarrow$ $\text{LHS} \xrightarrow[T\to\infty]{} 0$.

$$\Rightarrow \boxed{\langle KE \rangle = \frac{1}{2} \langle PE \rangle} .$$

Virial Theorem

Many, many uses for this theorem !

How long does it take systems to become virialized? What is the relaxation time? What is the main mechanism?

Possibilities:

1) 2-body relaxation.
2) Violent Relaxation

Consider a system of N (collisionless) particles, of mass m, in a region of size R. & mass $M = Nm$

The typical time scale for formation of such a system

$$t_{dyn} \sim \frac{1}{\sqrt{G\rho}} \sim \frac{R^{3/2}}{\sqrt{GM}} \qquad \text{where } \rho \sim \frac{M}{R^3}.$$

Dynamical time scale.

The time taken to cross the system.

$$t_{cross} \sim \frac{R}{v} \sim \frac{R}{\sqrt{GM/R}} \sim \frac{R^{3/2}}{\sqrt{GM}} \sim t_{dyn} \qquad \text{were } v^2 \sim \frac{GM}{R}.$$

Crossing time scale

For a typical galaxy $N \sim 10^{11}$, $M = 10^{11} M_\odot$
$R \sim 10$ kpc. $\uparrow$ star.

$t_{dyn} \sim t_{cross} \sim 5 \times 10^{7}$ yrs.

Note age of universe $t \sim 10^{10}$ yrs.

Before considering 2-body relaxation due to gravitational interactions, let us make sure that physical collisions of stars are irrelevant.

Mean free time of collision $\sim \frac{1}{n \sigma v} \sim \frac{R^3}{N \sigma v}$

radius of star.

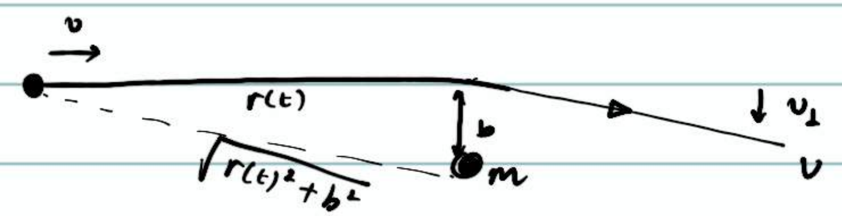
$$\left.\begin{array}{ll} r_\odot \sim 10^{9} \text{ m} & v \sim 300 \text{ km s}^{-1} \\ R \sim 10 \text{ kpc} & \\ M = 10^{11} M_\odot & \end{array}\right\} \Rightarrow \quad \sim 10^{19} \text{ yrs.}$$

$\gg$ age of the universe!

Gravitational 2-body interactions could play a role in the relaxation. Let us estimate the time scale for such two body interactions to wipe out memory of initial conditions (this is what we mean by relaxation / virialization)

Two body encounters

Consider a mass m, scattering from another mass m,



First consider weak scattering : (No significant change in kinetic energy)

$$\dot{v}_\perp \approx \frac{-Gm\,b}{(v^2t^2+b^2)^{3/2}}$$ when we assumed $r(t) \approx vt$.

$$\Rightarrow \quad v_\perp(t) \approx \frac{Gm}{b}\frac{t}{\sqrt{v^2t^2+b^2}} \xrightarrow[vt \gg b]{} \frac{Gm}{bv}$$

So each weak scattering produces a change in $v_\perp \sim \dfrac{Gm}{bv}$

Digression to Strong scattering

For strong scattering $v_\perp \sim v$

Hence $b \lesssim \frac{Gm}{v^2}$ $\Rightarrow$ cross section for strong scattering $\sim b^2$.

How likely are strong scatterings? The typical time scale for such interactions

$$t_{relax(s)} \sim \frac{1}{n\sigma v} \sim \frac{R^3}{N\left(\frac{Gm}{v^2}\right)^2 v} = \frac{N R^3 v^3}{(GM)^2} \qquad \because M = Nm$$

$$= \left(\frac{R}{v}\right) N \qquad \because \frac{GM}{R} \sim v^2$$

$$\therefore \quad t_{relax(s)} \sim \left(\frac{R}{v}\right) N \sim t_{cross} N$$

2 body relaxation time scale (strong)

For $N \sim 10^{11}$ $\quad t_{relax(s)} \gg t_{cross}, t_{dyn}, t_{universe}$.

So strong scattering encounters are exceedingly rare as well and cannot account for relaxation.

Let us go back to weak scattering.

Lecture 23

Review:

barred
↓

- Galaxies
  - Spiral — S0, S1 . . . , SB0, SB1 . . .
  - Ellipticals — E0, E1 . . .
  - irregulars
  - AGNs

- Galaxy Clusters.

- Gravitational Dynamics: - Virial Theorem
  - Relaxation time scales.

$\langle K.E \rangle = \frac{1}{2} \langle P.E \rangle$.

N particles, $M = Nm$.

$t_{dyn} \sim \frac{1}{\sqrt{G\rho}}$.

$t_{cross} \sim \left(\frac{R}{v}\right)$

$t_{coll} \sim \frac{1}{n\sigma v}$ $\quad \sigma \sim r^2$ where r = radius of particle.

$t_{relax(s)} \sim \frac{1}{n\sigma v}$ $\quad \sigma \sim b^2$; $b \sim \frac{Gm}{v^2}$

$\sim t_{cross} N$.

For $N \gg 1$, $t_{relax(s)} \gg t_{cross}$.

& $t_{relax(s)} \gg t_{universe}$! , Note $t_{relax(s)} \ll t_{coll}$.

From the last class, recall that

Each weak two body encounter produces $v_\perp \sim \frac{Gm}{bv}$

However they will be in random directions.

So $\sum v_\perp \simeq 0$. What about $\sum v_\perp^2$?

$$\sum v_\perp^2 \approx \int \underbrace{\left(\frac{Gm}{bv}\right)^2}_{v_\perp^2} \times \underbrace{n}_{\#\text{ density}} \times \underbrace{\left(vt \times 2\pi b\, db\right)}_{\text{volume}}.$$

$$= 2\pi \left(\frac{Gm}{v}\right)^2 \frac{Nm}{R^3} \cdot v\, t \int_{b_{min}}^{b_{max}} \frac{db}{b}.$$

Weak encounter $\Rightarrow$ $b_{min} \sim \frac{Gm}{v^2}$

Size of system $\sim R \Rightarrow b_{max} \sim R$

$$\therefore \sum v_\perp^2 \approx 2\pi \left(\frac{Gm}{v}\right)^2 \frac{N}{R^3}\, v \cdot t \ln\left(\frac{Rv^2}{Gm}\right)$$

$$= 2\pi \frac{G^2m^2}{R^3} \frac{N}{v} \ln\left(\frac{Rv^2}{Gm}\right) t$$

$$\therefore \quad t_{relax(w)} \sim \frac{R^3 v^3}{2\pi\, G^2 m^3 N \, \ln\left(\frac{Rv^2}{Gm}\right)}$$

$$\approx \frac{N\, R^3 v^3}{2\pi\,(GM)^2} \, \frac{1}{\ln\left(N \frac{Rv^2}{GM}\right)} \qquad M \approx Nm$$

$$t_{relax(w)} \sim \frac{t_{cross}}{6} \, \frac{N}{\ln N} \sim \frac{t_{rel(s)}}{6 \ln N}$$

$\frac{GM}{v^2} \sim R$.

$t_{relax(s)} \sim N\left(\frac{R}{v}\right)$

2 body relaxation time scale (weak).

$$t_{relax(w)} < t_{relax(s)}$$

But t_{dyn}, t_{cross}, $t_{universe} << t_{relax(s)}$, $t_{relax(w)}$

*. So two body relaxation cannot work!

So what causes such a system to relax?

[What does not work: (1) direct physical collisions
(2) strong scattering 2 body
(3) weak scattering 2 body

A partial answer is provided by a process called violent relaxation.

The time and space dependent fluctuations in the gravitational potential of the system can be quite efficient in mixing up the orbits (and mixing the energies of the particles).

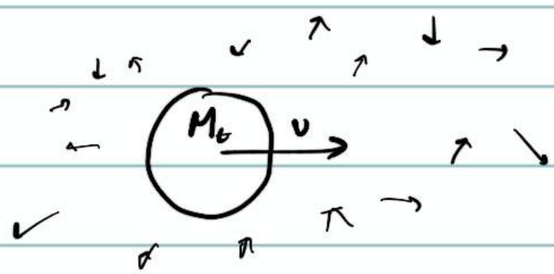
Detailed estimates show $t_{relax(v)} \sim$ few t_{dyn}.

Thus violent relaxation might work!

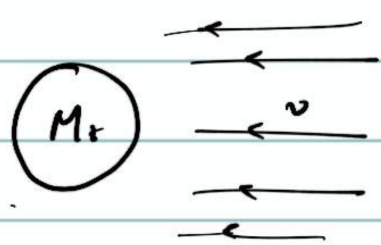
So far we have been discussing equal mass particles. What about the motion of a star or galaxy through a cloud of smaller particles? [imagine galaxies in a cluster].

This leads to an interesting phenomenon called dynamical friction.

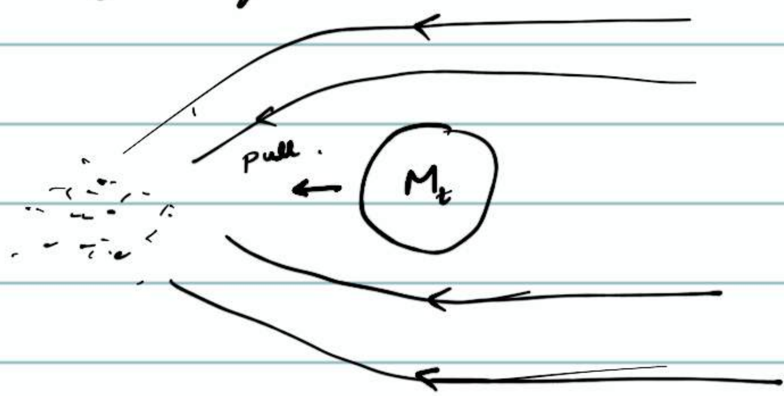
Consider an object of mass M_t moving at a velocity v through a gas of particles



In the frame of the star, the particles are coming towards it with a velocity $-v$



Because of M_t's gravity, the particles will get focussed behind the object



This extra density behind the object pulls on it.

Thus in the frame of the roughly stationary particles, the larger object is slowed down! This effect is called dynamical friction.

$$\left|\frac{dv}{dt}\right| \propto \frac{G^2 M_t \rho}{v^2}$$ [dim analysis] + proportionality checks.

$$t_{dyn.\,fric} << t_{relax(w)}$$

and can be quite small! Dynamical friction is relevant for slowing down & collisions of galaxies. Might also be relevant for massive galaxies sinking to the center of clusters!

More about violent relaxation & dynamical friction on next page.

Discussion in context of clusters of Galaxies.

Violent Relaxation. One of the most important of the aforementioned processes is known as *violent relaxation*. This process very quickly establishes a virial equilibrium in the course of the gravitational collapse of a mass concentration. The reason for it are the small-scale density inhomogeneities within the collapsing matter distribution which generate, via Poisson's equation, corresponding fluctuations in the gravitational field. These then scatter the infalling particles and, by this, the density inhomogeneities are further amplified. The fluctuations of the gravitational field act on the matter like scattering centers. In addition, these field fluctuations change over time, yielding an effective exchange of energy between the particles. In a statistical average, all galaxies obtain the same velocity distribution by this process. As confirmed by numerical simulations, this process takes place on a time-scale of $t_{\rm cross}$, i.e., roughly as quickly as the collapse itself.

Dynamical Friction. Another important process for the dynamics of galaxies in a cluster is *dynamical friction*. The simplest picture of dynamical friction is obtained by considering the following. If a massive particle of mass m moves through a statistically homogeneous distribution of massive particles, the gravitational force on this particle vanishes due to homogeneity. But since the particle itself has a mass, it will attract other massive particles and thus cause the distribution to become inhomogeneous. As the particle moves, the surrounding "background" particles will react to its gravitational field and slowly start moving towards the direction of the particle trajectory. Due to the inertia of matter, the resulting density inhomogeneity will be such that an overdensity of mass will be established along the track of the particle, where the density will be higher on the side opposite to the direction of motion (thus, behind the particle) than in the forward direction (see Fig. 6.9). By this process, a gravitational field will form that causes an acceleration of the particle against the direction of motion, so that the particle will be slowed down. Because this "polarization" of the medium is caused by the gravity of the particle, which is proportional to its mass, the deceleration will also be proportional to m. Furthermore, a fast-moving particle will cause less polarization in the medium than a slow-moving one because each mass element in the medium is experiencing the gravitational attraction of the particle for a shorter time, thus the medium becomes less polarized. In addition, the particle is on average farther away from the density accumulation on its backward track, and thus will experience a smaller acceleration if it is faster. Combining these arguments, one obtains for the dependence of this dynamical friction

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \propto -\frac{m\,\rho\,\boldsymbol{v}}{|\boldsymbol{v}|^3}\,, \tag{6.29}$$

where ρ is the mass density in the medium. Applied to clusters of galaxies, this means that the most massive galaxies will experience the strongest dynamical friction, so that they are subject to a significant deceleration through which they move deeper into the potential well. The most massive cluster galaxies should therefore be concentrated around the cluster center, so that a spatial separation of galaxy populations with respect to their masses occurs (mass segregation). If dynamical friction acts over a sufficiently long time, the massive cluster galaxies in the center may merge into a single one. This is one possible explanation for the formation of cD galaxies.

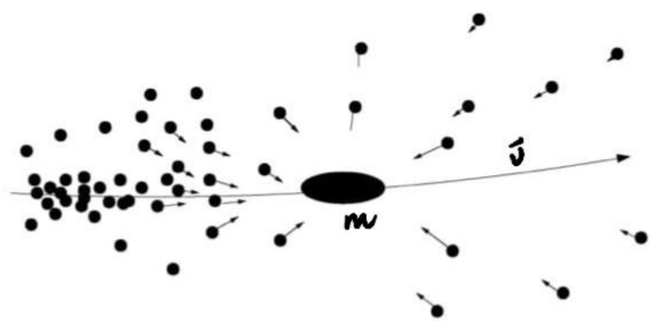


Fig. 6.9. The principle of dynamical friction. The gravitational field of a massive particle (here indicated by the large symbol) accelerates the surrounding matter towards its track. Through this, an overdensity establishes on the backward side of its orbit, the gravitational force of which decelerates the particle

Dynamical friction also plays an important role in other dynamical processes in astrophysics. For example, the Magellanic Clouds experience dynamical friction on their orbit around the Milky Way and thereby lose kinetic energy. Consequently, their orbit will become smaller over the course of time and, in a distant future, these two satellite galaxies will merge with our Galaxy. In fact, dynamical friction is of vital importance in galaxy merger processes which occur in the evolution of the galaxy population, a subject we will return to in Sect. 9.6.

From Peter Schneider's Textbook: Extragalactic Astronomy & Cosmology

The dynamical friction time scale:

$$t_{dyn.fric} \sim \frac{v^3}{2\pi G^2 M_t \rho \ln N}$$

ρ = density of surrounding particles.

Caution: Unless dealing with globular clusters, gastrophysics has a significant impact on evolution, interactions etc.

involves gas physics beyond just gravity

Dynamics in elliptical galaxies

$\varepsilon = 1 - \frac{b}{a}$ = ellipticity

v_{rot} = rotational velocity of stars/gas.
- measure from the shift in doppler lines

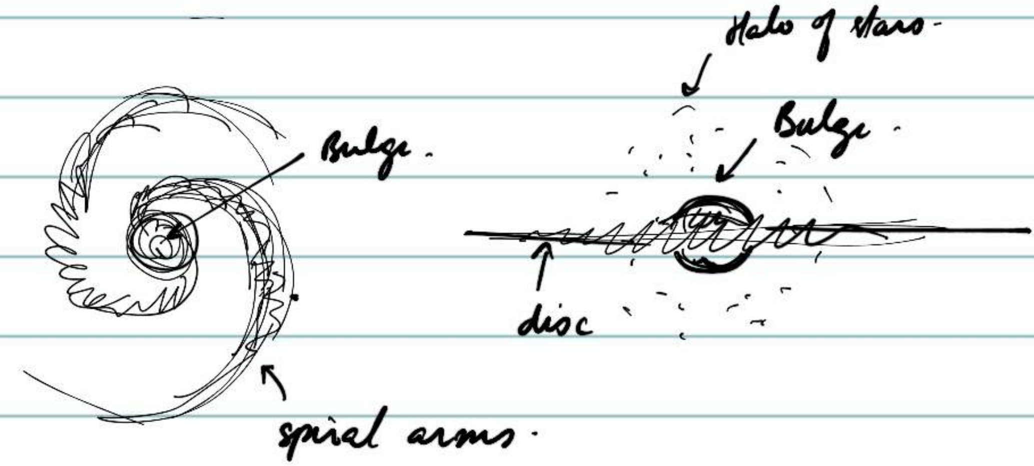
σ_v = velocity dispersion
- measured by broadening of the lines

For ellipticals, assuming a isotropic velocity distribution

$$\Rightarrow \left(\frac{v_{rot}}{\sigma_v}\right)_{iso} \approx \sqrt{\frac{\varepsilon}{1-\varepsilon}}.$$

However measured $|v_{rot}| \ll \sigma_v$!
Hence it is likely that the velocity distribution is anisotropic &/or something else is needed for large σ_v !

Rotation curves for spirals



$v_{rot}(r)$ measure from motion of stars & 21 cm line from neutral hydrogen (doppler shifts).

Gravitational dynamics & rotation curves

Review:

$$\nabla^2 \Phi = 4\pi G \rho$$

gravitational potential (Φ); mass density (ρ)

In a spiral galaxy, the central bulge, as well as halo of stars can be approximated as being spherically symmetric.

For spherical symmetry: $\rho(\vec{r}) = \rho(r)$

$\Phi(\vec{r}) = \Phi(r)$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \rho(r)$$

For circular orbits, the velocity

$$v_c^2(r) = r\frac{\partial \phi}{\partial r} = \frac{GM(r)}{r}$$ where

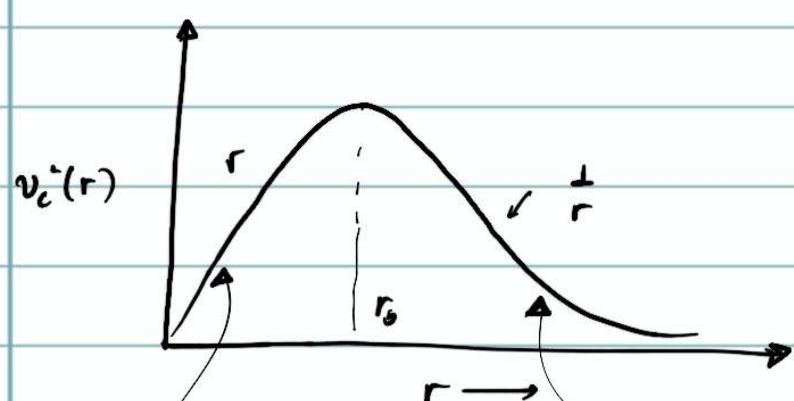
$M(r) = 4\pi\int dr'\, r'^2 \rho(r')$ = mass enclosed with the region r.

Assuming a constant density for the central bulge

$$\rho(r) = \begin{cases} \rho_0 & r < r_0 \\ 0 & r > r_0 \end{cases}$$ (very crude!).

$$v_c^2(r) = \frac{GM(r)}{r} \propto \begin{cases} r & r < r_0 \\ \frac{1}{r} & r > r_0 \end{cases}$$ [$\because$ mass enclosed does not change after r_0]

Bulge.



For milky way $r_0 \sim 1$ kpc.

depends on details of ρ this does not!

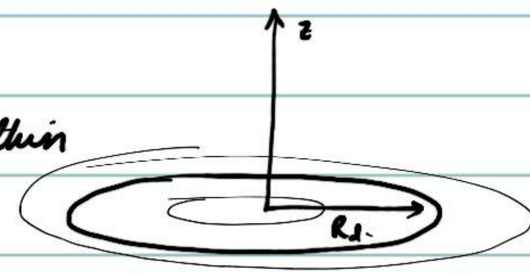
For the halo of stars, the density also falls of rather steeply. For example, in the milky way $\rho_{halo} \sim r^{-3.5} \Rightarrow v_c^2(r) \propto r^{-3/2}$ due to halo stars.

The disc of the milky way is obviously not spherically symmetric. We can approximate it by a axisymmetric disc (ignore spiral arms for the moment).

Axisymmetric disc.

$\rho(\vec{r}) = \Sigma(R)\,\delta(z)$

infinitely thin



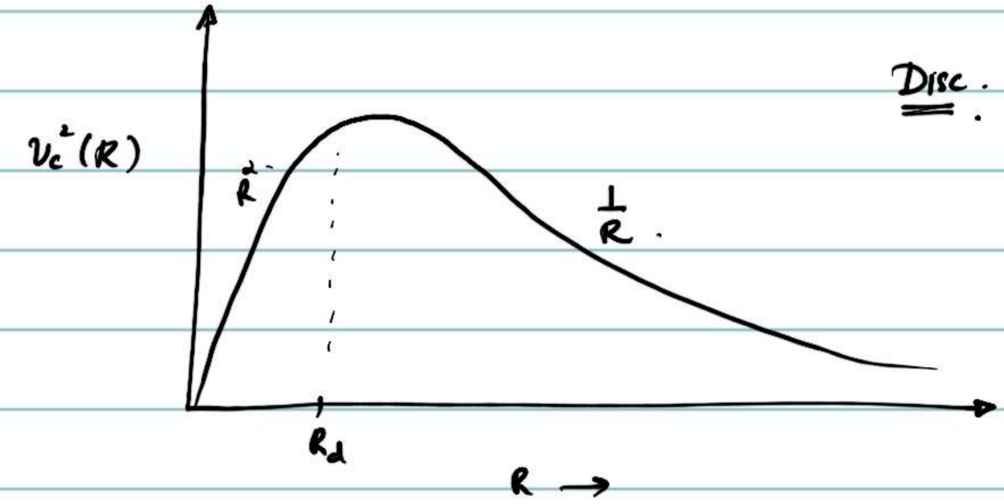
Let $\Sigma(R) = \Sigma_d\, e^{-R/R_d}$ (for milky way, $R_d \sim 3\,\text{kpc}$)

The circular velocity in the plane of the disc would be given by

$$v_c^2(R) = R\frac{\partial \Phi}{\partial R} = 4\pi G \Sigma_0 R_d\, y^2 \left[I_0(y)K_0(y) - I_1(y)K_1(y) \right]$$

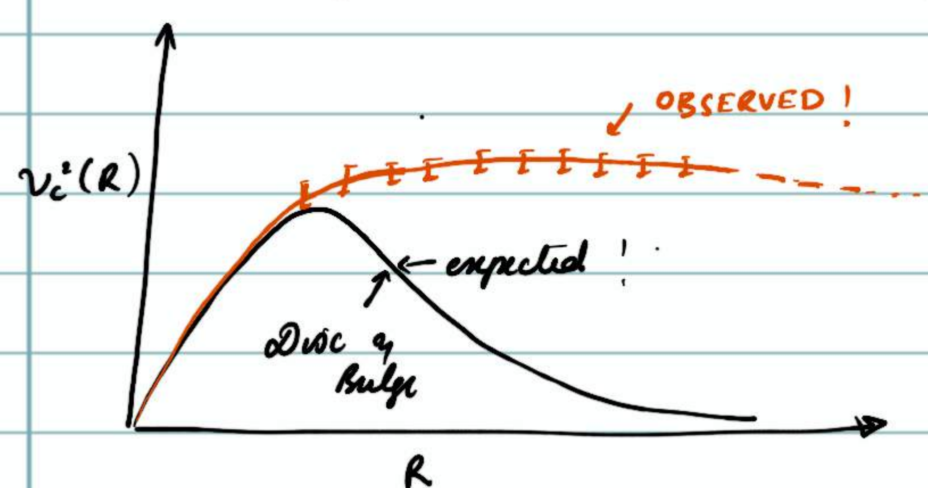
$y = R/2R_d$; K & I = Bessel functions.

& $M(R) = 2\pi \Sigma_0 R_d^2 \left[1 - e^{-R/R_d}\left(1 + \frac{R}{R_d}\right) \right]$



So from the bulge, (stellar) halo & disc we get a a rotation curve that falls off as $v^2 \propto 1/R$ beyond the region where most of the visible mass is.

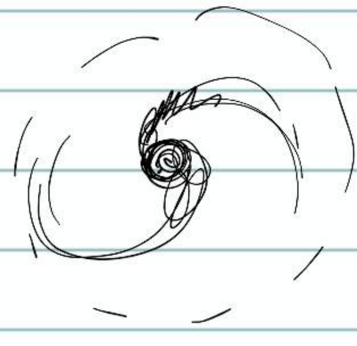
However, for a large sample of galaxies.

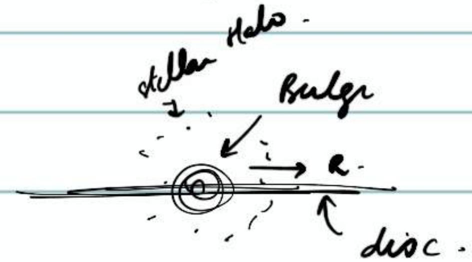


Lecture 24

Review :

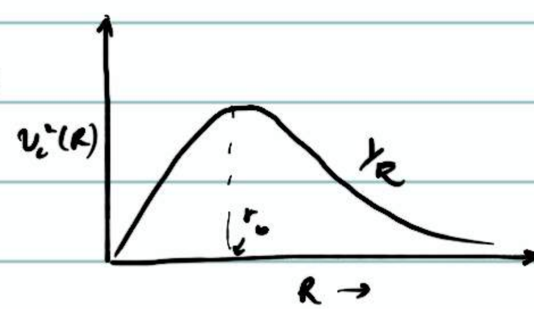
Rotation curves for spiral galaxies.





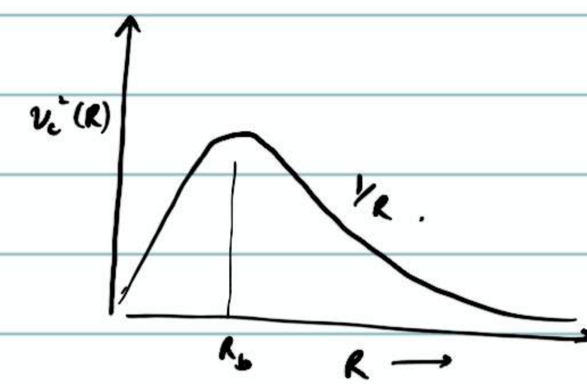
Bulge :
$$v_c^2(R) = -R\frac{\partial \Phi}{\partial R} = \frac{GM(R)}{R}$$

$$\rho = \begin{cases} \rho_b & r \le r_b \\ 0 & r > r_b \end{cases}$$



Thin
Disc :
$$v_c^2(R) = -R\frac{\partial \Phi}{\partial R}.$$

$$\rho = \Sigma(R)\,\delta(z)$$
$$\Sigma(R) = \Sigma_d\, e^{-R/R_d}.$$

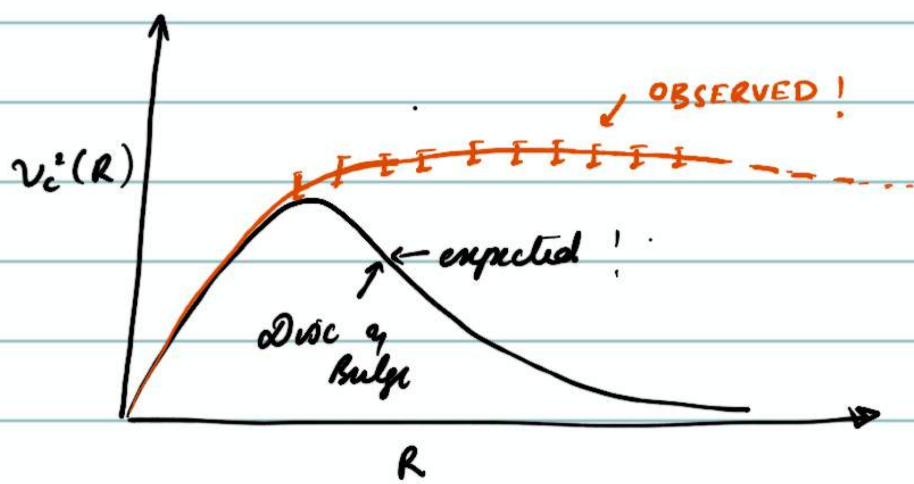


Stellar Halo : $\rho \propto r^{-3.5}$, $v_c^2(R) \propto R^{-3/2}$

$\propto 1/R$ beyond the visible.

So from the bulge, (stellar) halo & disc we get a a rotation curve that falls off as $v^2 \propto 1/R$ beyond the region where most of the visible mass is.

However, for a large sample of galaxies.



How to get $v_c^2(R) \sim$ const ?
$R > R_d$

for spherical symmetry

$$v_c^2(r) = \frac{GM(r)}{r} = \text{const} \Rightarrow M(r) = 4\pi \int dr' \, r'^2 \rho(r') \propto r$$

$$\Rightarrow \rho(r) \propto \frac{1}{r^2}$$

Thus an additional "invisible" component with $\rho(r) \propto \frac{1}{r^2}$ would work!

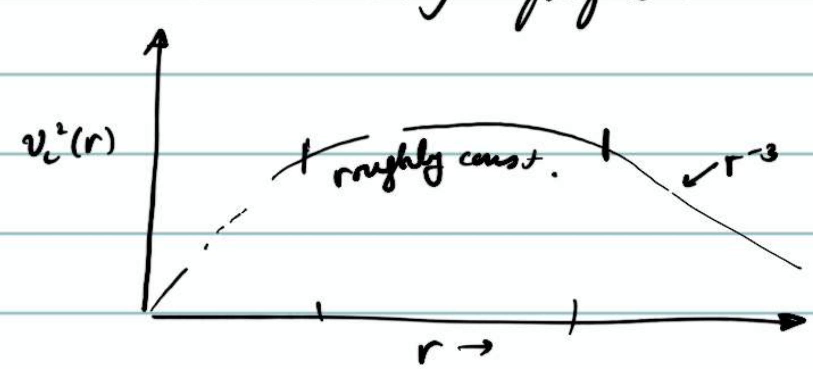
Example $\rho(r) = \dfrac{\rho_0}{1 + \left(\frac{r}{a}\right)^2}$ would work with appropriate choice of a & ρ_0.

A good choice based on numerical simulation of dark matter

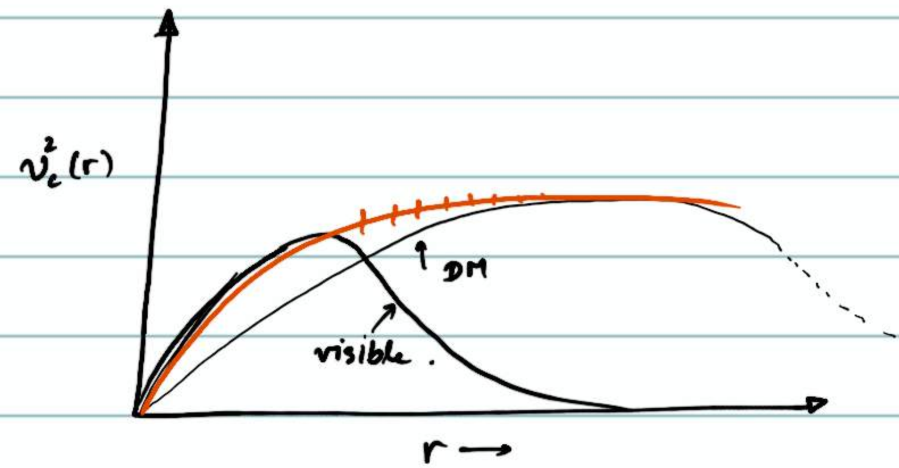
$$\rho(r) = \frac{\rho_0}{\left(\frac{r}{a}\right)\left(1 + \left(\frac{r}{a}\right)^2\right)}$$

(needs a bit more modification at $r \to 0$).

For this density profile.

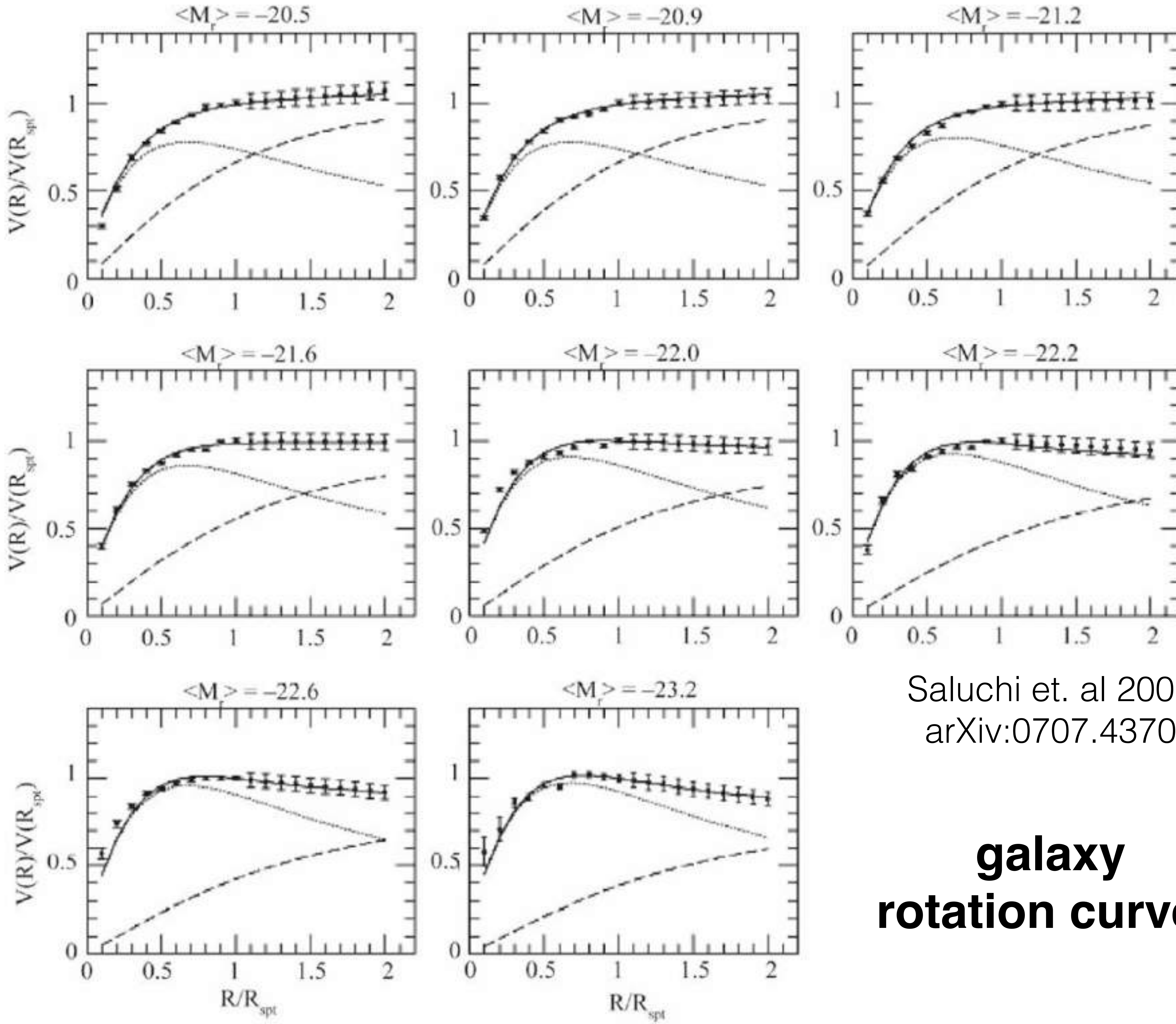


Thus we can get agreement with the observations by introducing "dark matter".



Similar discrepancy and fits possible for many, many galaxies!

Refer to slides for the lecture.



Saluchi et. al 2007
arXiv:0707.4370

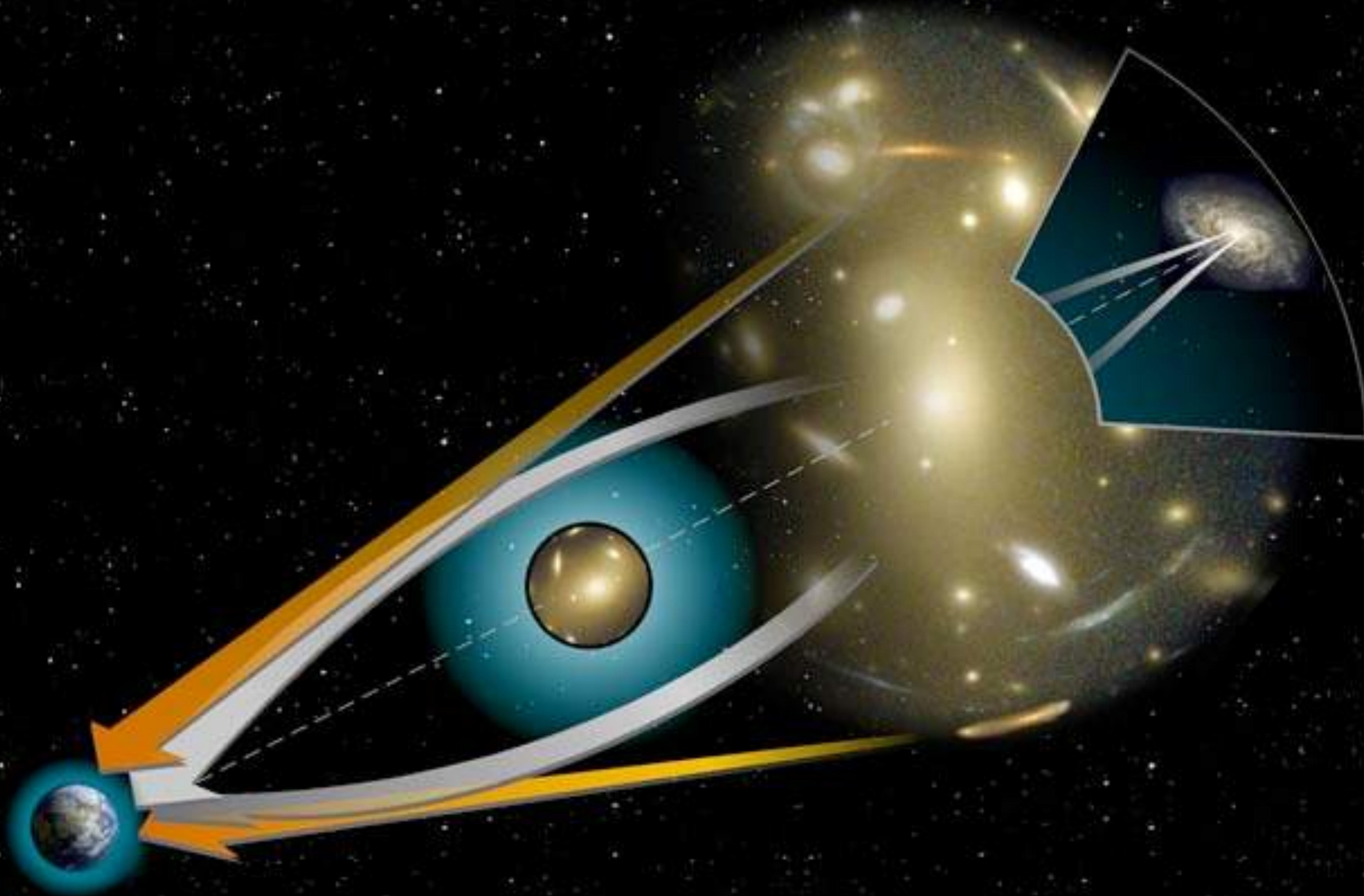
galaxy rotation curves

dark matter halo: simulations

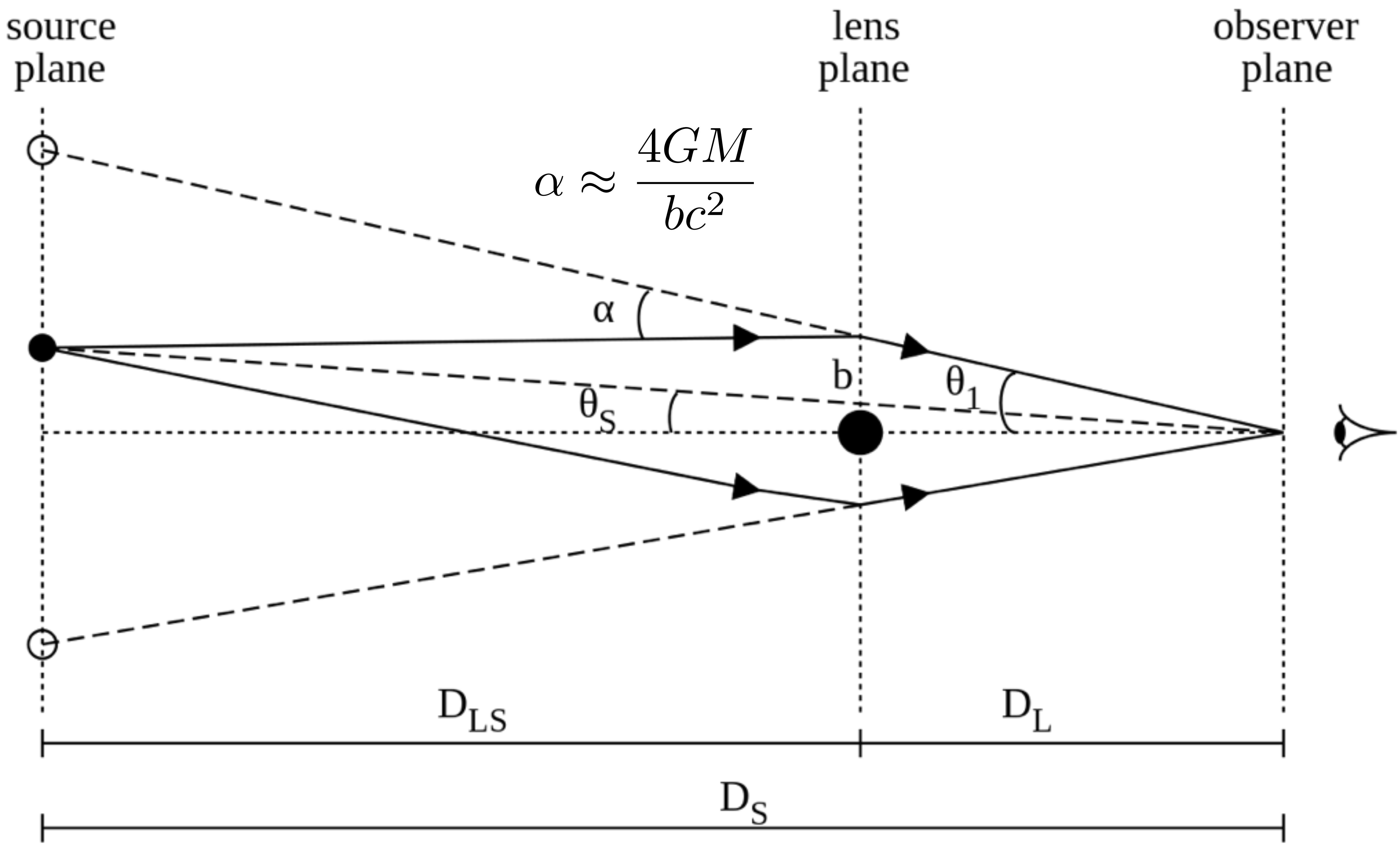
Milky Way sized halo, Aquarius simulation, 200 Million particles

Digression: Astrophysical Case for Dark Matter

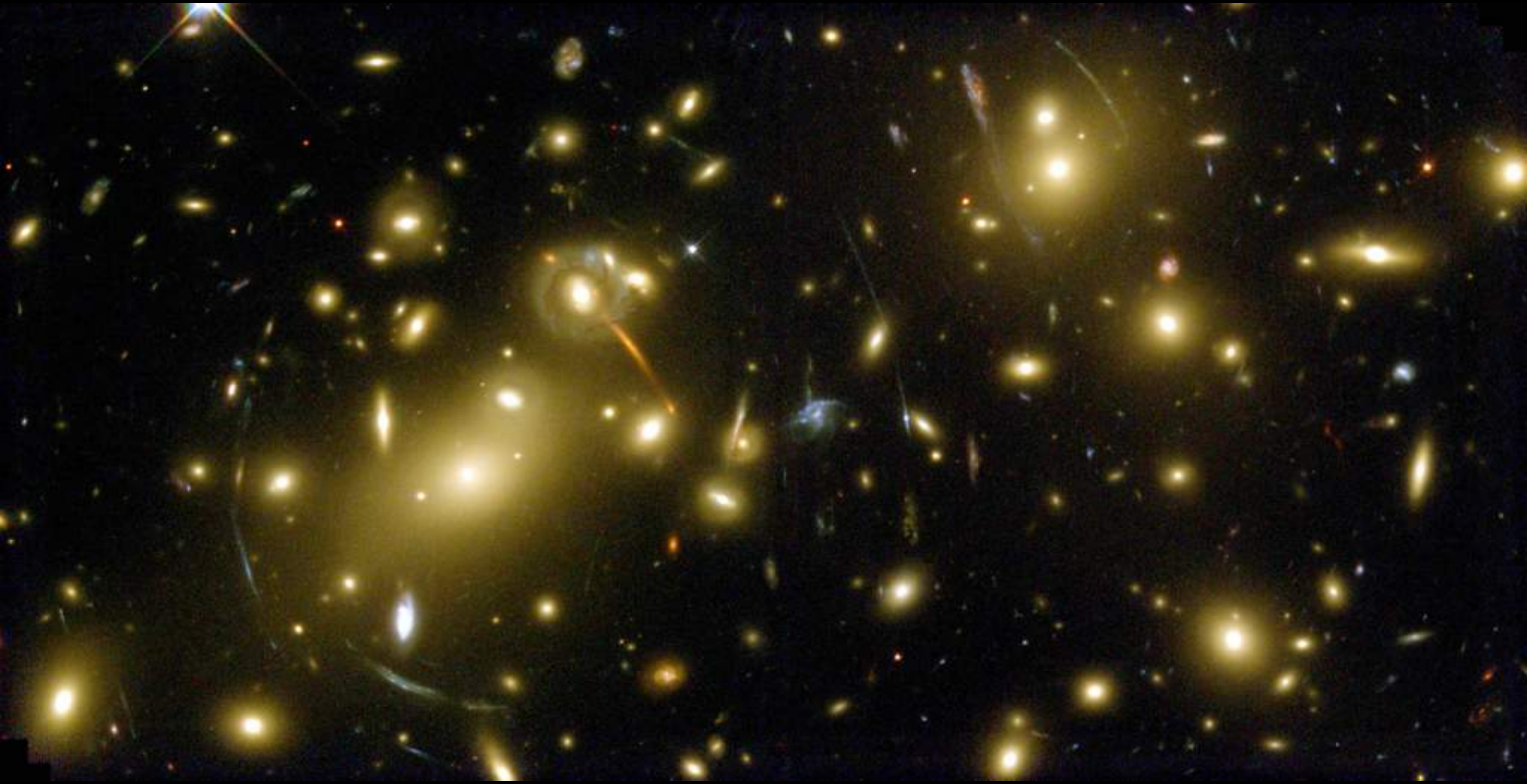
gravitational lensing



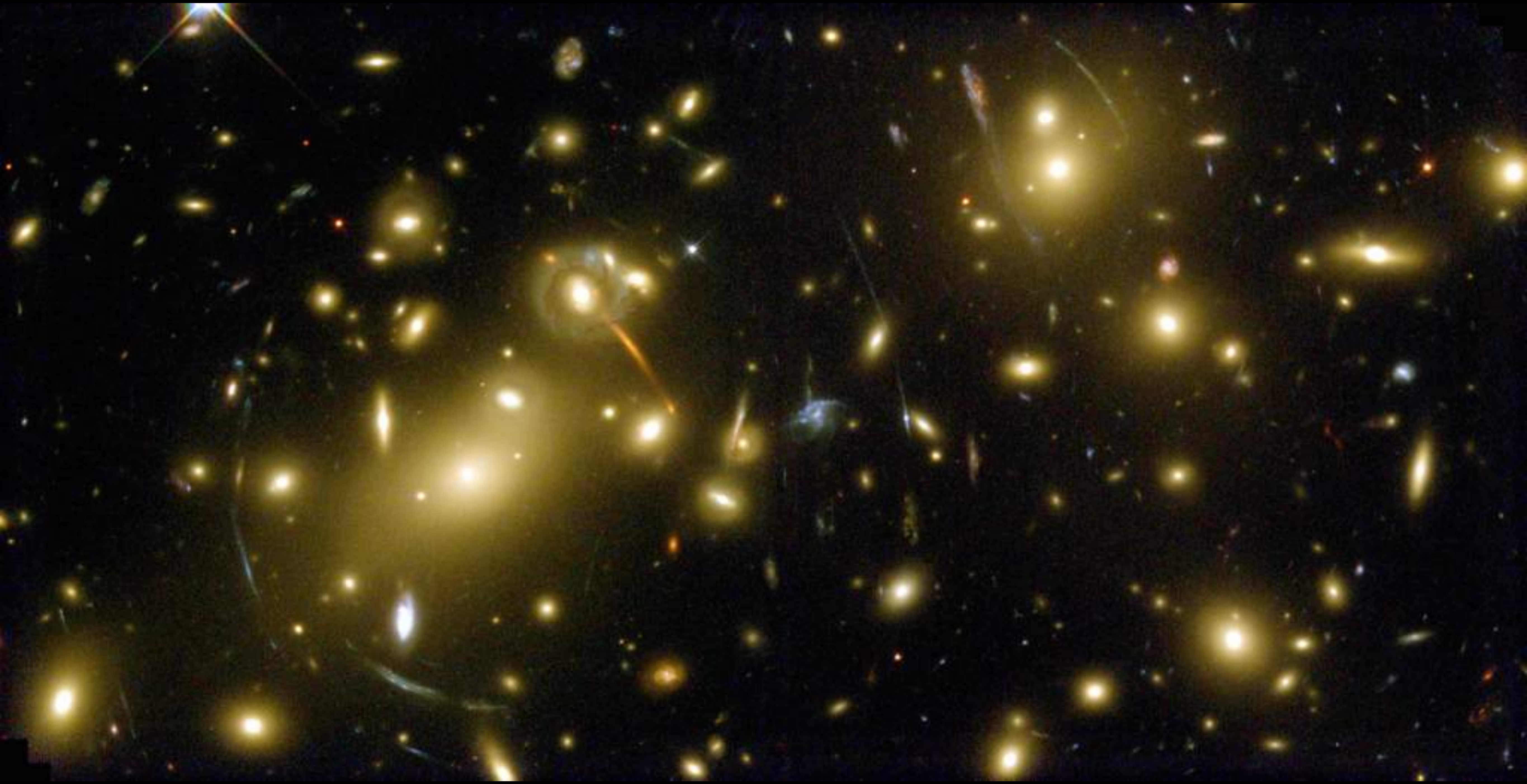
gravitational lensing

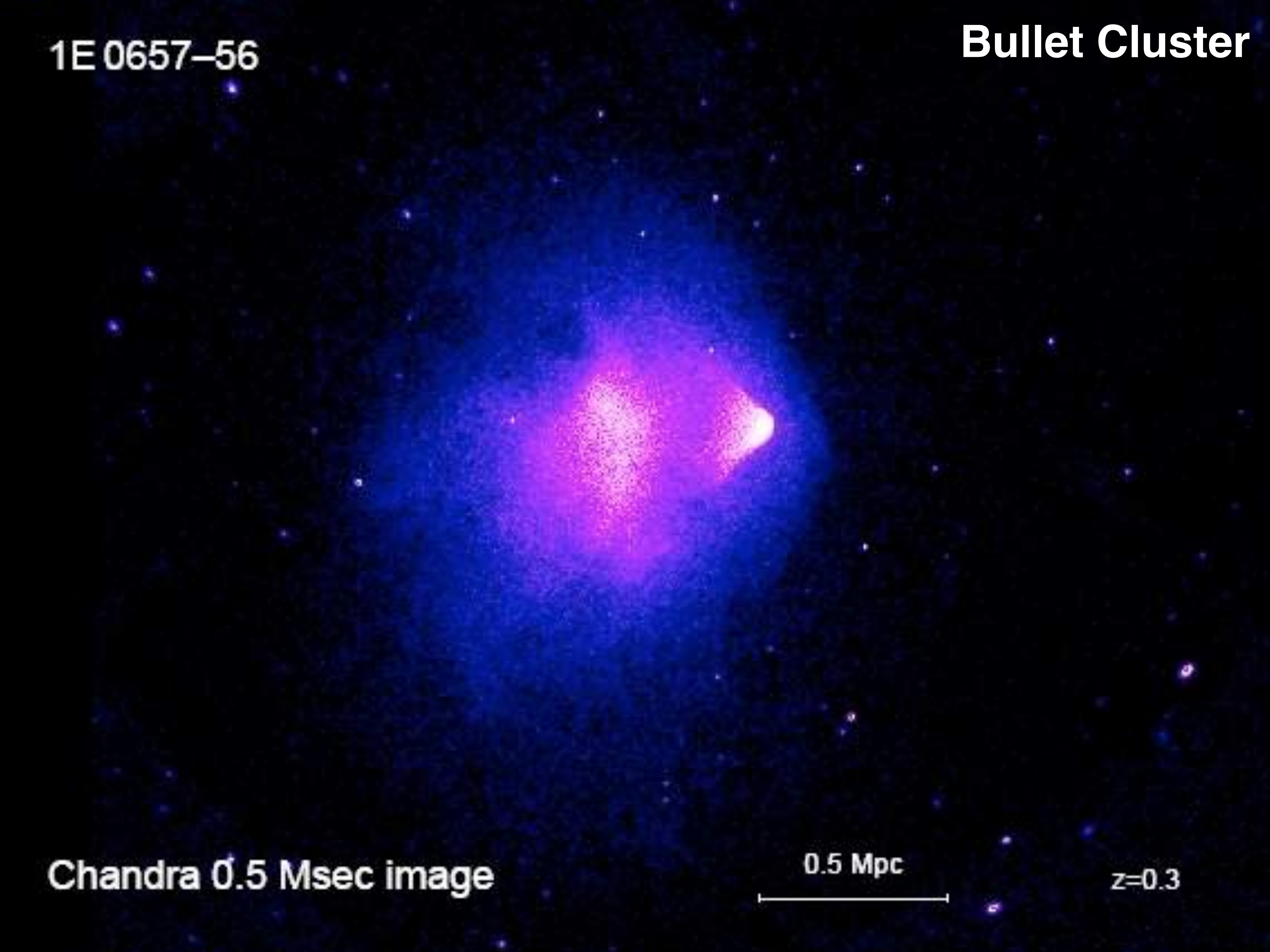


distortion of images



too much distortion!

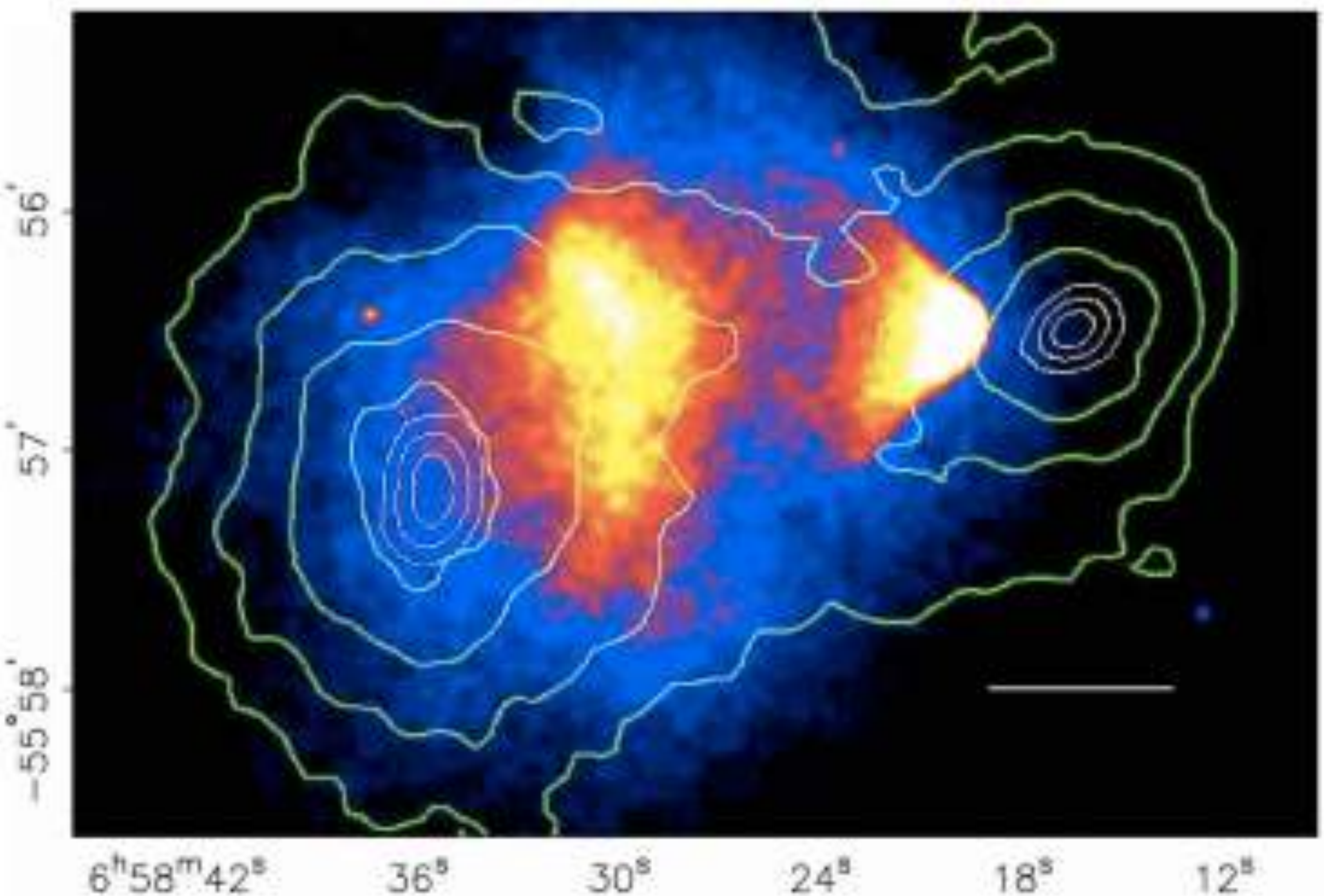




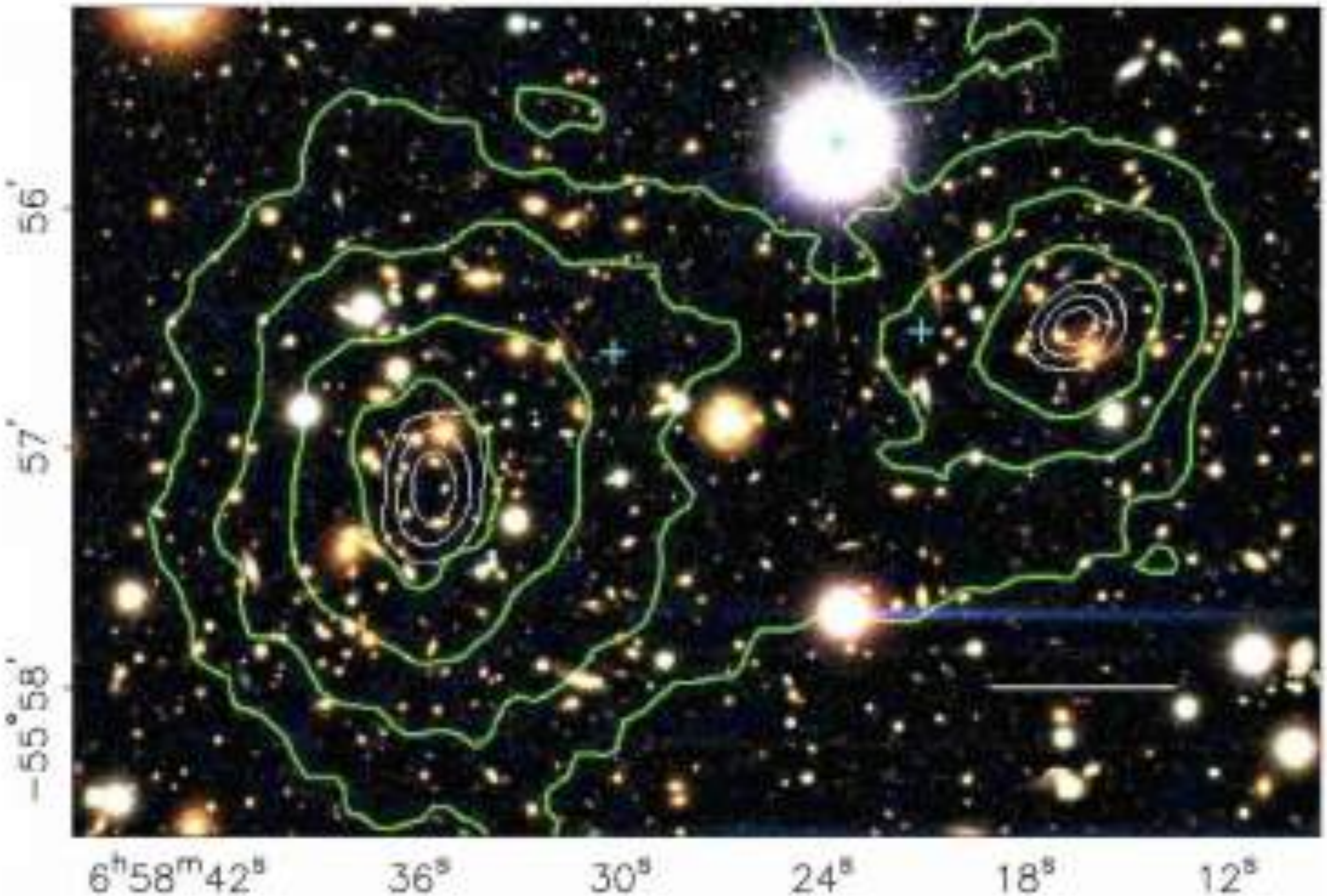
1E 0657–56
Bullet Cluster
Chandra 0.5 Msec image
0.5 Mpc
z=0.3

contours: mass distribution from lensing

Clowe et. al 2006



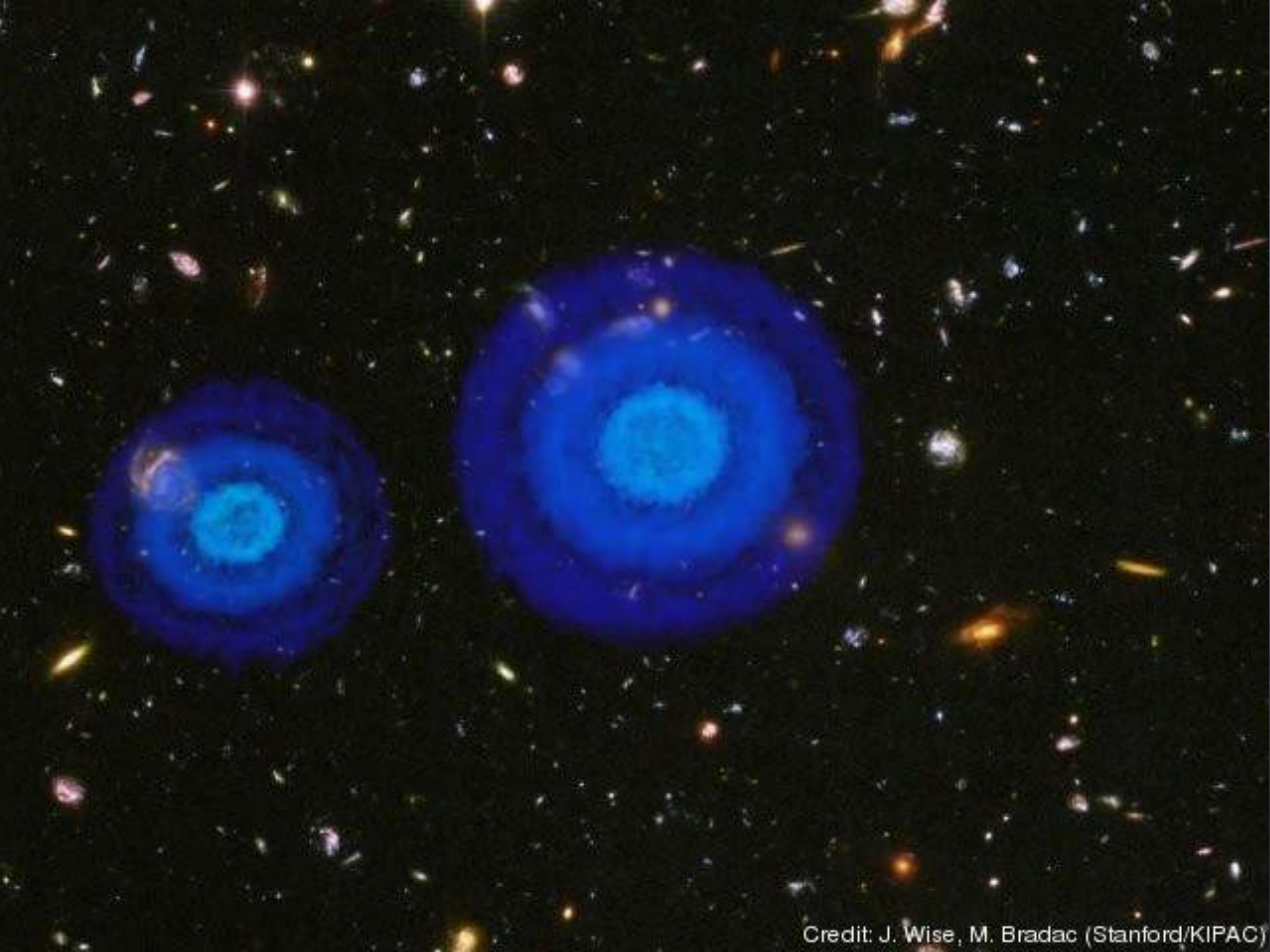
contours: mass distribution from lensing



optical

NOT enough “visible matter”

Mass discrepancy = 10 x baryonic matter



Credit: J. Wise, M. Bradac (Stanford/KIPAC)

important evidence of dark matter ?

some debate about large collision velocity


hot stuff evaporates

galaxy clusters are too hot!

need more invisible mass!

closer to home …

Rotation Curve for the Milky Way

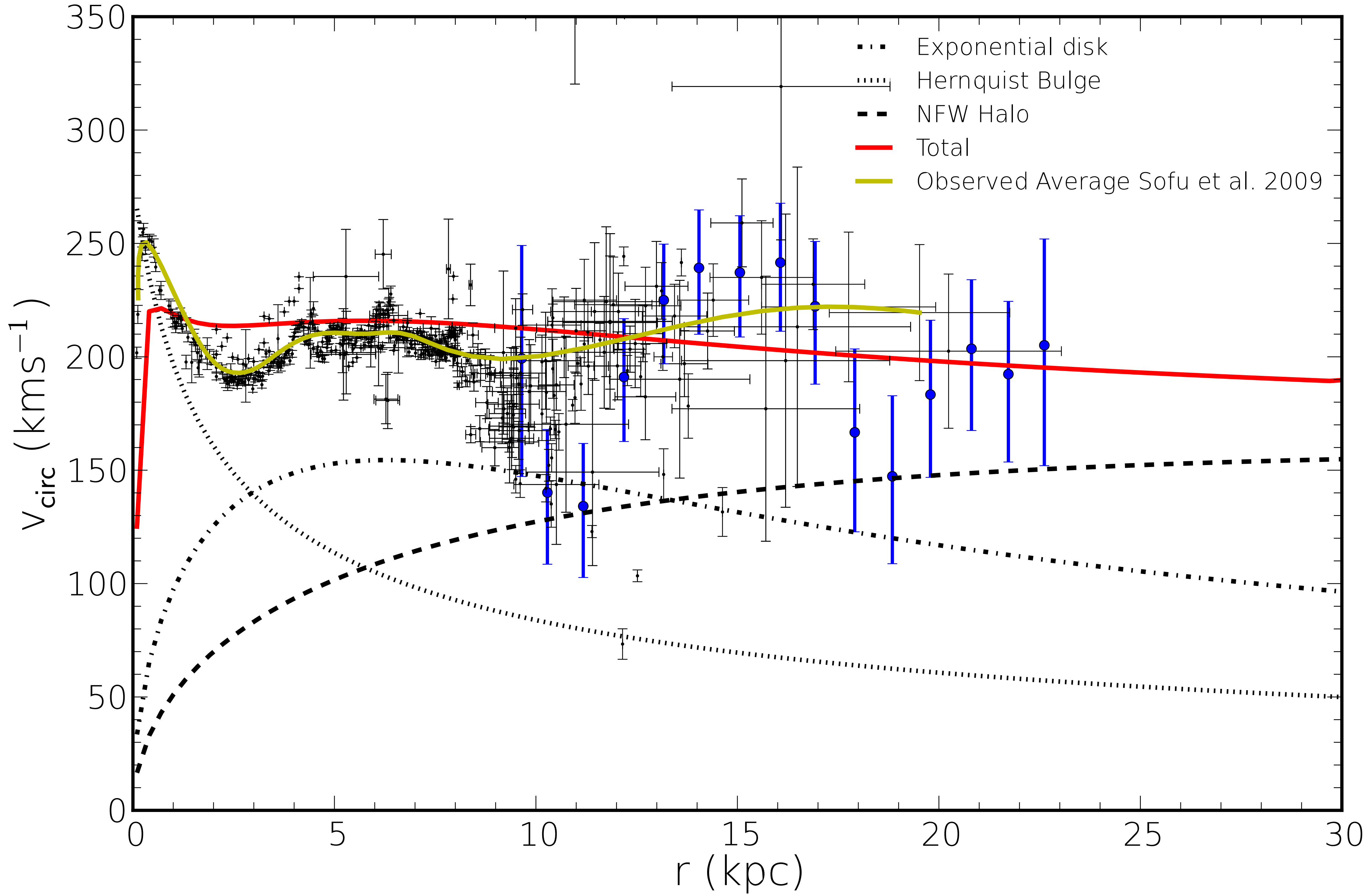
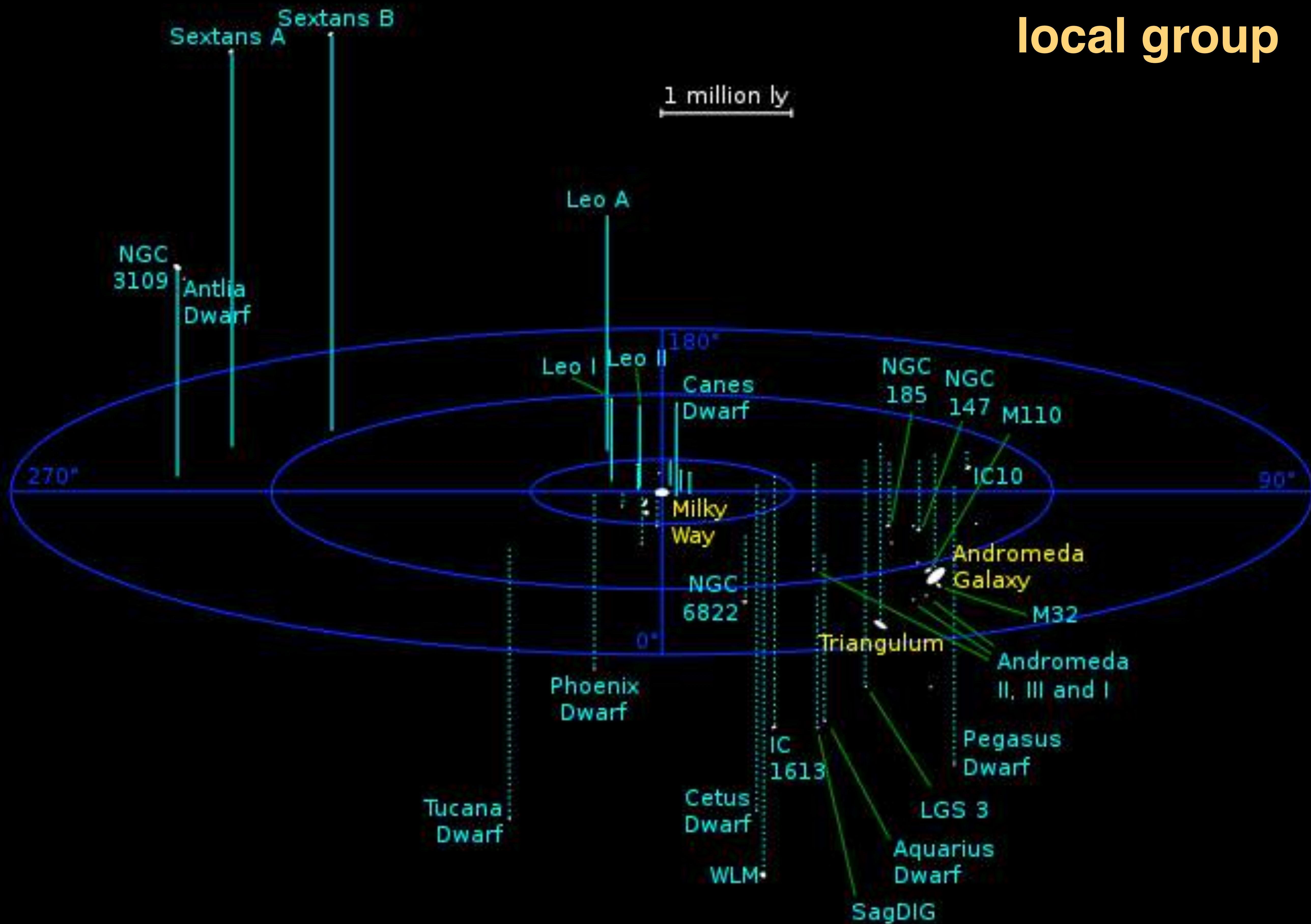


Figure 10. Circular velocity curve of the Galaxy and their individual components along a galactocentric distance (r). The blue marker represents the value of $v_{\rm circ}$ obtained in the CME bins in r. Red solid line is our fit of the total potential. Black dotted and dotted-dashed lines are the fixed disk and the bulge circular velocity profile for set of adopted values of masses and scale radii. Dashed line is the fitted NFW profile. Black dots with error bars are the collated $v_{\rm circ}$ values given by Sofue et al. (2009) whereas yellow solid line is the average of the given observed values.

local group



Sextans B
Sextans A
1 million ly
Leo A
NGC 3109
Antlia Dwarf
180°
Leo I
Leo II
Canes Dwarf
NGC 185
NGC 147
M110
270°
IC10
90°
Milky Way
Andromeda Galaxy
NGC 6822
M32
0°
Triangulum
Andromeda II, III and I
Phoenix Dwarf
Pegasus Dwarf
IC 1613
Tucana Dwarf
Cetus Dwarf
LGS 3
Aquarius Dwarf
WLM
SagDIG


local group dwarf spheroidal galaxy: NGC 147
near andromeda

https://arxiv.org/abs/0708.0010v1

velocity dispersion in dwarf spheroidals

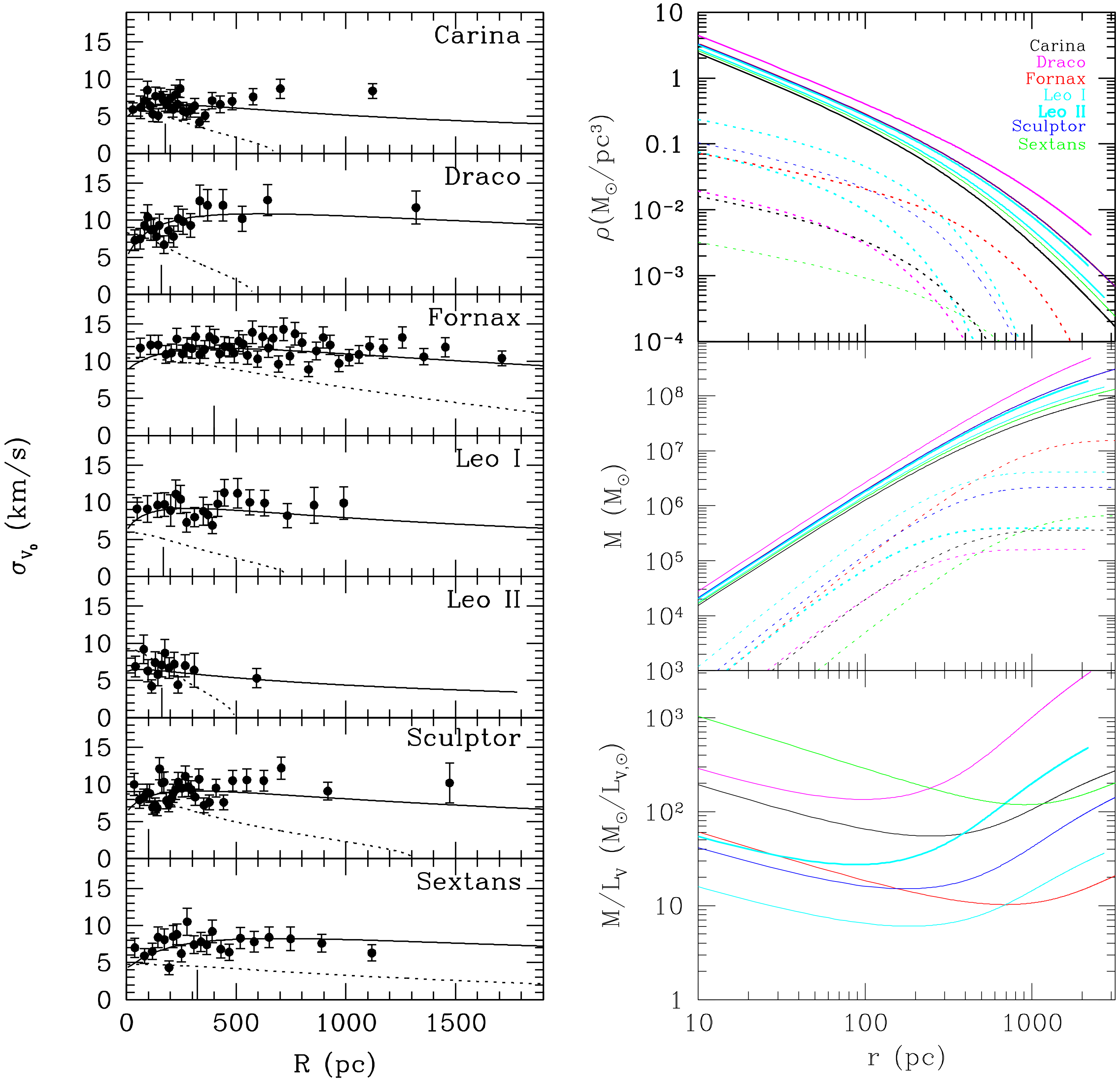
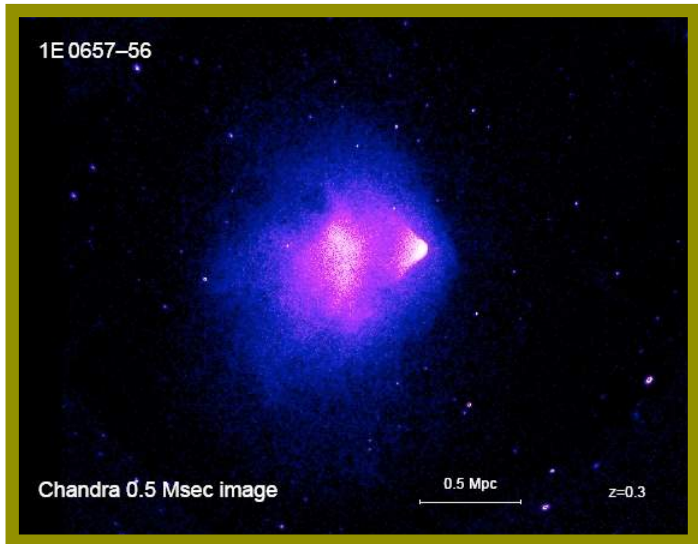
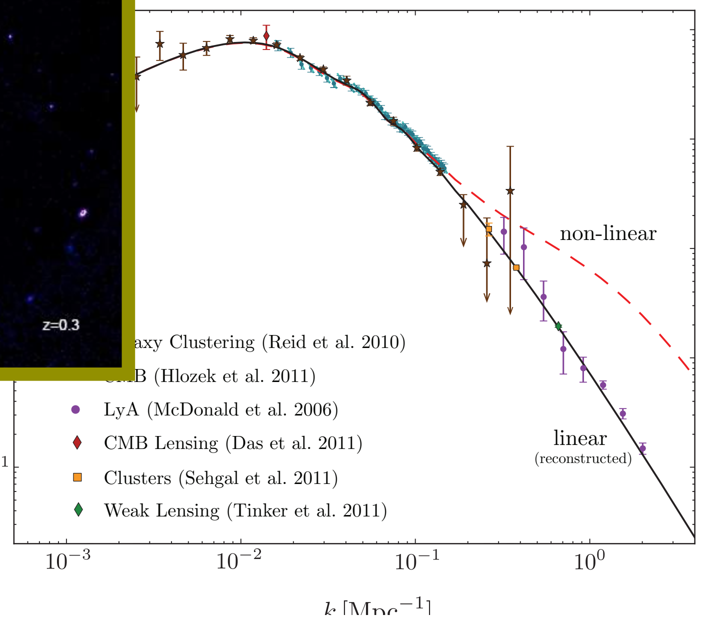
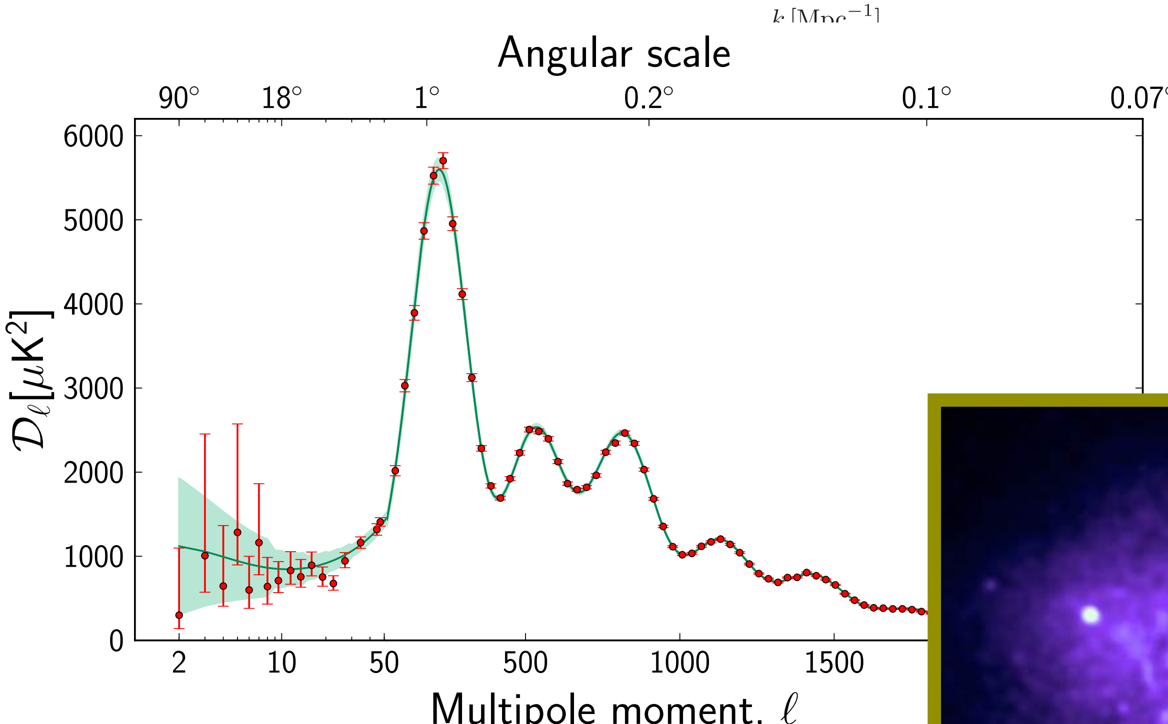


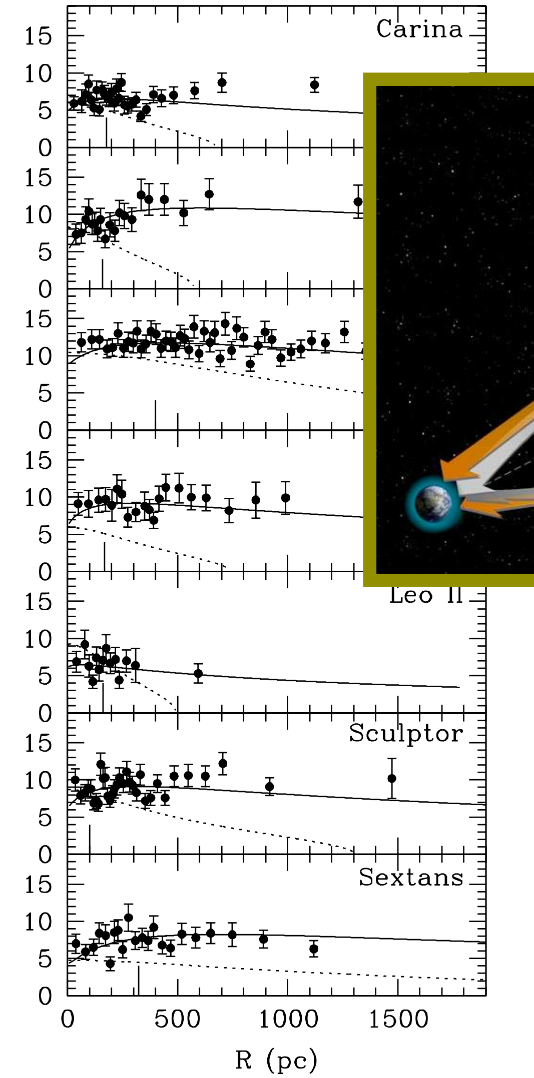
FIG. 2.— *left:* Projected velocity dispersion profiles for seven Milky Way dSph satellites. Overplotted are profiles corresponding to mass-follows light (King 1962) models (dashed lines; these fall to zero at the nominal "edge" of stellar distribution), and best-fitting NFW profiles that assume constant velocity anisotropy. Short, vertical lines indicate luminous core radii (IH95). Distance moduli are adopted from Mateo (1998). *right:* Solid lines represent density, mass and M/L profiles corresponding to best-fitting NFW profiles. Dotted lines in the top and middle panels are baryonic density and mass profiles, respectively, following from the assumption that the stellar component (assumed to have M/L=1) has exponentially falling density with scale length given by IH95.

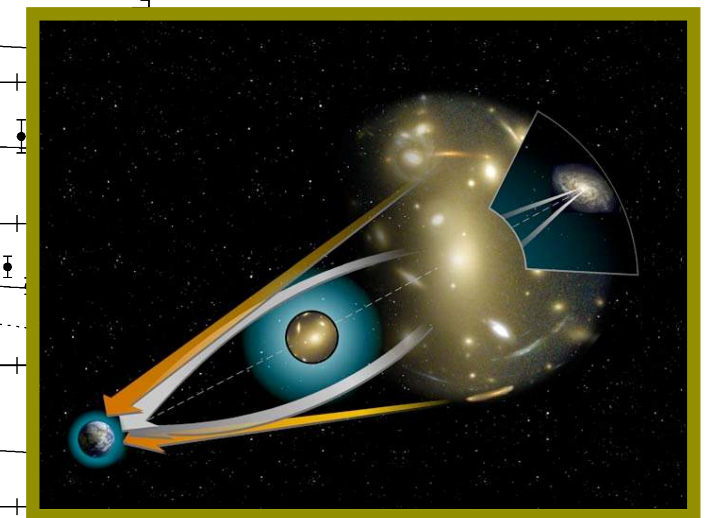
Astrophysical/Cosmological Case For Dark Matter

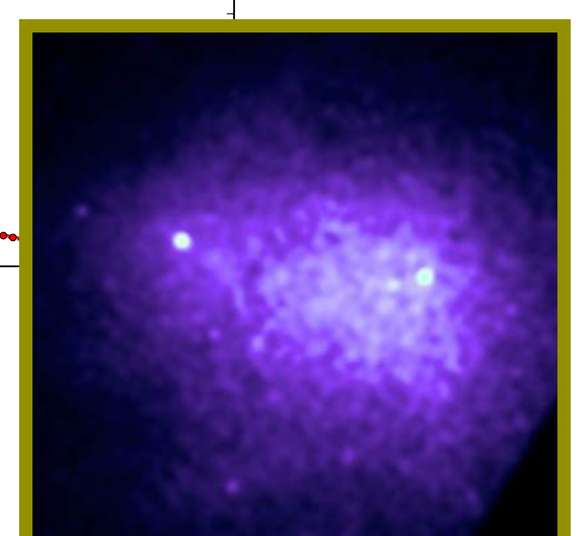


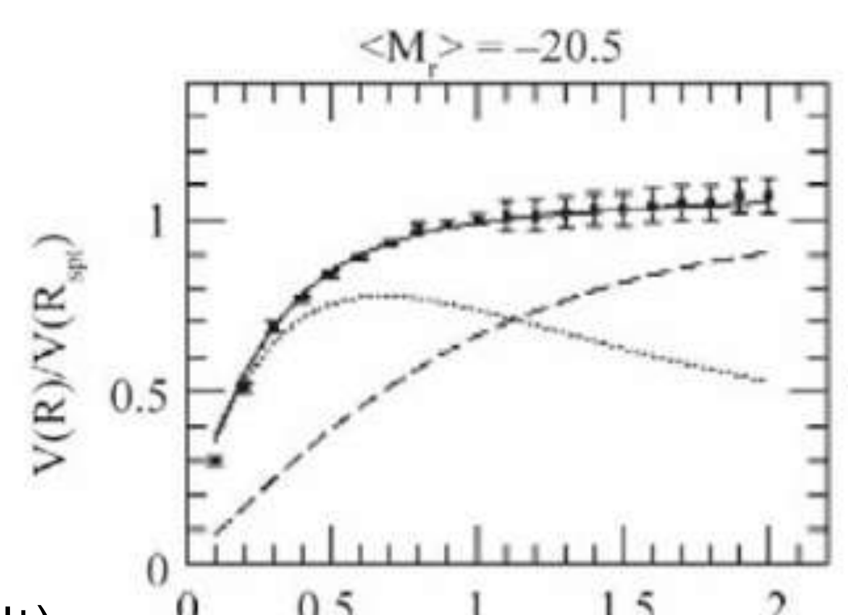












or some non-trivial modification of gravity … (CMB is difficult)

NO non-gravitational detection yet!

- **Indirect detection** — decay/ annihilation products)
- **Direct detection** — accelerators (LHC), underground detectors (Xenon, CDMS)

Also, work continues on modifications of gravity …

Lecture 25

Plan: 1) a taste of modified gravity theories
2) Scaling Relations
3) SMBHs.

Review:

- gravitational dynamics - virial theorem
- relaxation mechanisms & time scales.

Ellipticals - velocity dispersion
Spirals - rotation curves.

missing matter (dark).

gravitational lensing Cluster temp. Bullet Cluster
dwarf spheroidals Large scale structure CMB & BBN.

BBN, CMB $\Rightarrow$ missing matter not baryonic!

Most of the evidence = gravitational effects.
Search = direct detection. • CDMS, Xenon. etc
indirect detection. • annihilation etc.

Could Newton / Einstein be wrong?

1) **MOND** : Modified Newtonian dynamics. (Milgrom)

"Toy" model.

Newton : $F = ma$ Einstein ~ Newton

$$\Rightarrow \frac{GMm}{r^2} = ma$$

$$\Rightarrow a = \frac{GM}{r^2}$$

$$\Rightarrow v^2 = \frac{GM}{r}$$ [source of trouble for rotation curves]

Milgrom : $f\left(\frac{a}{a_0}\right)\frac{GMm}{r^2} = ma$ $\qquad f\left(\frac{a}{a_0}\right) \overset{?}{=} \left(1 + \frac{a_0}{a}\right)$.

modification.

$$\Rightarrow v^2 = \frac{GM}{r}\left(1 + \frac{a_0}{a}\right)$$

For $a \ll a_0$

$$\approx \frac{GM}{r}\left(\frac{a_0}{a}\right)$$

But $a = \frac{v^2}{r} \propto \frac{1}{r}$

$$\approx \frac{GM}{\cancel{r}}\,\frac{a_0}{\frac{v^2}{\cancel{r}}}$$

$a_0 \approx 10^{-10}\ m/s^{-2}$

$$\Rightarrow v^2 \approx \sqrt{GMa_0}$$

$a_{SE} \sim 10^{-3}\ m/s^2$.

sun-earth.

constant! as needed.

- Different function f are needed for different systems
- Still need additional invisible matter for clusters,
- Bullet Cluster difficulties - Why effect displaced from source?
- Detailed comparison with CMB missing/inconsistent!

More sophisticated models exist - Relativistic [TeVeS ↑ Bekenstein] example.

- but introduces additional fields ↖ Dark!

<u>In General</u>: Hard to modify GR without ruining other predictions that are successful.

The Quest continues! (especially because of DE).

Scaling Relations for Galaxies

Spirals
Tully-Fischer

$$L \propto v_{\mathrm{rot(max)}}^4$$

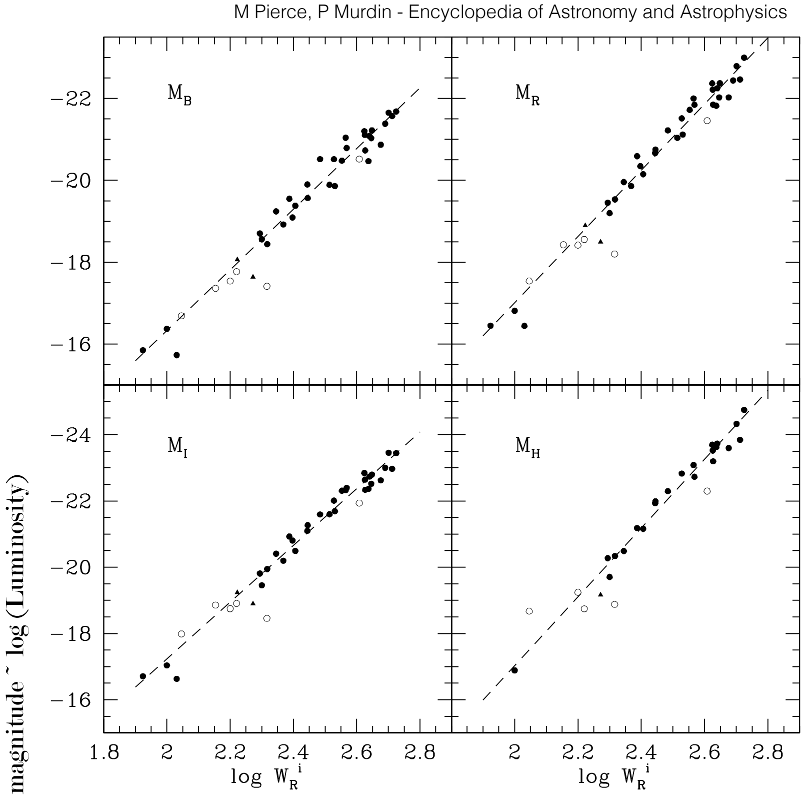
Ellipticals:
Faber-Jackson

$$L \propto \sigma_0^4$$

SMBHs
$M - \sigma$

$$M_{\mathrm{BH}} \approx 10^{8.32} M_\odot \left(\frac{\sigma}{200\,\mathrm{km\,s^{-1}}} \right)^{5.64}$$

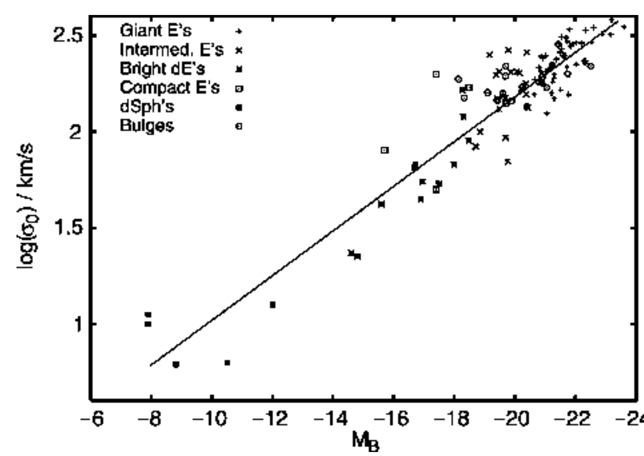
M Pierce, P Murdin - Encyclopedia of Astronomy and Astrophysics



~ log(projection corrected rotational velocity)

Figure 1. *B*, *R*, *I* and *H* band calibrations of the Tully–Fisher relation. Solid circles are galaxies with distances measured using Cepheids, solid triangles are galaxies with distances estimated via surface brightness fluctuation measurements within dE companions and open circles are systems thought to be group members with at least one galaxy with a Cepheid distance, and therefore thought to be at a similar distance. The dashed line is a least-squares fit to the solid points in each panel.

Schneider 2006



magnitude ~ log (Luminosity)

Fig. 3.22. The Faber–Jackson relation expresses a relation between the velocity dispersion and the luminosity of elliptical galaxies. It can be derived from the virial theorem

McConnell & Ma 2012

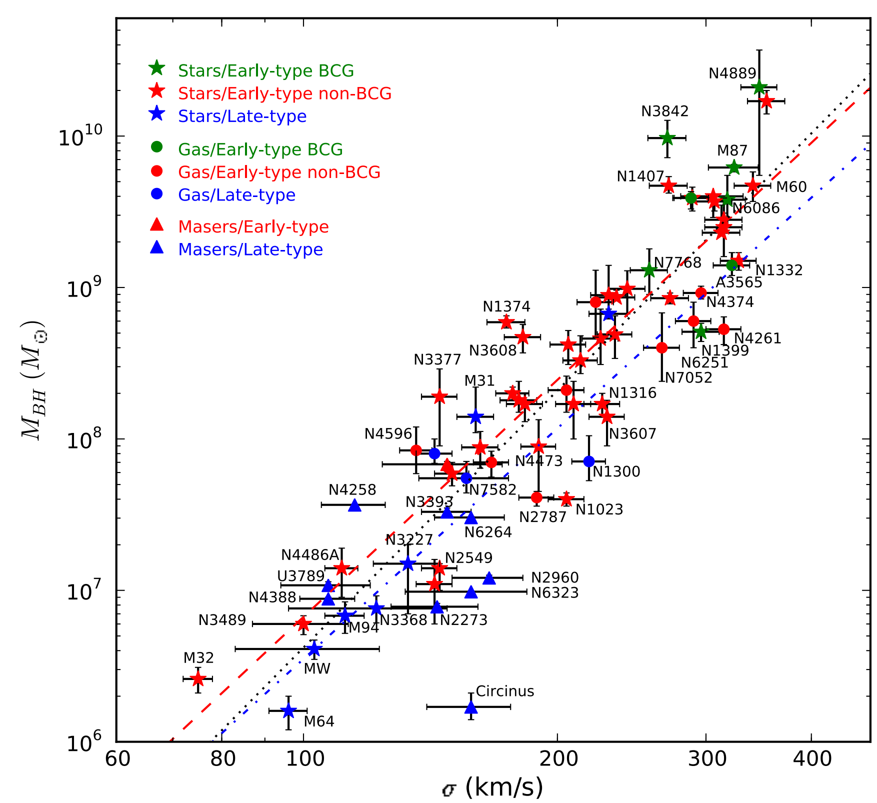


FIG. 1.— The $M_\bullet - \sigma$ relation for our full sample of 72 galaxies listed in Table A1 and at `http://blackhole.berkeley.edu`. Brightest cluster galaxies (BCGs) that are also the central galaxies of their clusters are plotted in green, other elliptical and S0 galaxies are plotted in red, and late-type spiral galaxies are plotted in blue. NGC 1316 is the most luminous galaxy in the Fornax cluster, but it lies at the cluster outskirts; the green symbol here labels the central galaxy NGC 1399. M87 lies near the center of the Virgo cluster, whereas NGC 4472 (M49) lies ~ 1 Mpc to the south. The black-hole masses are measured using the dynamics of masers (triangles), stars (stars) or gas (circles). Error bars indicate 68% confidence intervals. For most of the maser galaxies, the error bars in $M_\bullet$ are smaller than the plotted symbol. The black dotted line shows the best-fitting power law for the entire sample: $\log_{10}(M_\bullet/M_\odot) = 8.32 + 5.64 \log_{10}(\sigma/200\,\mathrm{km\,s^{-1}})$. When early-type and late-type galaxies are fit separately, the resulting power laws are $\log_{10}(M_\bullet/M_\odot) = 8.39 + 5.20 \log_{10}(\sigma/200\,\mathrm{km\,s^{-1}})$ for the early-type (red dashed line), and $\log_{10}(M_\bullet/M_\odot) = 8.07 + 5.06 \log_{10}(\sigma/200\,\mathrm{km\,s^{-1}})$ for the late-type (blue dot-dashed line). The plotted values of σ are derived using kinematic data over the radii $r_{\mathrm{inf}} < r < r_{\mathrm{eff}}$.

Scaling relations :

1. Tully Fisher (Spirals) $L \propto v_{max}^4$
2. Faber Jackson (Ellipticals) $L \propto \sigma_0^4$ (approximate).
3. $M-\sigma$ (SMBHs). $M \propto \sigma^{5-6}$

The virial theorem often plays an important role. To understand aspects of the above relations we also need to introduce "surface brightness".

Surface brightness $I = \frac{dF}{d\Omega}$: $\frac{\text{flux}}{\text{solid angle}}$ (at observer).

flux $\propto \frac{1}{d^2}$, solid angle $\propto \frac{1}{d^2}$ ← distance.

$\Rightarrow$ I is independent of distance from the source!

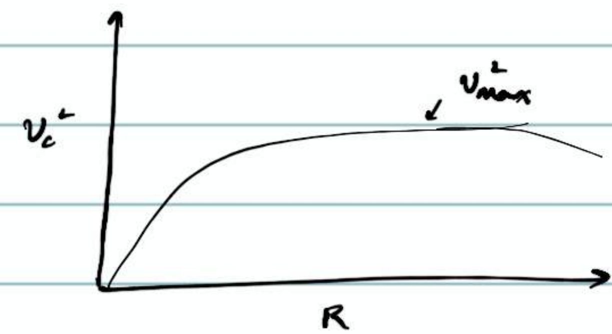
$\langle I \rangle \equiv$ avg surface brightness within a radius R of the source disc.

$\langle I \rangle = \frac{L}{R^2}$ ← luminosity.

Explaining Tully Fischer (spirals).

$L \propto v_{max}^4$ ← Seen in spirals.

For a dark-matter dominated spiral



$$v_{max}^2 = \frac{GM(R)}{R}$$

M = total mass, dominated by DM.

(How should the mass increase with radius?)

Now $L = \left(\frac{L}{M}\right) M = \left(\frac{M}{L}\right)^{-1} \frac{R}{G} v_{max}^2$ ← manipulate to get things in the with $\frac{M}{L}$ & $\langle I \rangle$

But $\langle I \rangle = \frac{L}{R^2} \Rightarrow R = \frac{L}{\langle I \rangle R}$

$$\therefore \quad L = \left(\frac{M}{L}\right)^{-1} \frac{L}{\langle I \rangle R} \frac{v_{max}^2}{G}$$

$$\propto \left(\frac{M}{L}\right)^{-2} \frac{1}{\langle I \rangle} \frac{v_{max}^2}{G^2} \frac{GM}{R}$$ ← $\frac{M}{L} \approx$ const., $\langle I \rangle \approx$ const

$$\propto \left(\frac{M}{L}\right)^{-2} \frac{1}{\langle I \rangle} v_{max}^4$$

Luminosity from stars. $L \propto M_* \Rightarrow \frac{M}{M_*} = \text{const} = \frac{M}{L} = \text{const}.$

If $\frac{M}{L} \approx \text{const}$ & $\langle I \rangle \approx \text{const}$ $\Big\}$ seems true for spirals.

then we get $L \propto v_{max}^4$ (See slides for data).

Measure rotation $v_{max} \rightarrow$ infer L
Measure flux
$\Big\} \rightarrow$ infer distance.

Explaining Faber - Jackson Relation (Ellipticals).

$L \propto \sigma_0^4$

$\uparrow$ $\sigma_0 \equiv$ velocity dispersion at the center of ellipticals.

Similar reasoning to Tully Fisher. However, not such a great fit. $L \propto \sigma_0^{2.65} R_e^{0.65}$ fits better

R_e: half of the light comes from here.

* Not easy to understand $\left(\frac{M}{L}\right) \propto M^a L^b$?.

$M - \sigma$ relation :

$M_\bullet \equiv$ mass of central blackhole in galaxies.
$\sigma_e \equiv$ velocity dispersion in the bulge.

$$M_\bullet = 1.35 \times 10^{8.6} M_\odot \left(\frac{\sigma_e}{200 \text{ km s}^{-1}} \right)^\alpha \qquad \alpha \sim 5.46$$

Surprising because $M_{bulge} \approx 10^2 - 10^3 M_\bullet$

Why is there a correlation ? ← not solved!

—x—

3) What role do SMBHs play in galaxy formation & evolution

2) How do we measure / infer their masses ?

1) How do we know there are SMBHs ?

Some scales of SMBHs in galaxies : $10^6 - 10^{10}\ M_\odot$

Imaging BH ?

1. $r_{sc} \equiv \frac{2GM_\bullet}{c^2} = \left(\frac{M_\bullet}{M_\odot}\right) 3\text{ km}. \sim 10^6 - 10^{10}\text{ km}$

$\sim 10^{-8} - 10^{-4}\text{ pc}.$

For a few $\times 10^6 M_\odot$ BH at our galactic center

$$\theta_{sc} \sim \frac{10^{-7}\text{ pc}}{10^4\text{ pc}} \sim 10^{-11}\text{ rad} \sim 10^{-6}\text{ arc sec}.$$

$\theta_{Hubble} \sim 10^{-1}$ arc sec. , VLBI 3 mm, 10^{-4} arc sec.

$\theta_{Moon} \sim 1800$ arc sec.

Difficult to image BHs schwarzchild radius in other galaxies! [Our galaxy, there is hope! Event Horizon Telescope]

2. Inferring the existence from dynamics:

r_{BH} = blackhole's radius of influence

$$M_{Bulge}(r < r_{BH}) = 2M_\bullet$$

$$r_{BH} \simeq \frac{GM_\bullet}{\sigma^2}$$ σ^2 = velocity dispersion of bulge.

$$r_{BH} \simeq 0.4 \text{ pc} \left(\frac{M_\bullet}{10^6 M_\odot}\right)\left(\frac{\sigma}{100 \text{ kms}^{-1}}\right)^{-2}$$

for $r < r_{BH}$, the BH influences the dynamics of stars.

$$\theta_{BH} \sim 0.1'' \left(\frac{M_\bullet}{10^6 M_\odot}\right)\left(\frac{\sigma}{100 \text{ kms}^{-1}}\right)^{2}\left(\frac{D}{\text{Mpc}}\right)^{-1}$$

HST ok, nearby and massive needed.

Observe stellar kinematics to infer mass of <u>BHs</u>.

Direct method, relatively clean but cannot work for far off BHs.

If $\sigma^2 \propto \frac{1}{r} \Rightarrow$ BH at the center.

or rising as you go deeper & deeper.

& $\sigma^2 \Rightarrow M$ that cannot be contained within the radius.

What else could it be argument.

Other techniques: 1) Reverberation mapping

2) Energy output!

active galactic nuclei.

Lecture 26

Some scales of SMBHs in galaxies : $10^6 - 10^{10}\, M_\odot$

Imaging BH?

1. $r_{sc} \equiv \frac{2GM_\bullet}{c^2} = \left(\frac{M_\bullet}{M_\odot}\right) 3\text{ km} \sim 10^6 - 10^{10}$ km

$\sim 10^{-8} - 10^{-4}$ pc.

For a few $\times 10^6 M_\odot$ BH at our galactic center

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What else could it be argument.

Other techniques: 1) Reverberation mapping

2) Energy output!

active galactic nuclei.

3. BHs as central Engines in Galaxies

AGN ≡ active galactic nuclei.

① Some AGNs $L \sim 10^{47}$ ergs/s. $L_\odot \sim 10^{33}$ ergs/s.

② Outflows - radio emitting lobes $\gtrsim 1$ Mpc
⇒ AGN active $\sim 10^7$ yrs.

① & ② ⇒ $\underline{\underline{E \gtrsim 10^{61} \text{ ergs}}}$.

③ variability ~ 1 day ⇒ $\underset{\text{size of source}}{R} \lesssim \underline{10^{15} \text{ cm}}$.

What can cause such a huge output from such a small region?

Nuclear fusion? (ignore gravity, just burn like a star)

- efficiency $\varepsilon \lesssim 0.8\%$

$\therefore \quad E \lesssim 0.008\, Mc^2$

For $E \sim 10^{61}$ ergs $\Rightarrow M \sim 10^9\, M_\odot$
from a $R \sim 10^{15}$ cm radius!

Note that $r_{sc} \sim 10^{15}$ cm for $M \sim 10^9 M_\odot$
So ignoring gravity not warranted!

Turns out the KE gained from infall & then used to generate heat $\rightarrow$ radiation is more efficient, $\varepsilon \sim 6-29\%$!

Our Galaxy: The milky way

http://www.chromoscope.net/

OUR GALAXY

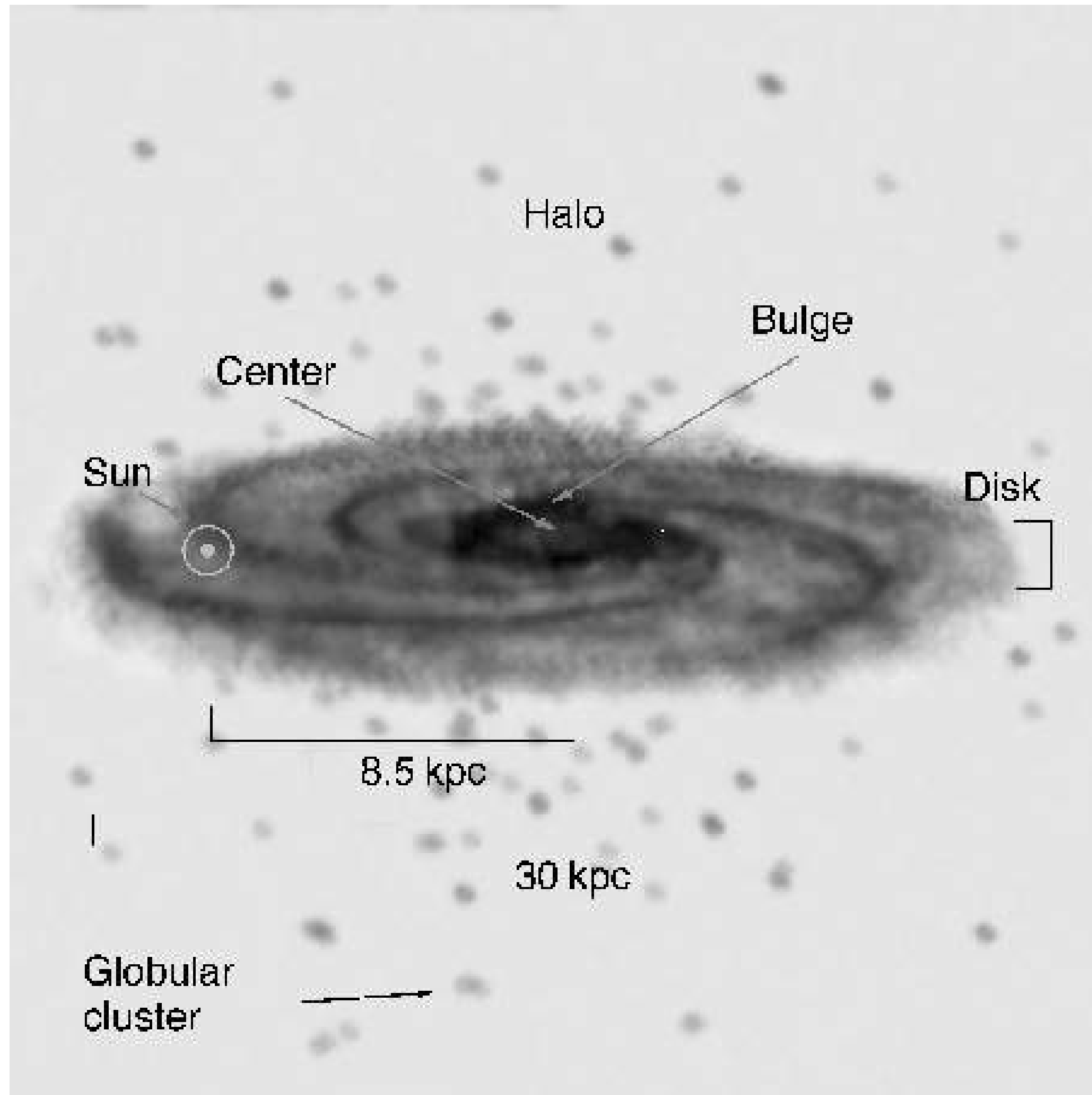


Fig. 1.3. Schematic structure of the Milky Way consisting of the disk, the central bulge with the Galactic center, and the spherical halo in which most of the globular clusters are located. The Sun orbits around the Galactic center at a distance of about 8 kpc

Galactic disc: distribution of stars

$$n(R, z) = n_0 \left(\mathrm{e}^{-|z|/h_{\mathrm{thin}}} + 0.02\mathrm{e}^{-|z|/h_{\mathrm{thick}}}\right) \mathrm{e}^{-R/h_R}$$

$$h_{\mathrm{thin}} \approx 325\,\mathrm{pc}$$

$$h_{\mathrm{thick}} \approx 1.5\,\mathrm{kpc}$$

$$h_R \approx 3.5\,\mathrm{kpc}$$

thin disc: younger stars

thick disc: older stars

$$n \approx 0.2\,\mathrm{stars\,pc}^{-3}$$

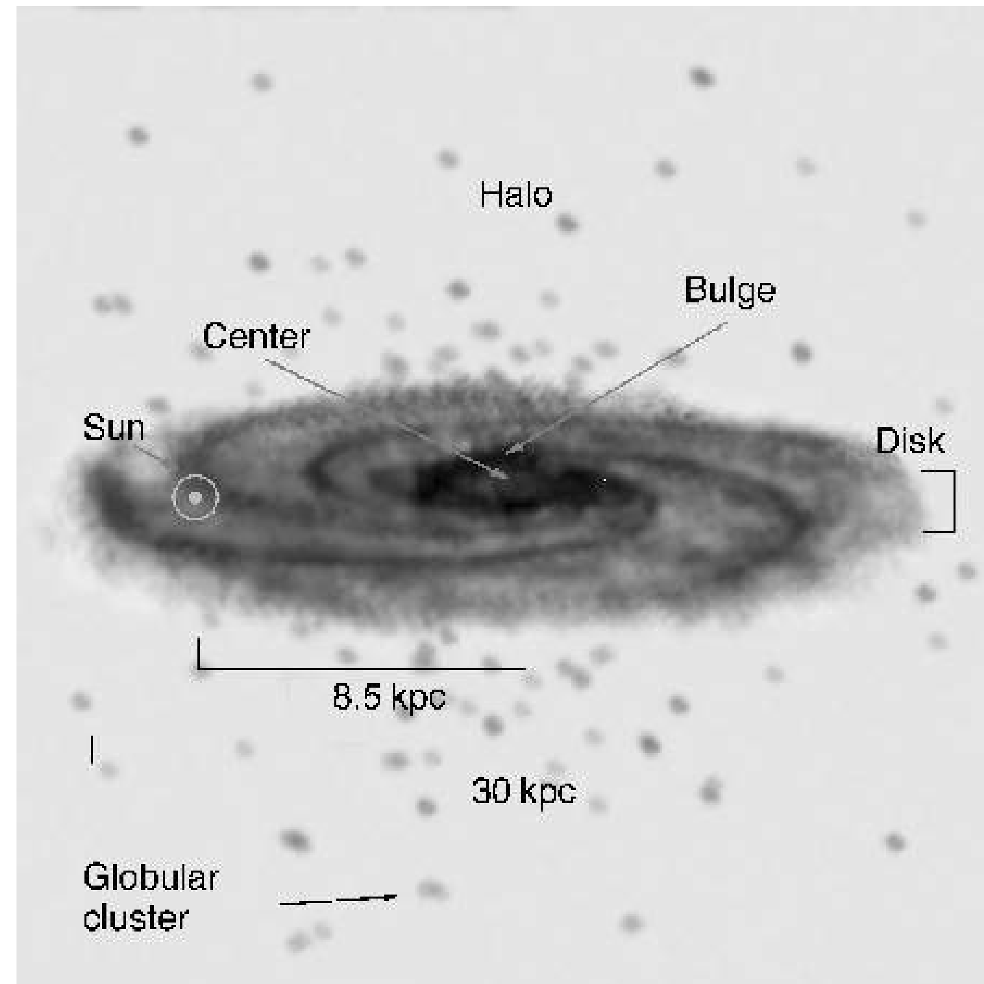


Fig. 1.3. Schematic structure of the Milky Way consisting of the disk, the central bulge with the Galactic center, and the spherical halo in which most of the globular clusters are located. The Sun orbits around the Galactic center at a distance of about 8 kpc

Galactic disc: Luminosity

$$\frac{M}{L_B} \approx 3\frac{M_\odot}{L_\odot}$$ in thin disk

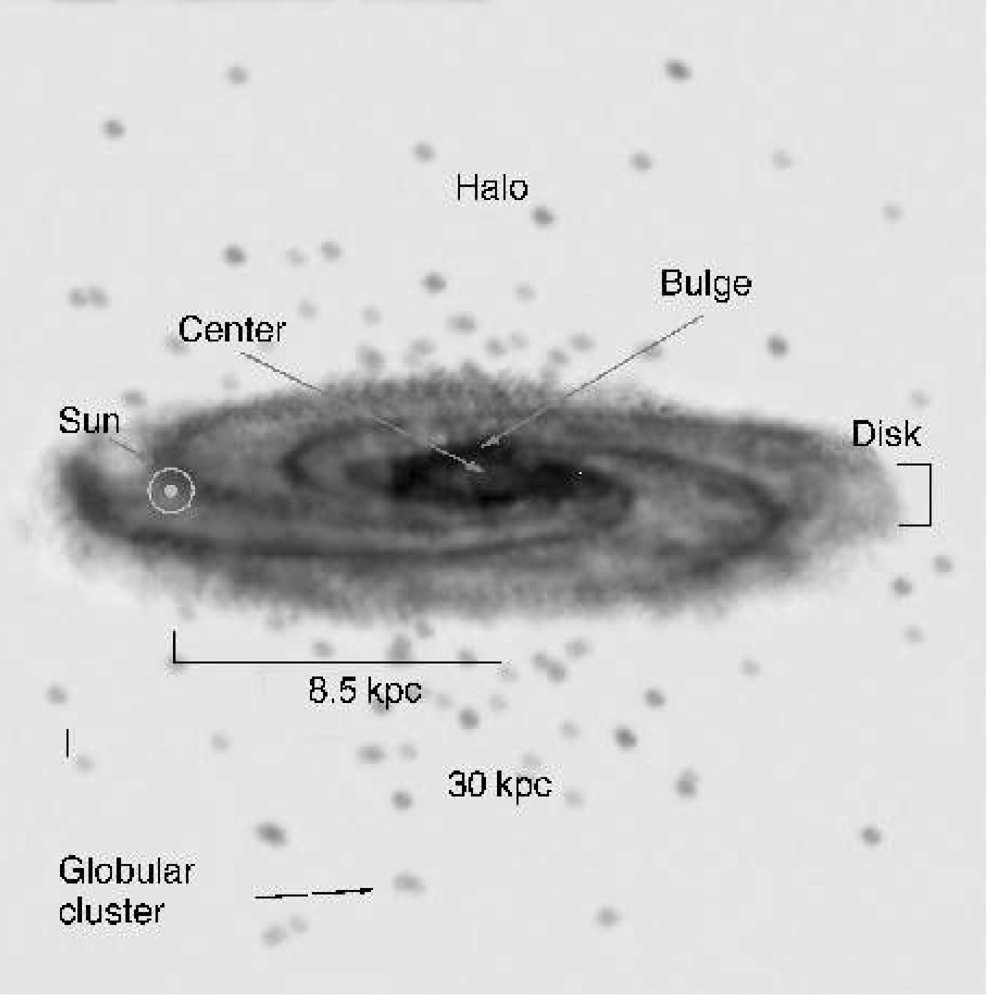
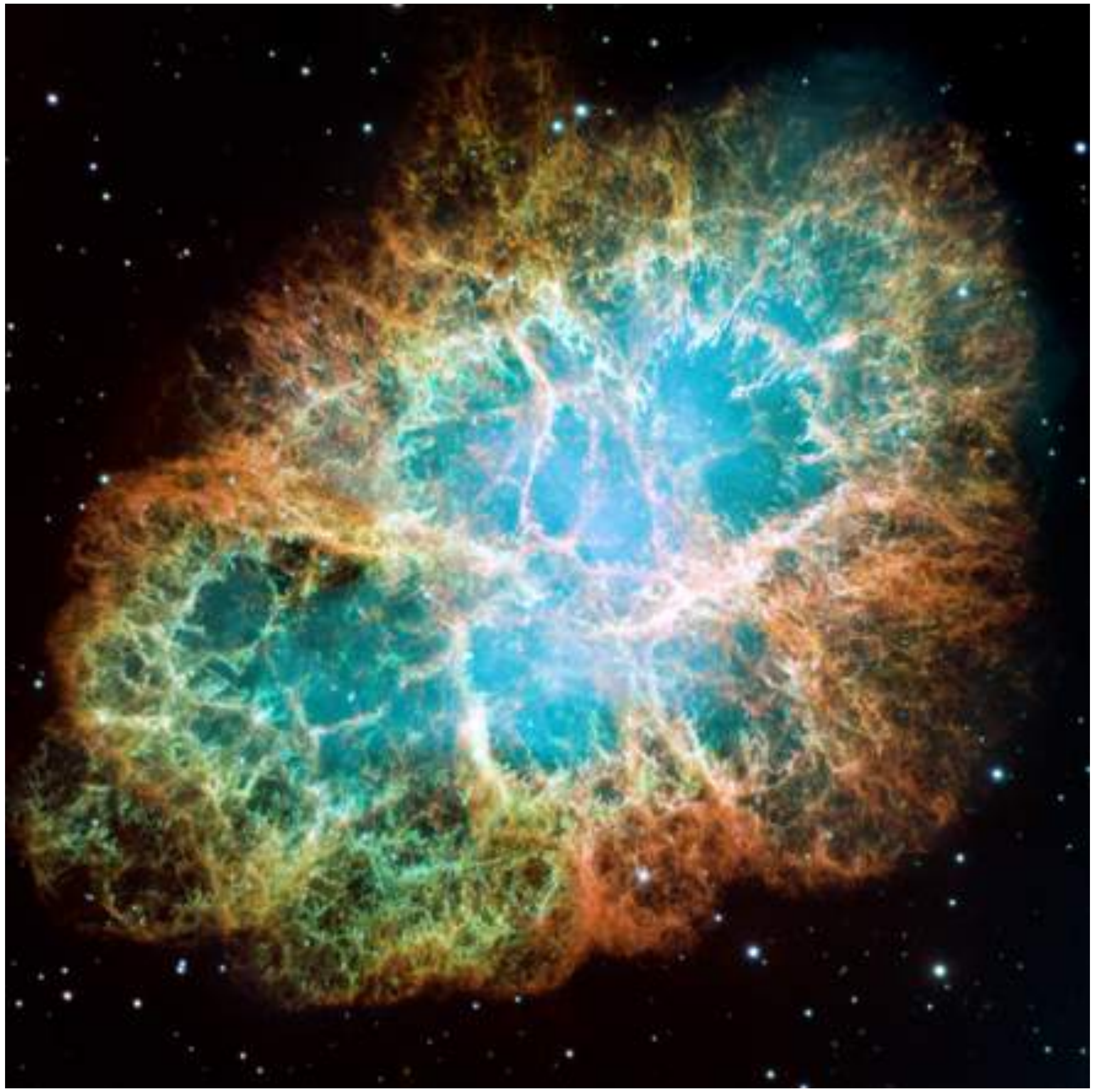


Fig. 1.3. Schematic structure of the Milky Way consisting of the disk, the central bulge with the Galactic center, and the spherical halo in which most of the globular clusters are located. The Sun orbits around the Galactic center at a distance of about 8 kpc

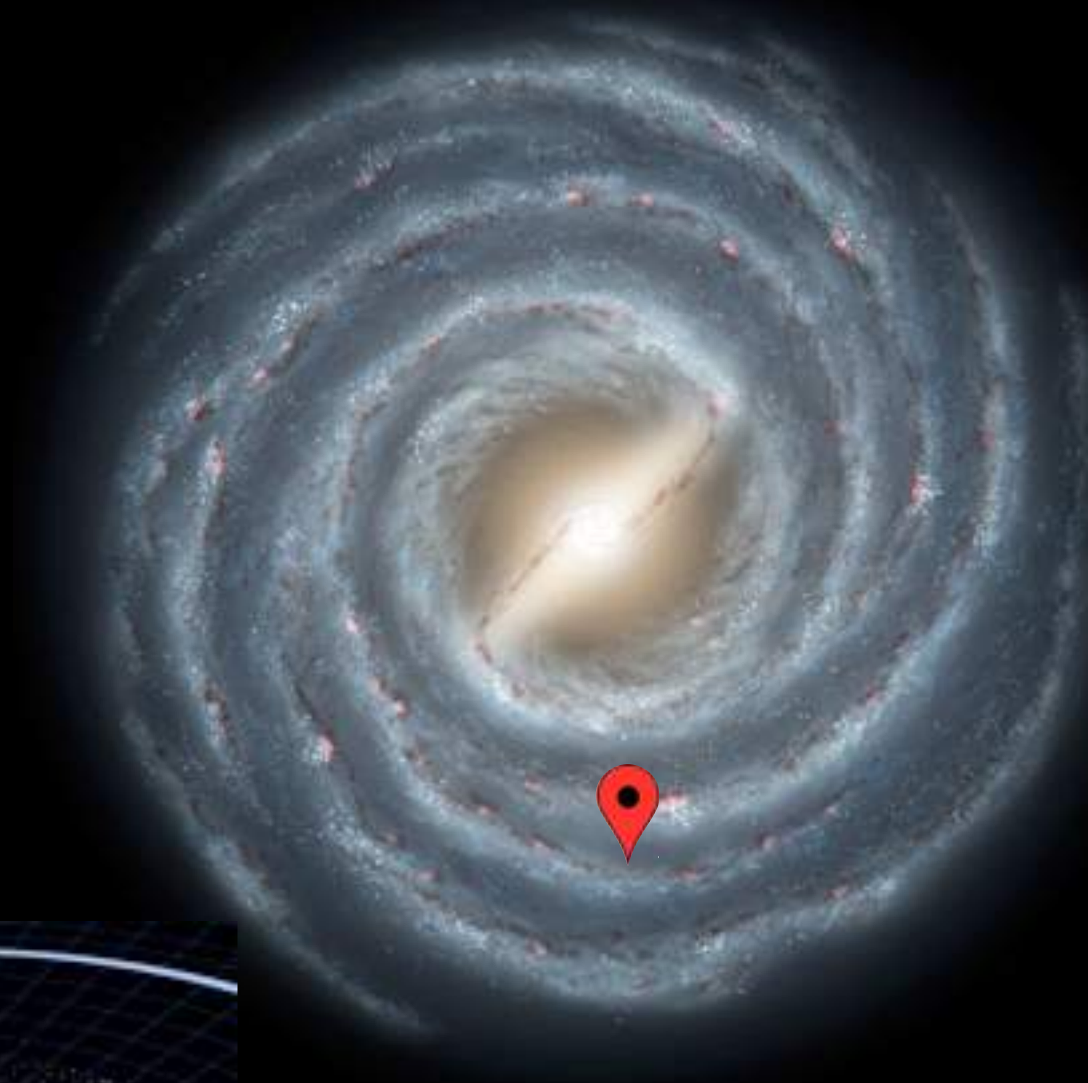
Galactic disc: Chemical Composition

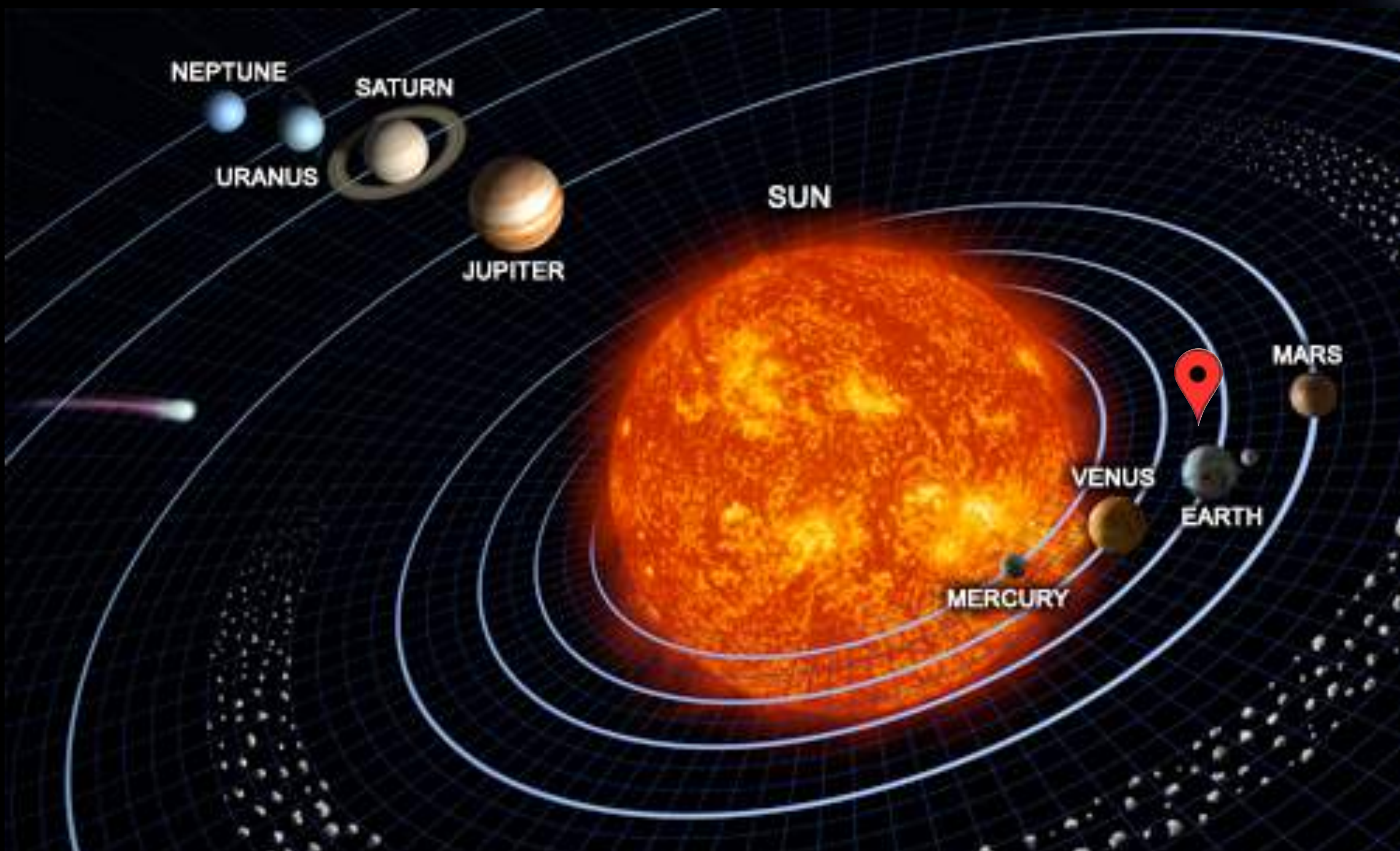
supernovae enrich the ISM with iron

new stars have more iron



leads to us



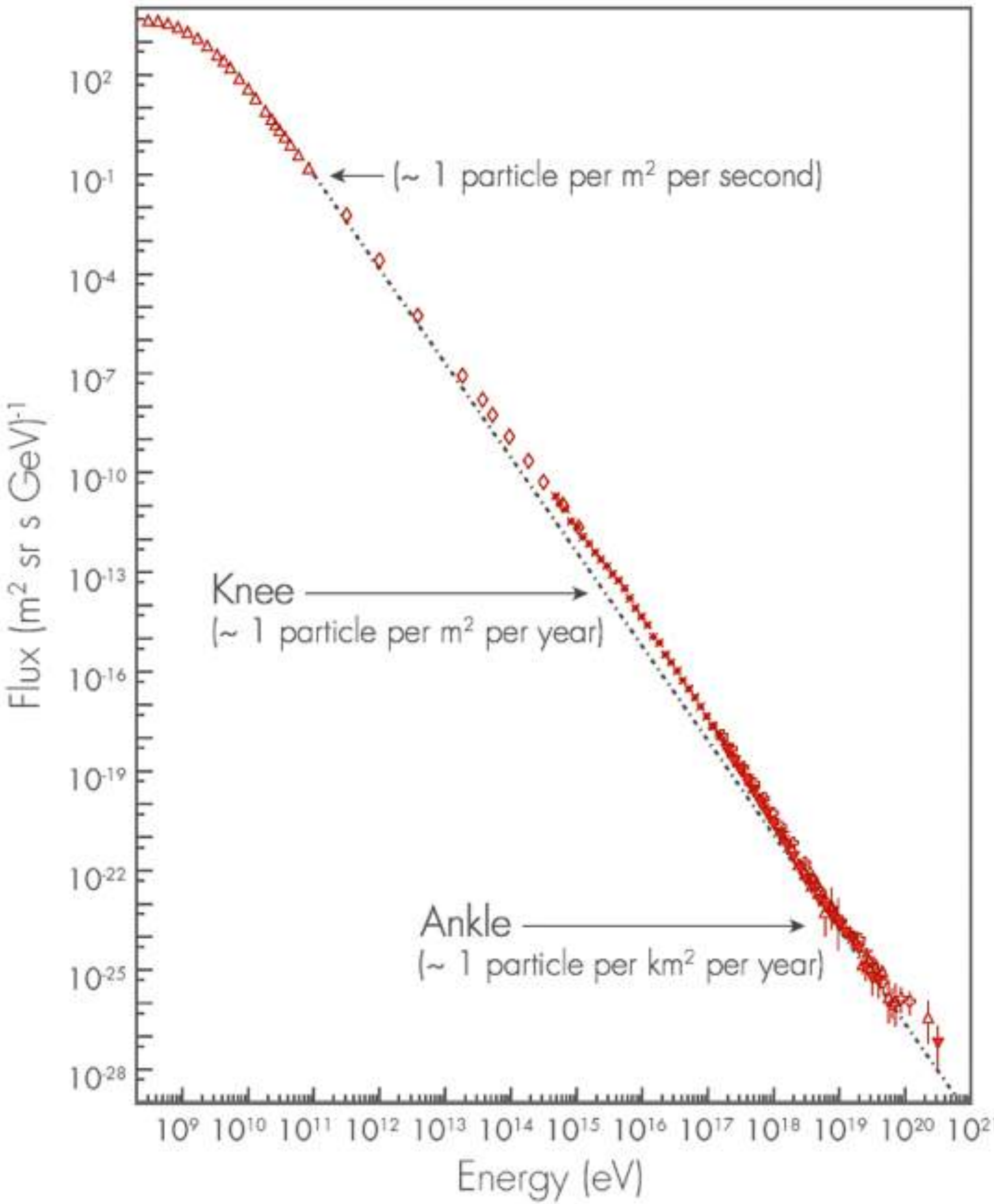

NEPTUNE
SATURN
URANUS
SUN
JUPITER
MARS
VENUS
EARTH
MERCURY

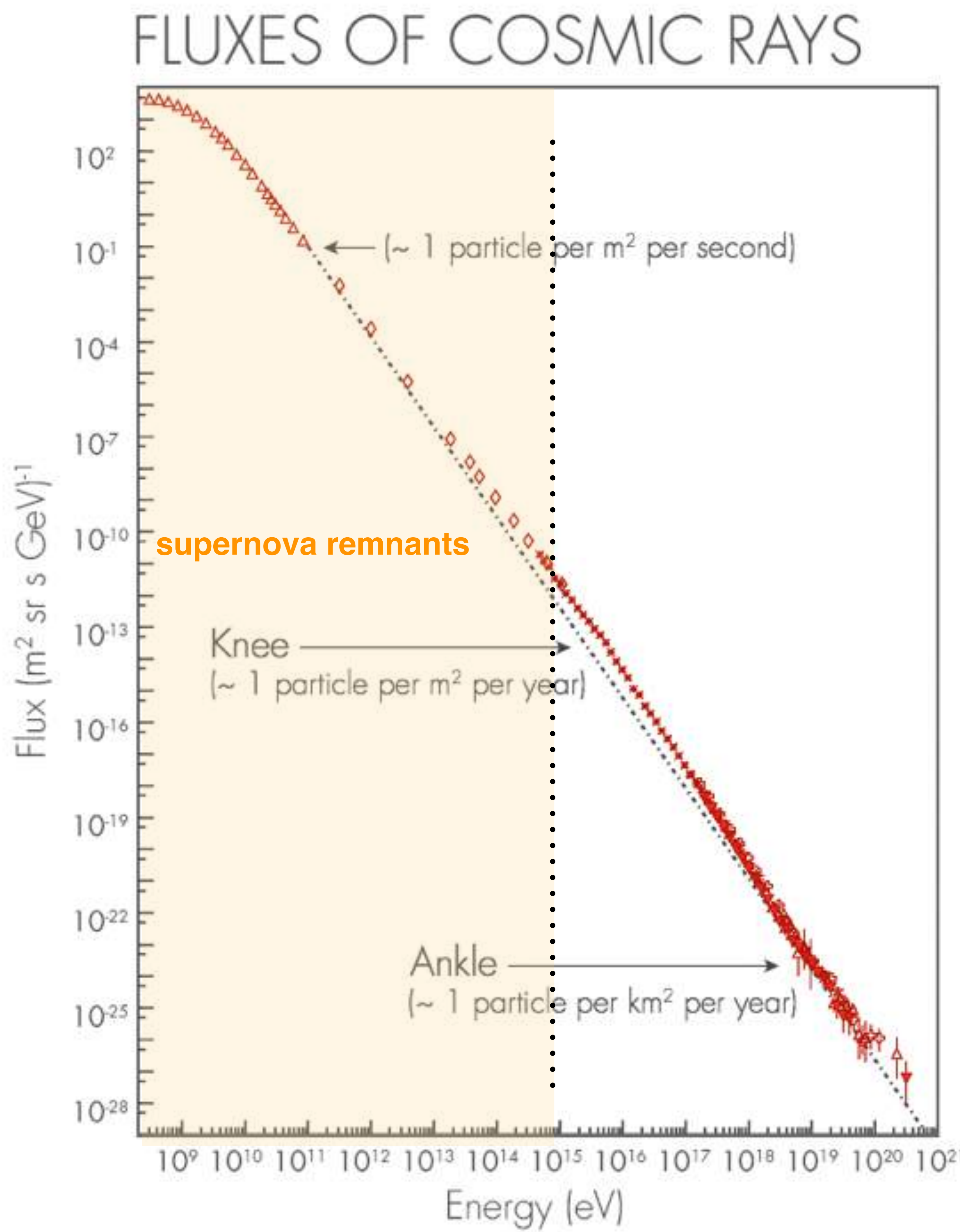
Multi-messenger astronomy

* EM waves *

- cosmic rays
- neutrinos
- gravitational waves

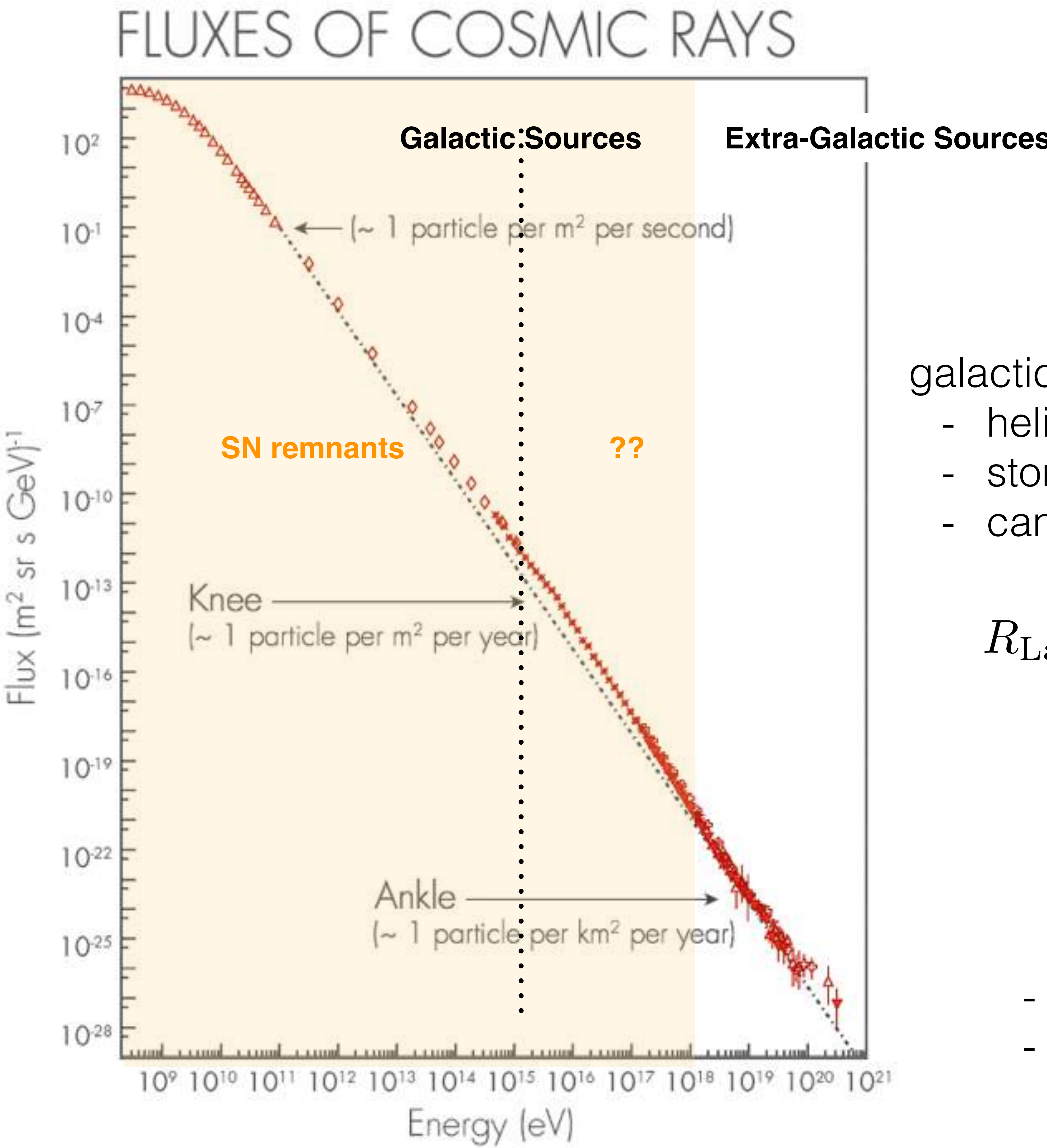
FLUXES OF COSMIC RAYS





Fermi acceleration by shocks

Galactic & Extra Galactic Sources



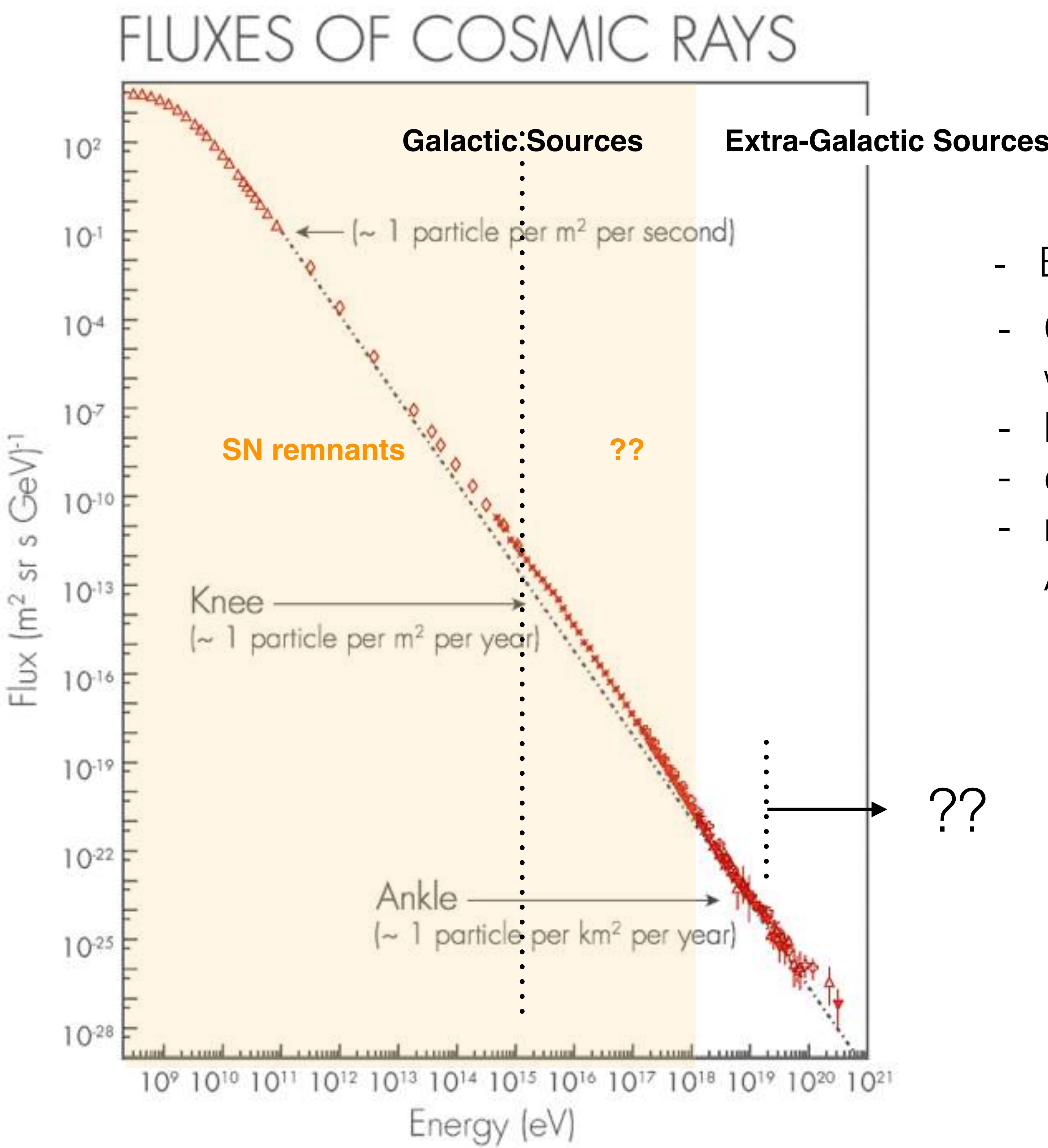
galactic magnetic fields: 10^{-10} T

- helical particle paths
- stored 10^7 yrs
- can confine up to 10^{18} eV

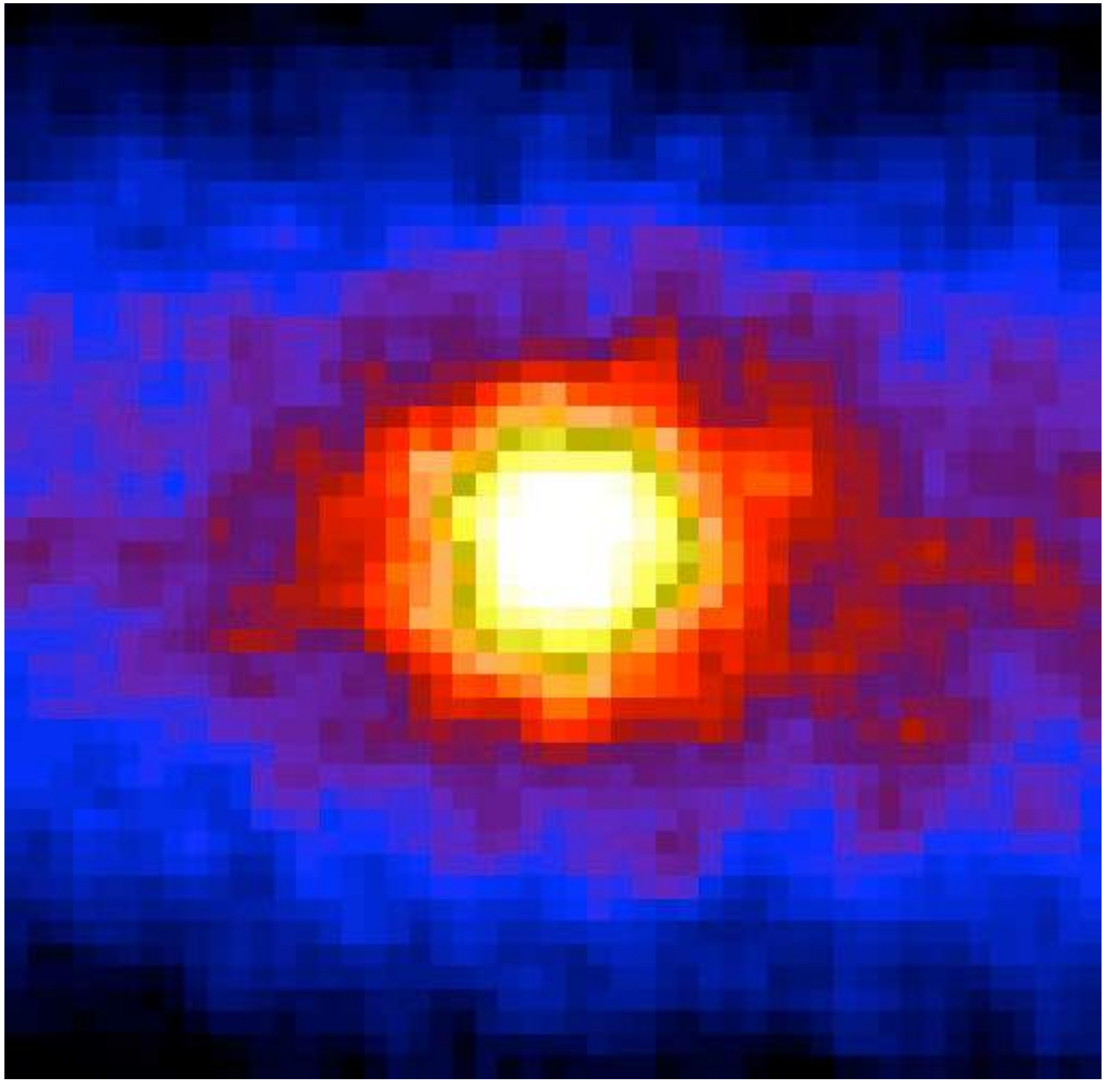
$$R_{\text{Larmor}} = \frac{E}{ecB}$$

- $E > 10^{18}$ eV
- AGNs are likely sources

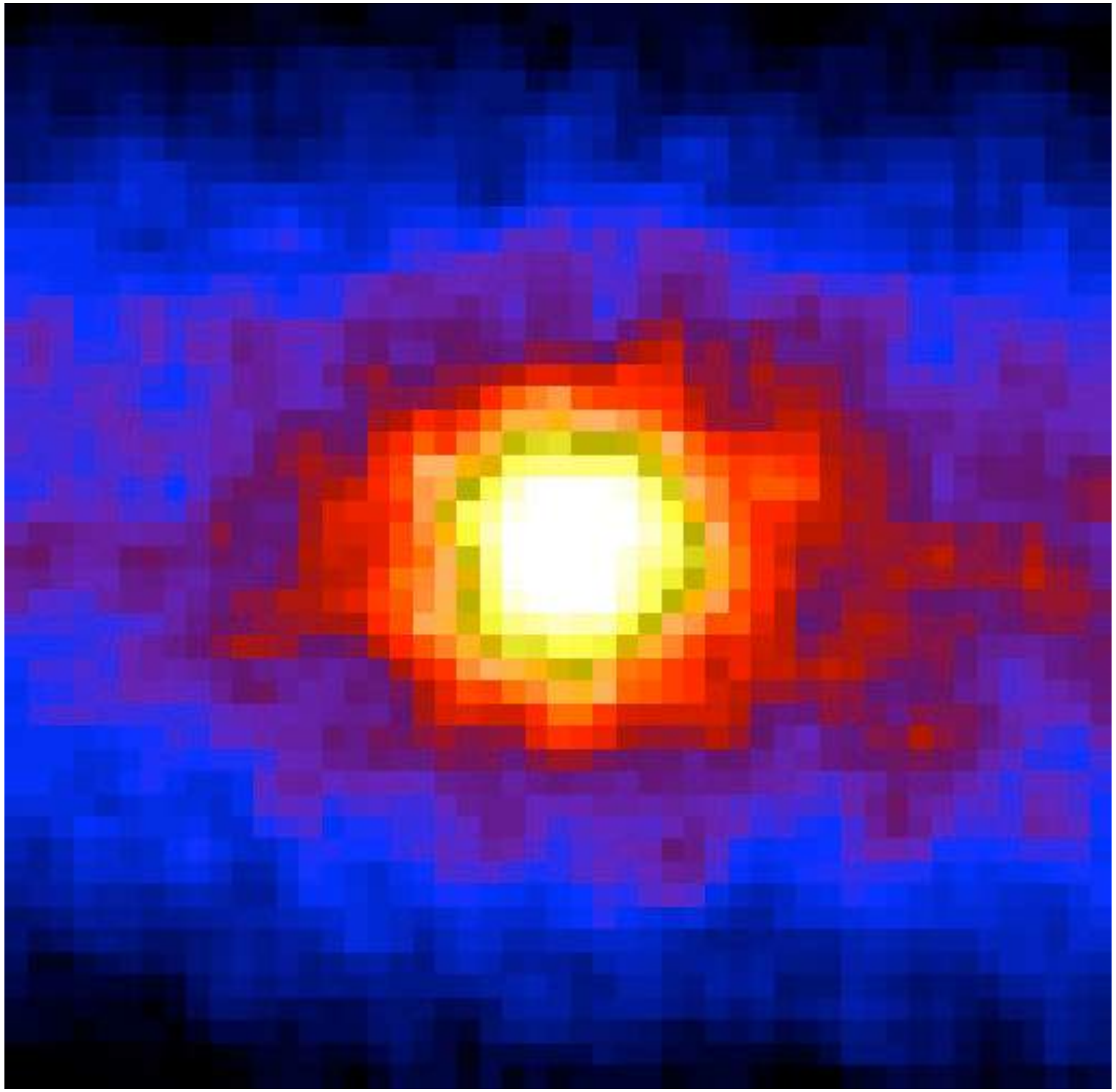
Galactic & Extra Galactic Sources



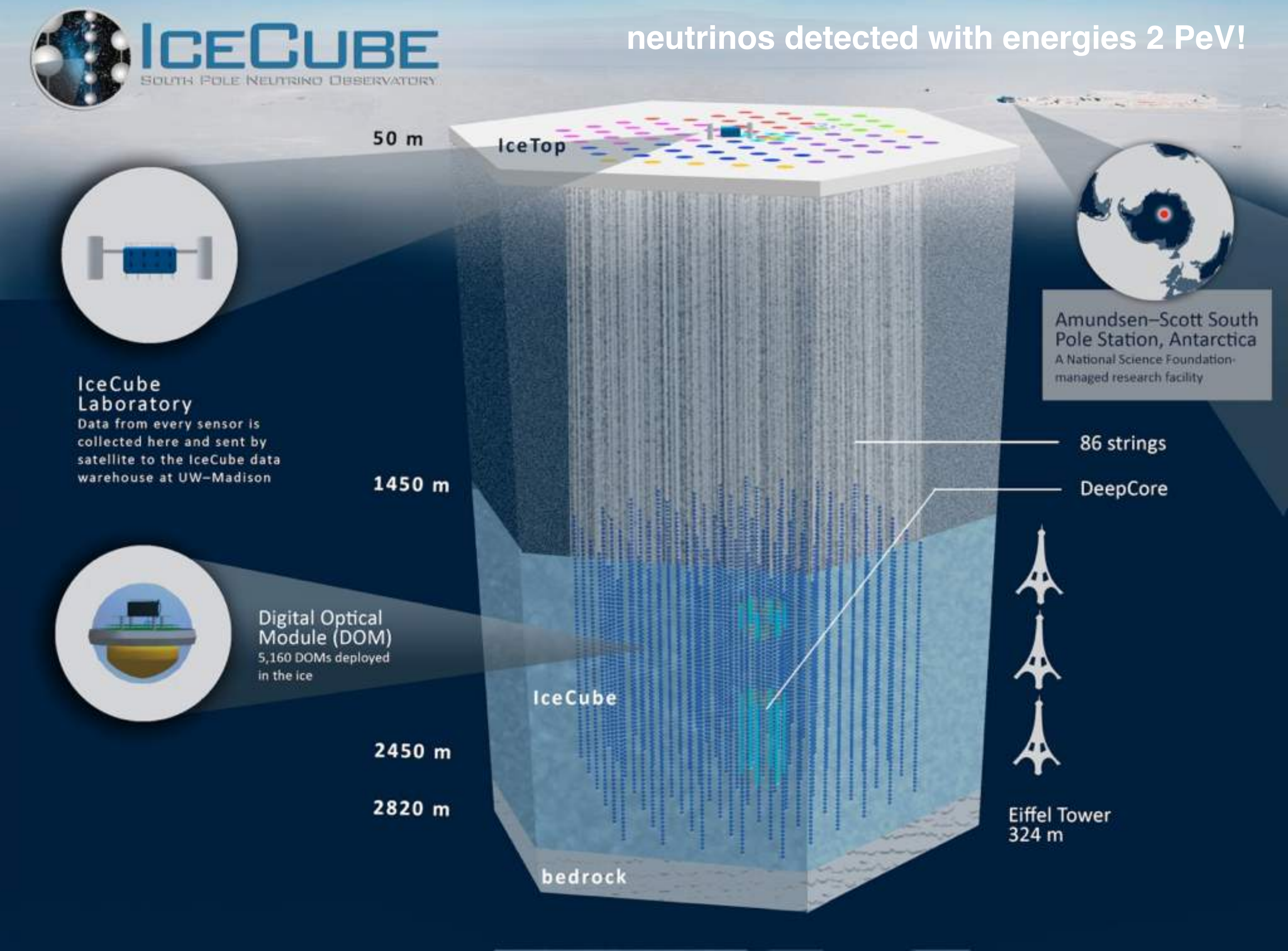
- E > 10^{19} eV !!!
- GZK cutoff: interaction with CMB
- lose energy to pions
- d < 10 Mpc
- not many suitable nearby AGNs!



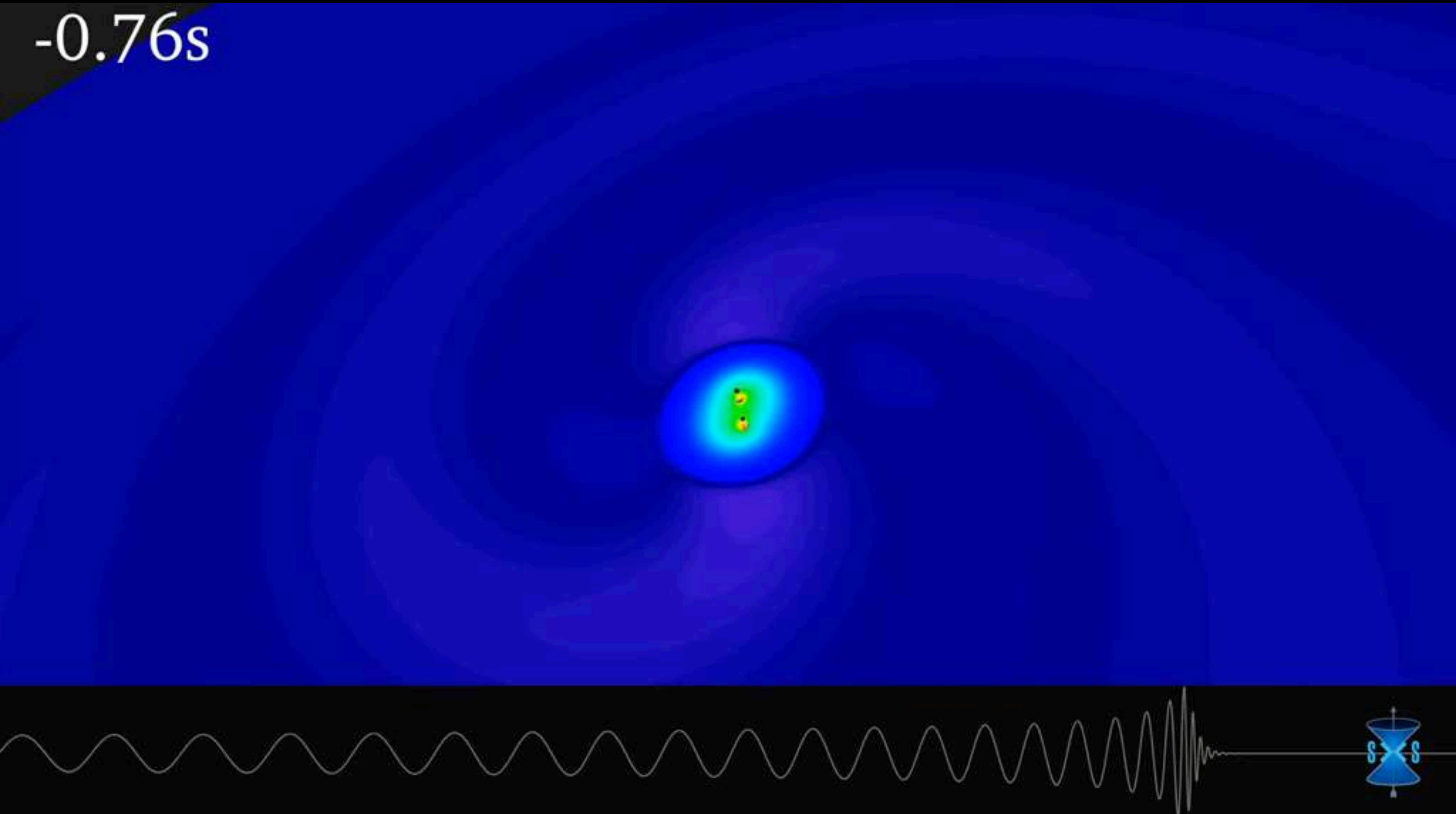
sun through the earth — seen via neutrinos!



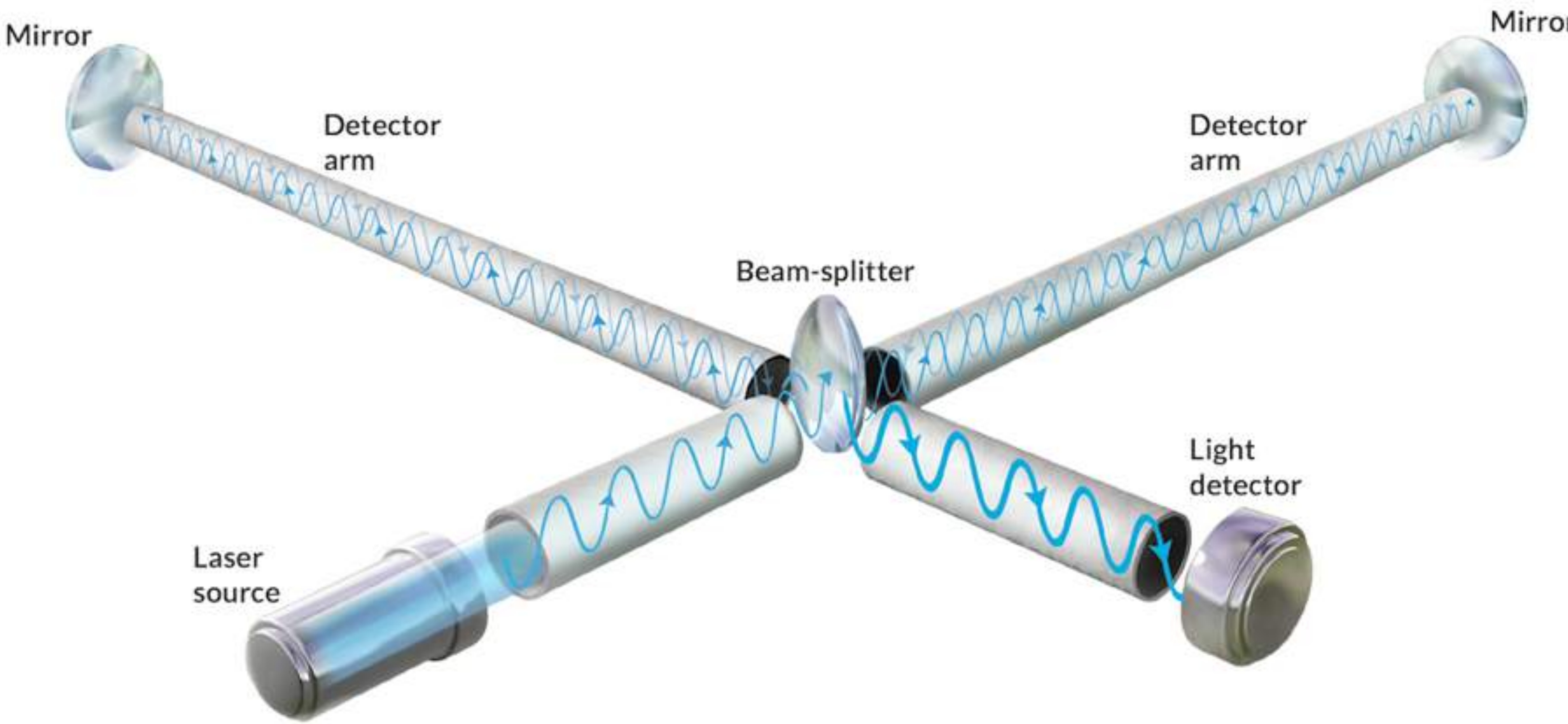
neutrinos detected with energies 2 PeV!

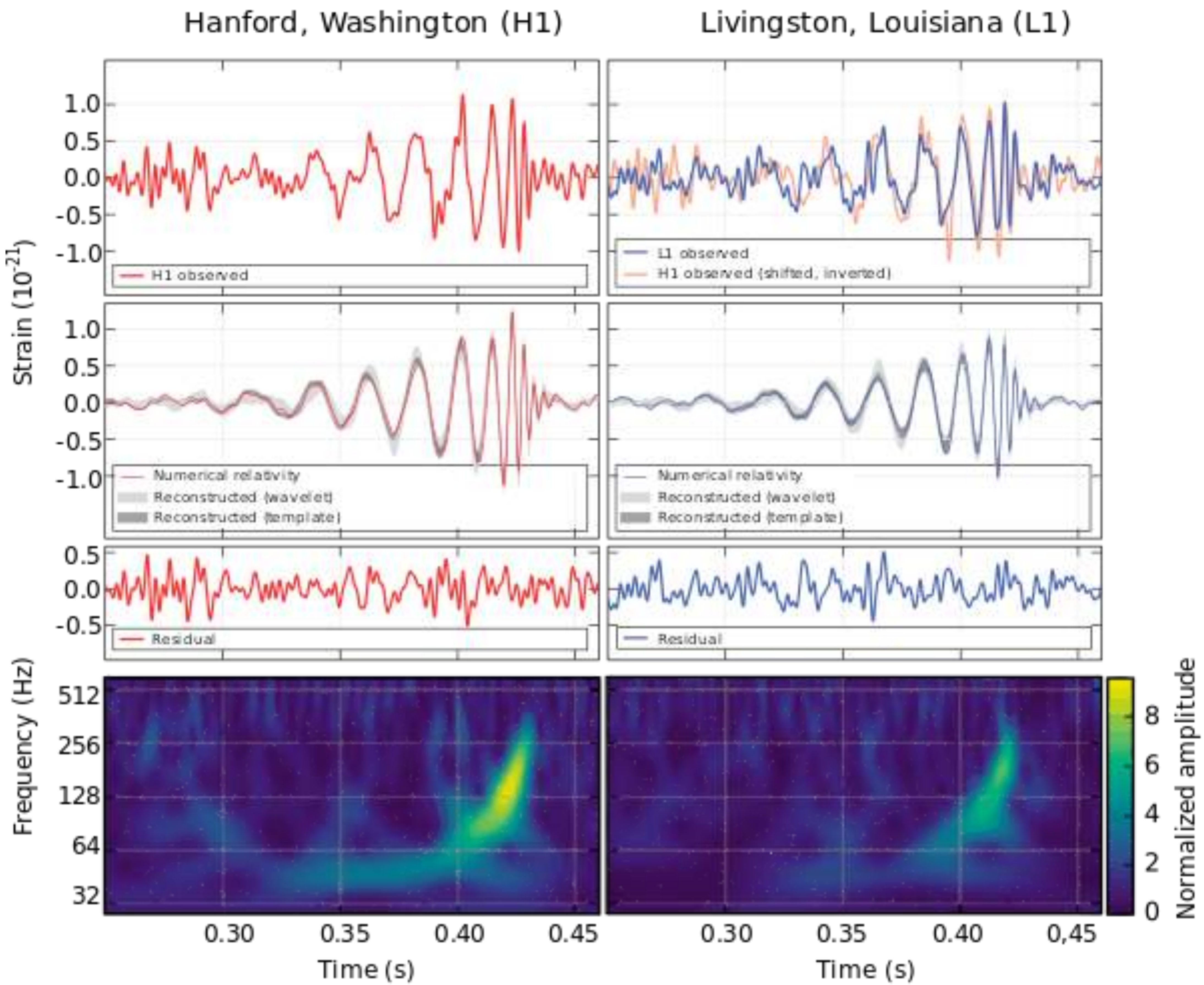


Neutrino Astronomy ?


-0.76s

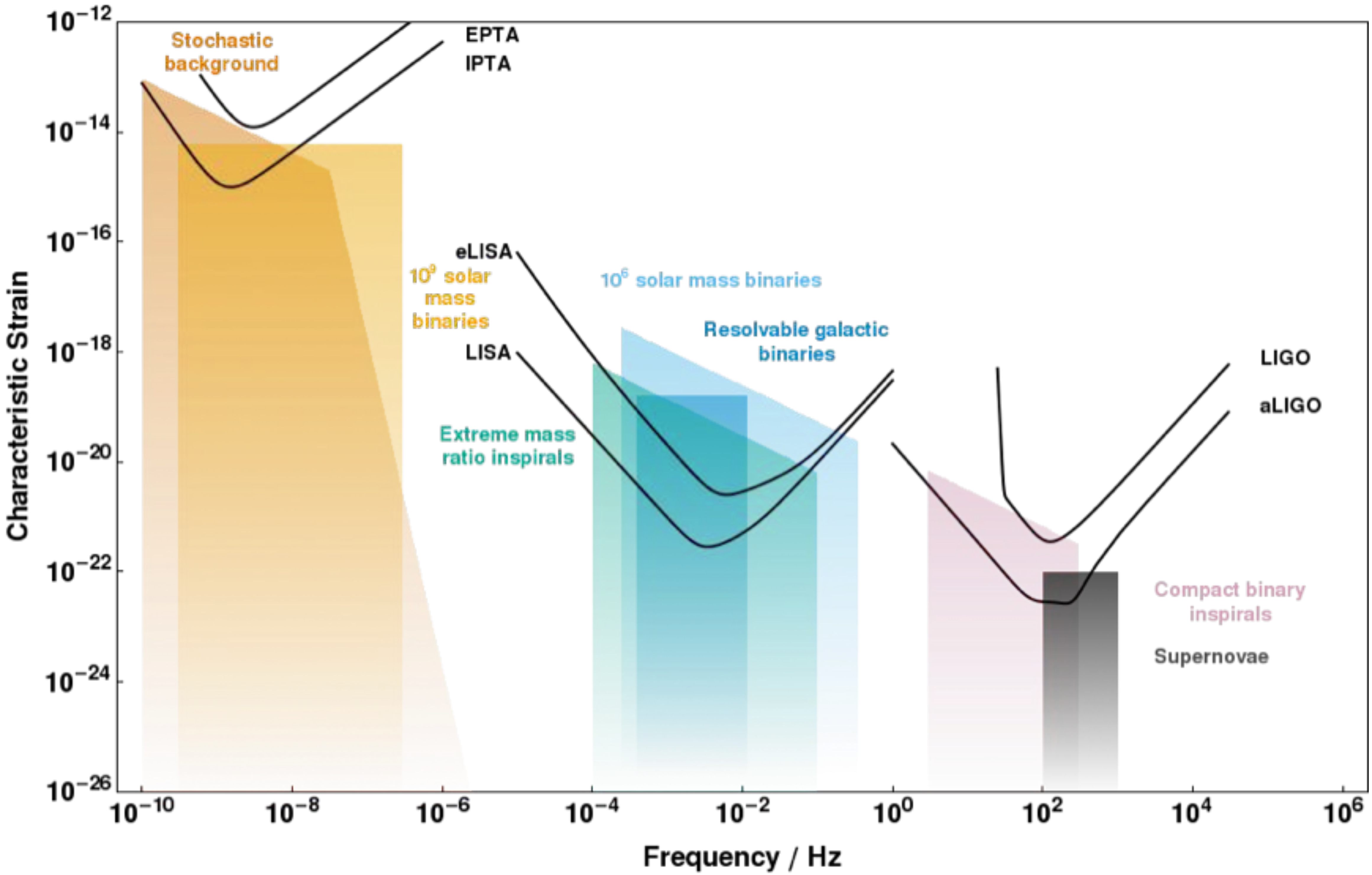

LIGO, Caltech, MIT, NSF


Mirror
Mirror
Detector arm
Detector arm
Beam-splitter
Light detector
Laser source

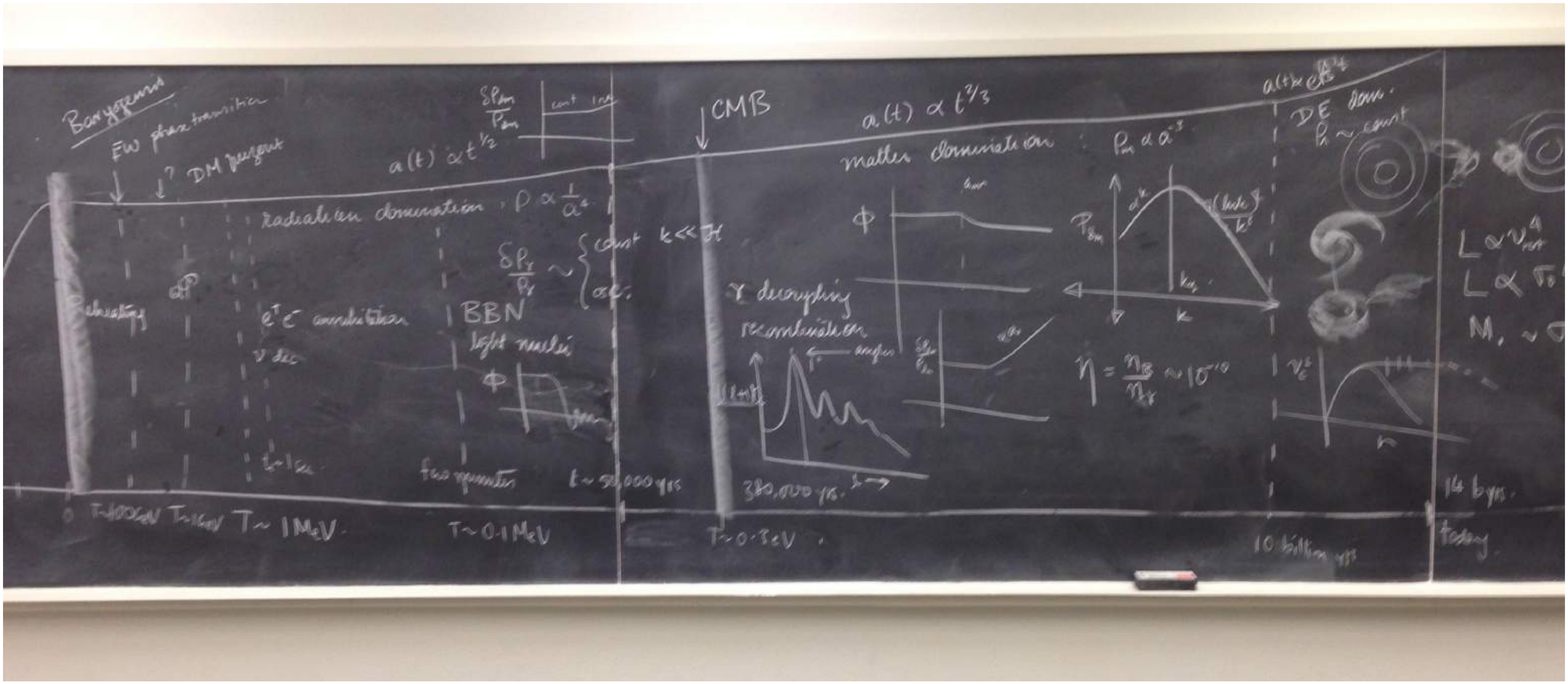


Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
Normalized amplitude
Time (s)

gravitational wave astronomy ?



http://rhcole.com/apps/GWplotter/



Baryogenesis
EW phase transition
DM freezeout
Reheating
$a(t) \propto t^{1/2}$
$\frac{\delta\rho_{dm}}{\rho_{dm}}$
radiation domination, $\rho \propto \frac{1}{a^4}$
e^+e^- annihilation
ν dec
$t \sim 1 sec$
BBN
light nuclei
few minutes
$t \sim 50,000$ yrs
$T \sim 1 MeV$
$T \sim 0.1 MeV$
CMB
$a(t) \propto t^{2/3}$
matter domination
$\rho_m \propto a^{-3}$
γ decoupling
recombination
380,000 yrs.
$T \sim 0.3 eV$
$\eta = \frac{n_B}{n_\gamma} \sim 10^{-10}$
k
DE dom.
$\rho_\Lambda \sim$ const
10 billion yrs
14 byrs.
today
$L \propto v_{rot}^4$