Lecture 10

Plan: - Equilibrium — n, ρ, P
- Effective relativistic d.o.f
- Entropy conservation.

* Application - Neutrino decoupling

Equilibrium expressions for n, ρ, P

$$n = \frac{g}{(2\pi)^3} \int d^3p \, \frac{1}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

$$= \frac{g}{2\pi^2} \int dp \, \frac{p^2}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

Relativistic : $m \gg T$

non-relativistic : $m \ll T$.

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases}$$

$$n \approx g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T}$$

T^3 available from dim. analysis

Boltzmann supressed

Similarly, for the energy density.

$$\rho = \frac{g}{(2\pi)^3}\int d^3p \, \frac{\sqrt{p^2+m^2}}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

$m \ll T$ (Relativistic) $\qquad\qquad$ $m \gg T$. (non relativistic)

$$\rho \approx \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$$

$$\rho \approx mn = gm\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

Finally, for the pressure.

$$P = \frac{g}{(2\pi)^3}\int d^3p \, \frac{p^2}{3\sqrt{p^2+m^2}} \, \frac{1}{\exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] \pm 1}$$

$m \ll T$ (Relativistic). $\qquad\qquad$ $m \gg T$ (non-relativistic)

$$P \approx \frac{1}{3}\rho$$

$$P \approx nT \ll \rho = mn$$

		# density n	energy density ρ	Pressure P
$T \gg m$	Bosons	$\frac{\zeta(3)}{\pi^2} g T^3$	$\frac{\pi^2}{30} g T^4$	$\frac{1}{3}\rho$
	Fermions	$\frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3$	$\frac{7\pi^2}{240} g T^4$	$\frac{1}{3}\rho$
			↑ relativistic	
			↓ non-relativistic	
$T \ll m$	Bosons OR Fermions	$g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$	mn	nT $(\ll \rho)$

Consider two species of bosonic particles with masses m_1 & m_2, in thermal equilibrium at a temperature T. Consider $m_1 \ll T$ & $m_2 \gg T$ then

$$\rho_{tot} = \rho_1 + \rho_2 \approx \frac{\pi^2}{30} g T^4 + m_2 g \left(\frac{m_2 T}{2\pi}\right)^{3/2} \underbrace{e^{-m_2/T}}_{\text{small!}}$$

$$\approx \frac{\pi^2}{30} g T^4$$

Hence the energy density is dominated by relativistic ($m \ll T$) particles.

- Note that if $T \gg m_1, m_2$ then at that temperature both would contribute
- If $m_1 \ll m_2$, then as the temperature falls in an expanding universe, there will be a temperature where m_1 contributes but m_2 does not.

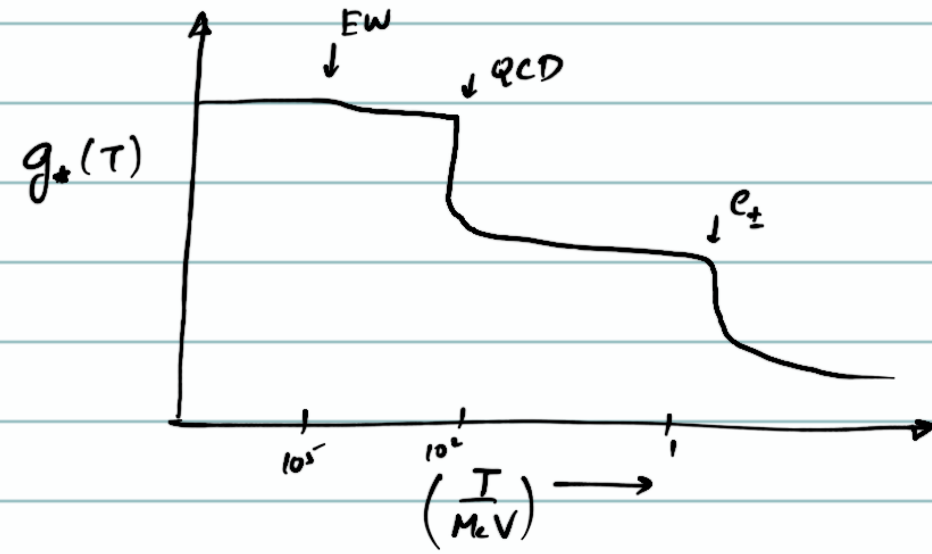
It is useful and convenient to define the "effective relativistic degrees of freedom"

$$\rho(T) = \sum_{m \ll T} \rho_i + \sum_{m \gg T} \rho_i \approx \sum_{m \ll T} \rho_i \equiv \frac{\pi^2}{30} g_*(T) T^4$$

where $g_*(T) \equiv g_b + \frac{7}{8} g_f$ = effective relativistic d.o.f.

total bosonic d.o.f. (for g_b) ... fermionic d.o.f. (for g_f)

Why is $g_*(T)$? because as the temperature falls, a given species might become non-relativistic and stop contributing to the energy density.



for standard model. (SM)

For $T > 10^5$ MeV, all SM particles are relativistic.

$g_b = 28$ $\Rightarrow$ $g_* \approx 106.75$
$g_f = 90$

Today ($T_0 \sim 10^{-4}$ eV) $g_* \approx$ few [photons & maybe neutrinos].

Note if species decouple from each other ie ($\Gamma \lesssim H$), then they might each still have a FD or BE distribution and could be at different temperatures. (T_i)

More generally.

$$g_*(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum g_i \left(\frac{T_i}{T}\right)^4$$

Conservation of Entropy (Equilibrium).

Want to show $\frac{dS}{dt} = 0$.

For $\frac{\mu}{T} \to 0$, note that since the distribution function only depends on E/T, we get

① $$\frac{\partial P}{\partial T} = \frac{\rho + P}{T}$$ (see proof later).

Second, using the 2nd law of thermodynamics & ①, we get

$dS = \frac{dU + PdV}{T}$.

② $$dS = d\left[\frac{(\rho + P)V}{T}\right]$$

Finally using ①, ② & the continuity eq. $\frac{d\rho}{dt} = -3H(\rho+P)$ we get

✓ ③ $$\frac{dS}{dt} = 0$$

Proof of ①, ② & ③ next.

Proving $\frac{\partial P}{\partial T} = \frac{\rho + P}{T}$

$$P = \frac{g}{(2\pi)^3}\int d^3p\, f(p,T)\frac{p^2}{3E(p)} \qquad f(p) = \left[e^{\frac{\sqrt{p^2+m^2}}{T}} \pm 1\right]^{-1}$$

$$E(p) = \sqrt{p^2+m^2}$$

$$\rho = \frac{g}{(2\pi)^3}\int d^3p\, f(p,T)E(p)$$

Note $d^3p = p^2\,dp\,d\Omega$

Since f & E are independent of angles

$$\int d^3p \rightarrow 4\pi\int dp\, p^2$$

$$\therefore \quad P = \frac{g}{2\pi^2}\int_0^\infty dp\, p^2 f(p,T)\frac{p^2}{3E} \qquad \rho = \frac{g}{2\pi^2}\int dp\, p^2 f(p,T)E(p)$$

Also note that

$$\frac{\partial f}{\partial T} = -\frac{E}{T}\frac{\partial f}{\partial p}\frac{dp}{dE} = -\frac{E^2}{Tp}\frac{\partial f}{\partial p} \qquad \because EdE = pdp$$

$$\therefore \quad \frac{\partial P}{\partial T} = \frac{g}{2\pi^2}\int_0^\infty dp\, p^2 \, \frac{-E^2}{Tp}\left(\frac{\partial f}{\partial p}\right)\frac{p^2}{3E}$$

$$= -\frac{g}{6\pi^2}\frac{1}{T}\int_0^\infty dp\, p^3 E(p)\frac{\partial f}{\partial p}$$

$$\frac{\partial}{\partial p}\left[p^3 E f\right] = p^3 E(p)\frac{\partial f}{\partial p} + \frac{\partial}{\partial p}(p^3 E)f$$

$$\therefore \quad \frac{\partial P}{\partial T} = -\frac{g}{6\pi^2}\frac{1}{T}\cancel{\left[p^3 E f\right]_0^\infty} + \frac{g}{6\pi^2}\frac{1}{T}\int dp\left(3p^2E + \frac{p^4}{E}\right)f$$

$$= \frac{1}{T}\left[\frac{g}{2\pi^2}\int dp\, p^2 E f + \frac{g}{2\pi^2}\int dp\, p^2\frac{p^2}{3E}f\right]$$

$$\therefore \quad \boxed{\frac{\partial P}{\partial T} = \frac{1}{T}(\rho + P)}$$

② Proof of $dS = d\left[\frac{(\rho+P)}{T}\right]$

$$dS = \frac{dU + PdV}{T}$$

$$= \frac{d(\rho V) + PdV}{T}$$

$$= \frac{d[(\rho+P)V]}{T} - V\frac{dP}{T}$$

$$= \frac{d[(\rho+P)V]}{T} - \frac{V}{T^2}(\rho+P)\,dT \qquad \text{using } \frac{\partial P}{\partial T} = \frac{(\rho+P)}{T}.$$

$$= d\left[\frac{(\rho+P)V}{T}\right]$$

③ Proof of $\frac{dS}{dt} = 0$

$$\frac{dS}{dt} = \frac{d}{dt}\left[\frac{(\rho+P)V}{T}\right] \qquad V \propto a^3.$$

$$= \underbrace{\cancel{3H\frac{(\rho+P)V}{T}} + \cancel{\frac{\dot\rho V}{T}}}_{0} + \frac{\dot P V}{T} - \frac{(\rho+P)V}{T^2}\dot T$$

$[\because \ \dot\rho = 3H(\rho+P)]$

$$= \frac{V}{T}\Big[\underbrace{\dot P - \rho+P\frac{\dot T}{T}}_{0 \text{ because of ①}: \ \frac{\partial P}{\partial T} = \frac{\rho+P}{T}.}\Big] = 0$$

Implications of $\frac{d}{dt}\Big[\overbrace{\frac{(\rho+P)V}{T}}^{S}\Big] = 0$

Useful to define entropy density

$$s = \frac{S}{V} = \frac{\rho+P}{T}$$

$$\frac{dS}{dt} = 0 \Rightarrow \boxed{s \propto a^{-3}} \qquad (\because V \propto a^3)$$

IMPORTANT.

It turns out that this relationship is true for each species, even if they are at different temperatures T_i (some might have decoupled.)

ie $s_i = \frac{\rho_i + P_i}{T_i} \propto a^{-3}$

& $s = \sum_i \left(\frac{\rho_i + P_i}{T_i}\right) \propto a^{-3}$.

$\frac{\rho+P}{T}$ is dominated by relativistic species. Upon evaluation of the $\rho+P$ integrals we get.

$$s = \frac{2\pi^2}{45} g_*(T) T^3$$

effective relativistic d.o.f.

Implications of Entropy Conservation

$s \propto a^{-3} \Rightarrow \quad g_{*}(T)\, T^3 a^3 = \text{const}$.

If $g_{*s}(T)$ is not changing the temperature

$$\boxed{T \propto \frac{1}{a}}$$ ✓

Putting all this formalism to use:

(a) Neutrino decoupling.

Fermi's constant.

$\sigma \sim G_F^2 T^2$

$\Gamma = n\sigma v \sim n\sigma \quad G_F^2 T^5$

$G_F = 10^{-5}\ GeV^{-2}$

$$H \simeq \sqrt{\frac{g_* \pi^2}{90 m_{pl}^2}}\, T^2 \sim \frac{T^2}{m_{pl}}$$

← (get by dim analysis)

Decoupling happens when $\frac{\Gamma}{H} \lesssim 1$

To find when this happens.

$$\frac{\Gamma}{H} \sim 1 \Rightarrow \frac{G_F^2 T^5}{T^2/m_{pl}} \sim \left(\frac{T}{1 MeV}\right)^3$$

$\therefore$ $\boxed{T_{\nu\, dec} \simeq 1 MeV}$ $\Longleftrightarrow$ $\boxed{t_{\nu\, dec} \simeq 1 sec}$